International Journal of Mathematics Trends and Technology
ISSN: 2231 - 5373 / https://doi.org/10.14445/22315373/IJMTT-V68I4P508
Volume 68 Issue 4, 43-51, April 2022
© 2022 Seventh Sense Research Group®

On Vertex-Edge Eccentric Connectivity Index of Wheel related and Windmill Graphs

Shiladhar Pawar

Department of Studies in Mathematics
University of Mysore, Manasagangotri
Mysuru-570 006, India

Received Date: 13 March 2022 Revised Date: 15 April 2022 Accepted Date: 19 April 2022

Abstract-*In a graph $G = (V, E)$, a vertex $v \in V$, ve-dominates every edge incident to it as well as every edge adjacent to these incident edges. The vertex-edge degree of a vertex v, is denoted by $d_{ve}(v)$ and is the number of edges ve-dominated by v. In this paper, we introduce the vertex-edge eccentric connectivity index of a graph G, denoted by $\xi^c_{vee}(G)$ and is equal to the sum of the product of the connectivity and ve-degree of the vertices of G. We calculate the vertex-edge eccentric connectivity index of if certain graphs. More specifically, we obtain the vertex-edge eccentric connectivity index of some wheel related graphs and windmill graphs. Finally, we obtain some upper and lower bounds on $\xi^c_{vee}(G)$.*

Keywords- *ve-Degree of Vertex, Connectivity, Eccentricity Index.*
2010 Mathematics Subject Classification *: 05C05, 05C07, 05C35.*

1 *INTRODUCTION*

Throughout this paper, by $G = (V, E)$ we mean an undirected simple graph with vertex set V and edge set E. As usual, we denote the number of vertices and edges in a graph G by n and m respectively. The distance $d_G(u, v)$ between any two vertices $u, v \in V$ of G is equal to the length of a shortest path between u and v. For a vertex v of G, the eccentricity of v is $e(v) = \max\{d_G(v, u) : u \in V(G)\}$. The diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$ and the radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$. For a vertex $v \in V(G)$, the open neighborhood of v in G is denoted by $N(v)$ and is defined as $N(v) = \{u \in V(G) : d_G(u, v) = 1\}$ and the closed neighbourhood of v is defined as $N[v] = N(v) \cup \{v\}$. If $e(v) = rad(G) = diam(G)$, then G is called a self-centered graph. The wheel graph W_n with $n + 1$ vertices is defined to be the join of K_1 and C_n. The vertex corresponding to K_1 is known as the apex vertex while the vertices corresponding to C_n are known as rim vertices [12]. The helm graph H_n is a graph obtained from wheel graph W_n by attaching a pendant edge to each rim vertex. The helm

This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

graph contains three types of vertices, the vertex of degree n called apex, n pendant vertices and n rim vertices of degree four. Thus the helm graph H_n has $2n+1$ vertices and $3n$ edges. The gear graph G_n also sometimes known as a bipartite wheel graph, is a graph obtained from wheel graph W_n by adding a vertex between each pair of adjacent rim vertices. It contains three type of vertices, the vertex of degree n called apex, n vertices of degree three and n vertices of degree two. Thus gear graph G_n has $2n+1$ vertices and $3n$ edges. The Flower graph FL_n is the graph obtained from a helm graph by joining each pendant vertex to the apex of the helm graph. There are three types of vertices, the apex of degree $2n$, n vertices of degree four and n vertices of degree two. Thus the Flower graph FL_n has $2n+1$ vertices and $4n$ edges. The Sunflower graph SF_n is the graph obtained from the Flower graph FL_n by attaching n pendant edges to the apex vertex. Hence SF_n has four types of vertices, the apex vertex of degree $3n$, n vertices of degree four, n vertices of degree two and n pendant vertices. The French windmill graph F_n^m[7] is the graph obtained by taking $m \geq 2$ copies of the complete graph K_n, $n \geq 2$ with a vertex in common. That is $F_n^m = K_1 + \cup_{j=1}^{m} K_{n-1}$. Note that $F_2^m = K_{1,m}$. For any terminology or notation not mentioned here, we refer to[12]. A topological index of a G is a graph invariant number calculated from G. Various topological indices represent molecule structures and have got greater applications in chemistry. The Zagreb indices have been introduced, more than fifty years ago by Gutman and Trinajestic [9], in 1972, and studied by various authors in [1, 4, 10, 11, 13, 17]. Recently several graph invariants are defined based on vertex eccentricities and studied by so many authors. Analogously to Zagreb indices, Ghorbani et al.[8] and Vukičević et al.[16], defined the Zagreb eccentricity indices by replacing degrees by the eccentricity of vertices. A vertex v, ve-dominates every edge incident to v, as well as every edge adjacent with vertices in $N(v)$. That is, each edge incident to a vertex in $N[v]$. The concept of ve-degree of vertices in a graph G is defined by authors in[2]. Recently, Chellali et al.[3] studied properties of ve degrees of vertices in graphs. In [5], Ediz defined ve-degree atom-bond connectivity, ve-degree geometric-arithmetic, ve-degree harmonic and ve-degree sum-connectivity indices as parallel to their corresponding classical degree versions. Let us present some of the ve-degree based indices of graphs. The first ve-degree Zagreb alpha index[6] of G is defined as

