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Original Article

Some Representations of Clifford Algebras

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Abstract – *In this paper, we will construct Clifford algebras as equivalence classes of polynomials in* n *free variables* X_1, X_2, . . . X_n. *Therefore some representations for functions theory in these algebras are investigated.*

Keywords - *Clifford algebras, Hyper complex analysis, Cauchy-Riemann system.*

I. INTRODUCTION

So far, as we know, the theory of a holomorphic function has not only reached its fullness and beauty in terms of structure but also enriched many applications in different fields. In the theory of partial differential equations sense, the theory of a holomorphic function $w = u + iv$ is essentially the theory of the solution of the following Cauchy-Riemann system

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

The real part and imaginary part of the holomorphic function are harmonic functions. But not with any two harmonic functions u and v, then $u + iv$ is a holomorphic function: They must be pairs of harmonic functions associated together by a specified rule (conjugate rule). Here, the conjugate rule is the Cauchy-Riemann condition.

The ideas of complex analysis started in the middle of the 18th century [15], first of all in connected with the Swiss mathematician, Leonhard Euler, and its mainly results in the 19th century have introduced by AugustinLouis Cauchy, Georg Friedrich Bernhard Riemann and Karl Theodor Wilhelm Weierstrass. As more and more new problems emerge from the realities that need to be solved, more research has been done to expand the Cauchy-Riemann system (which is also an extension of the theory of a holomorphic function) [see 1,14,16-25]. Looking back at these expansions, one can see that the authors find several ways, linking the harmonic functions together. There are many authors investigating about "ar´eolaire derivative" to solve particular problems that can be mentioned in [10-13].

In order to introduce complex number, usually one defines the product of an ordered pair of real numbers (a,b) with sum and product rules as follows:

$$(a,b)+(c,d)=(a+c,b+d),$$
$$(a,b).(c,d)=(ac-bd,ad+bc).$$

Special complex numbers $(a,0)$ can be identified with the real