$$S^{\alpha} = S^{\alpha}(G) = \sum_{v \in V(G)} d_{ve}(v)^2.$$

Further the first ve-degree Zagreb beta index[6] of G is defined as

$$S^{\beta} = S^{\beta}(G) = \sum_{uv \in V(G)} d_{ve}(u) + d_{ve}(v).$$

In this paper, motivated by connectivity index, we introduce the ve-degree eccentric connectivity index of a graph. For a connected graph $G = (V, E)$, the ve-degree of a vertex v, denoted by $d_{ve}(v)$, is the number of different edges which are incident with vertices in the closed neighborhood $N[v]$ of v [2]. The $ve-$*degree eccentric connectivity index* of G is defined as

$$\xi^c_{vee}(G) = \sum_{v\in V(G)} d_{ve}(v)e(v).$$

2 COMPUTATION OF $\xi^c_{vee}(G)$ OF SOME WHEEL RELATED GRAPHS AND WINDMILL GRAPHS

Theorem 2.1. For $n \geq 3$, let W_n be the wheel with $n+1$ vertices. Then

$$\xi^c_{vee}(W_n) = 2n(n+5).$$

Proof: Let W_n be the wheel with v_0 as the apex vertex and $v_1, v_2, \ldots, v_n$ as the rim vertices. Then $e(v_0) = 1$ and $e(v_i) = 2$ for $1 \leq i \leq n$. Further $d_{ve}(v_0) = 2n$ and $d_{ve}(v_i) = n+4$ for $1 \leq i \leq n$. By definition, we have

$$\begin{aligned}\xi^c_{vee}(W_n) &= \sum_{i=0}^{n} d_{ve}(v_i)e(v_i) = d_{ve}(v_0)e(v_0) + \sum_{i=1}^{n} d_{ve}(v_i)e(v_i)\\ &= 2n.1 + \sum_{i=1}^{n}(n+4).2 = 2n + 2n(n+4) = 2n(n+5).\end{aligned}$$

■

Theorem 2.2. For $n \geq 3$, let G_n be the gear graph with $2n+1$ vertices. Then

$$\xi^c_{vee}(G_n) = 3n(n+12).$$

Proof: Let G_n be the gear graph with $2n+1$ vertices, and let v_0 be the apex vertex, $v_1, \ldots, v_n$ be the rim vertices with degree three and $u_1, \ldots, u_n$ be the rim vertices with degree two. Then $d_{ve}(v_0) = 3n$, $d_{ve}(v_i) = n+4$, for $1 \leq i \leq n$ and $d_{ve}(u_i) = 6$ for $1 \leq i \leq n$. Further $e(v_0) = 2$, $e(v_i) = 3$ for $1 \leq i \leq n$ and $e(u_i) = 3$ for $1 \leq i \leq n$. By definition, we have

$$\begin{aligned}\xi^c_{vee}(G_n) &= \sum_{v\in V(G_n)} d_{ve}(v)e(v)\\ &= d_{ve}(v_0)e(v_0) + \sum_{i=1}^{n} d_{ve}(v_i)e(v_i) + \sum_{i=1}^{n} d_{ve}(u_i)e(u_i)\end{aligned}$$

$$= 3n.2 + \sum_{i=1}^{n}(n+4).3 + \sum_{i=1}^{n} 6.3$$
$$= 6n + 3n(n+4) + 18n = 3n(n+12).$$

■

Theorem 2.3. For $n \geq 3$, let H_n be the helm graph with $2n+1$ vertices. Then

$$\xi_{vee}^{c}(H_n) = n(3n+43).$$

Proof: Let H_n be the helm graph with $2n+1$ vertices and let v_0 be the apex vertex, $v_1, \ldots, v_n$ be the rim vertices with degree four and $u_1, \ldots, u_n$ be the pendant vertices. Then $d_{ve}(v_0) = 3n$, $d_{ve}(v_i) = n+7$ for $1 \leq i \leq n$ and $d_{ve}(u_i) = 4$ for $1 \leq i \leq n$. Further $e(v_0) = 2$, $e(v_i) = 3$ for $1 \leq i \leq n$ and $e(u_i) = 4$ for $1 \leq i \leq n$. By definition, we have

$$\xi_{vee}^{c}(H_n) = \sum_{v \in V(H_n)} d_{ve}(v)e(v)$$
$$= d_{ve}(v_0)e(v_0) + \sum_{i=1}^{n} d_{ve}(v_i)e(v_i) + \sum_{i=1}^{n} d_{ve}(u_i)e(u_i)$$
$$= 3n.2 + \sum_{i=1}^{n} 3(n+7) + \sum_{i=1}^{n} 4.4$$
$$= 6n + 3n(n+7) + 16n = n(3n+43).$$

■

Theorem 2.4. Let FL_n be the Flower graph with $2n+1$ vertices. Then

$$\xi_{vee}^{c}(FL_n) = 8n(n+3).$$

Proof: Let FL_n be the Flower graph with $2n+1$ vertices and $4n$ edges and let v_0 be the apex vertex, $v_1, \ldots, v_n$ be the rim vertices with degree four and $u_1, \ldots, u_n$ be the extreme vertices with degree two. Then $d_{ve}(v_0) = 4n$, $d_{ve}(v_i) = 2n+7$ for $1 \leq i \leq n$ and $d_{ve}(u_i) = 2n+3$ for $1 \leq i \leq n$. Further $e(v_0) = 1$, $e(v_i) = 2$ for $1 \leq i \leq n$ and $e(u_i) = 2$ for $1 \leq i \leq n$. By definition, we have

$$\xi_{vee}^{c}(FL_n) = \sum_{v \in V(Fl_n)} d_{ve}(v)e(v)$$
$$= d_{ve}(v_0)e(v_0) + \sum_{i=1}^{n} d_{ve}(v_i)e(v_i) + \sum_{i=1}^{n} d_{ve}(u_i)e(u_i)$$

$$
\begin{aligned}
&= 4n.1 + \sum_{i=1}^{n}(2n+7).2 + \sum_{i=1}^{n}(2n+3).2 \\
&= 4n + 2n(2n+7) + 2n(2n+3) = 8n(n+3).
\end{aligned}
$$

∎

Theorem 2.5. Let SFL_n be the Sunflower graph with $3n+1$ vertices. Then

$$\xi_{vee}^{c}(SFL_n) = n(18n+25).$$

Proof: Let SFL_n, be the Sunflower graph with $3n+1$ vertices and let v_0 be the apex vertex, $v_1, \ldots, v_n$ be the rim vertices of degree four, $u_1, \ldots, u_n$ be the extreme vertices of degree two and $w_1, \ldots w_n$ be the vertices of degree one. Then $d_{ve}(v_0) = 5n$, $d_{ve}(v_i) = 3n+7$, $d_{ve}(u_i) = 3n+3$, $d_{ve}(w_i) = 3n$, $e(v_0) = 1$, $e(v_i) = e(u_i) = e(w_i) = 2$, for $1 \leq i \leq n$. Be definition, we have

$$
\begin{aligned}
\xi_{vee}^{c}(SFL_n) &= \sum_{v \in V(Sf_n)} d_{ve}(v)e(v) \\
&= d_{ve}(v_0)e(v_0) + \sum_{i=1}^{n} d_{ve}(v_i)e(v_i) + \sum_{i=1}^{n} d_{ve}(u_i)e(u_i) \\
&\quad + \sum_{i=1}^{n} d_{ve}(w_i)e(w_i) \\
&= 5n.1 + \sum_{i=1}^{n}(3n+7).2 + \sum_{i=1}^{n}(3n+3).2 + \sum_{i=1}^{n}(3n).2 \\
&= 5n + 2n(3n+7) + 2n(3n+3) + 2n.3n = n(18n+25).
\end{aligned}
$$

∎

Theorem 2.6. Let $m, n \geq 2$ and let F_n^m be the French windmill graph. Then

$$\xi_{vee}^{c}(F_n^m) = m\frac{(n-1)}{2}\Big(2n^2 + 4mn - 5n - 4m + 4\Big).$$

Proof: Let F_n^m be theFrench windmill with v_0 as the central common vertex and let $v_{1j}, \ldots,$ $v_{(n-1)j}$ be the vertices of the j^{th} copy of K_{n-1} for $1 \leq j \leq m$. Then $d_{ve}(v_0) = \frac{mn(n-1)}{2}$, $d_{ve}(v_{ij}) = \frac{n(n-1)}{2} + (m-1)(n-1)$, $e(v_0) = 1$, $e(v_{ij}) = 2$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. By the definition, we have

$$\xi_{vee}^{c}(F_n^m) = \sum_{v \in V(F_n^m)} d_{ve}(v)e(v)$$

$$= d_{ve}(v_0)e(v_0) + \sum_{i=1}^{n-1}\sum_{j=1}^{m} d_{ve}(v_{ij})e(v_{ij})$$
$$= \Big(\Big(\frac{n(n-1)}{2}\Big)m\Big).1 + \sum_{i=1}^{n-1}\sum_{j=1}^{m}\Big(\frac{n(n-1)}{2} + (m-1)(n-1)\Big)2$$
$$= \Big(\frac{n(n-1)}{2}\Big)m + 2m(n-1)\Big(\frac{n(n-1)}{2} + (m-1)(n-1)\Big)$$
$$= \Big(\frac{n(n-1)}{2}\Big)m + m(n-1)\Big(n^2 - n + 2mn - 2n - 2m + 2\Big)$$
$$= m\frac{(n-1)}{2}\Big(n + 2(n^2 - n + 2mn - 2n - 2m + 2)\Big)$$
$$= m\frac{(n-1)}{2}\Big(2n^2 + 4mn - 5n - 4m + 4\Big).$$

∎

Theorem 2.7. Let C_{n+1}^m be the Kulli cycle windmill graph with, $n \geq 3$ with a vertex K_1, in common. Then

$$\xi_{vee}^c(C_{n+1}^m) = \begin{cases} m\Big(2n + n^2(n+2)\Big), & \text{if n is even,} \\ mn\Big(2 + (n^2 + n - 2)\Big), & \text{if n is odd.} \end{cases}$$

Proof: Let C_{n+1}^m, be the Kulli cycle windmill graph with, $n \geq 3$ with a vertex K_1, in common, if $n \geq 5$, $m \geq 2$. Let v_0 be the central vertex (common) and C_n^j, $j = 1, 2 \cdots, m$ be the j^{th} copy of C_n in C_{n+1}^m, with vertex set $V_j = \{V_{1j}, V_{2j}, \cdots, V_{(n)j}\}$. Then $d_{ve}(v_0) = 2mn$, $d_{ve}(v_i) = 2n+4$, $i = 1, 2, \cdots, n$. $e(v_0) = 1$, $e(v_i) = \frac{n}{2}$, if n is even $i = 1, 2, \cdots, n$ and $e(v_i) = \frac{n-1}{2}$, if n is odd. $i = 1, 2, \cdots, n$. By the definition we have

Case I: If $n \geq 4$, with a vertex K_1, if n is even.

$$\xi_{vee}^c(C_{n+1}^m) = d_{ve}(v_0)e(v_0) + \sum_{i=1}^{n} d_{ve}(v_i)e(v_i),$$
$$= (2mn)(1) + m\sum_{i=1}^{n}(2n+4)\Big(\frac{n}{2}\Big),$$
$$= 2mn + m\Big(2n(n+2)\frac{n}{2}\Big),$$
$$= m\Big(2n + n^2(n+2)\Big).$$

Case II: If $n \geq 3$, with a vertex K_1, if n is odd.

$$\xi_{vee}^c(C_{n+1}^m) = d_{ve}(v_0)e(v_0) + \sum_{i=1}^{n} d_{ve}(v_i)e(v_i),$$

$$
\begin{aligned}
&= (2mn)(1) + m\sum_{i=1}^{n}(2n+4)\Big(\frac{n-1}{2}\Big),\\
&= 2mn + m\Big(2n(n+2)\Big(\frac{n-1}{2}\Big)\Big),\\
&= 2mn + mn\Big((n^2+n-2)\Big),\\
&= mn\Big(2+(n^2+n-2)\Big).
\end{aligned}
$$

■

3 BOUNDS ON THE VE-DEGREE ECCENTRIC CONNECTIVITY INDEX OF GRAPHS

Theorem 3.1. Let G be a connected graph with n vertices. Then

$$\xi^c_{vee}(G) \geq \frac{1}{rad(G)+diam(G)}\Big[S^{\alpha}(G) + rad(G)diam(G)M_1(G)\Big].$$

Proof: For $i = 1, \ldots, n$, let $a_i = e(v_i)$ and $b_i = d_{ve}(v_i)$, $M = diam(G)$ and $m = rad(G)$. Then we get that

$$\sum_{i=1}^{n} d^2_{ve}(v_i) + rad(G)diam(G)\sum_{i=1}^{n} e^2(v_i) \leq (rad(G)+diam(G))\sum_{i=1}^{n} e(v_i)d_{ve}(v_i)$$

which implies that,

$$S^{\alpha}(G) + rad(G)diam(G)M_1(G) \leq (rad(G)+diam(G))\phi(G).$$

Therefore,

$$\xi^c_{vee}(G) \geq \frac{1}{rad(G)+diam(G)}\Big[S^{\alpha}(G) + rad(G)diam(G)M_1(G)\Big].$$

■

Now, we have the following corollary from Theorem 3.1.

Corollary 3.2. Let G be a graph with n vertices whose $\Delta \leq n-1$ and $\delta \geq 1$. Then

$$\xi^c_{vee}(G) \geq \frac{1}{n}\Big[S^{\alpha}(G) + M_1(G)\Big].$$

Theorem 3.3. Let G be a connected graph with n vertices. Then

$$\xi^c_{vee}(G) \geq \frac{\phi(G)}{n}\Big[M_1(G) - 3\eta(G)\Big],$$

where $\eta(G)$ is the total number of triangles in G. The bounds is sharp on the cycle C_n, $n \geq 3$ and the star $K_{1,n-1}$, $n \geq 2$.

Proof: Chebyshev's inequality states the following:For any non increasing sequences $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$,

$$n \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \tag{3}$$

is true. Let G be a connected graph with $V(G) = \{v_1, \ldots, v_n\}$. Now, by setting $a_i = e(v_i)$ and $b_i = d_{ve}(v_i)$ for $1 \leq i \leq n$ in (3), we get that

$$n \sum_{i=1}^{n} e(v_i) d_{ve}(v_i) \geq \sum_{i=1}^{n} e(v_i) \sum_{i=1}^{n} d_{ve}(v_i)$$

$$\geq \phi(G) \Big[M_1(G) - 3\eta(G) \Big].$$

and so,

$$n \xi^c_{vee}(G) \geq \phi(G) \Big[M_1(G) - 3\eta(G) \Big].$$

Therefore

$$\xi^c_{vee}(G) \geq \frac{\phi(G)}{n} \Big(M_1(G) - 3\eta(G) \Big).$$

■

4 *CONCLUSION*

We have proposed new vertex edge eccentrical connectivity topological index of graph G. We calculated of some wheel related graphs and windmill graphs and also we calculated some upper and lower bounds on $\xi^c_{vee}(G)$.

References

[1] B. Borovicanin, K. C. Das, B. Furtula, and I. Gutman, *Bounds for Zagreb indices*, MATCH Commun. Math. Comput. Chem., **78(1)** (2017), 17-100.

[2] R. Boutrig, M. Chellali, T. W. Haynes, and S. T. Hedetniemi, *Vertex-edge domination in graphs,* Aequationes Math. **90(2)** (2016), 355-366.

[3] M. Chellali, T.W. Haynes, S.T. Hedetniemi, T.M. Lewis, *On ve-degrees and ev-degrees in graphs*, Discrete Mathematics **340** (2017), 31-38.

[4] K. C. Das and I. Gutman, *Some properties of the second Zagreb index*, MATCH Commun. Math. Comput. Chem., **52** (2004), 103-112.

[5] S. Ediz, *On ve-degree molecular topological properties of silicate and oxygen networks*, Int. J. Comput. Sci. Math. **9(1)** (2018) 1-12.

[6] S. Ediz, *Predicting some physicochemical properties of octane isomers: A topological approach using evdegree and ve-degree Zagreb indices*, Int. J. Sys. And Sci. Appl. Math. Vol. 2, No. 5, 2017, pp. 87-92.

[7] J. Gallian, Dynamic Survey DS6, *Graph Labeling*, Electronic J. Combin., DS6, (2007), 1-58.

[8] M. Ghorbani and M. A. Hosseinzadeh, *A New Version of Zagreb Indices,* Filomat, **21(1)** (2012), 93-100.

[9] I. Gutman, *Multiplicative Zagreb indices of trees*, Bull. Soc. Math. Banja. Luka., **18** (2011), 17-23.

[10] I. Gutman, B. Ruscic, N. Trinajstic and C. F. Wilcox, *Graph theory and molecular orbitals, XII. Acyclic polyenes*, J. Chem. Phys., **62** (1975), 3399-3405.

[11] I. Gutman and N. Trinajstic, *Graph theory and molecular orbitals. Total π-electron energy of Alternant hydrocarbons*, Chem. Phys. Lett., **17** (1972), 535-538.

[12] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, Mass. Menlo Park, Calif. London, 1969.

[13] V. R. Kulli, B. Chaluvaraju and H. S. Boregowda, *Some degree based connectivity indices of Kulli cycle windmill graph*, South Asain J. Maths. **6(6)** (2016), 263-268.

[14] P. Shiladhar, A. M. Naji and N. D.Soner, *Leap Zagreb indices of Some wheel related Graphs*, J. Comp. Math. Sci., **9(3)** (2018), 221-231.

[15] P. Shiladhar, A. M. Naji and N. D.Soner, *Computation of leap Zagreb indices of Some Windmill Graphs*, Int. J. Math. And Appl., **6(2-B)** (2018), 183-191.

[16] D. Vukicevic and A. Graovac, *Note on the Comparison of the First and Second Normalized Zagreb Eccentricity Indices,* Acta Chimica Slovenica, **57** (2010), 524-528.

[17] K. Xu and H. Hua, *A unified approach to extremal multiplicative Zagreb indices for trees, unicyclicand bicyclic graphs*, MATCH Commun. Math. Comput. Chem., **68** (2012), 241-256.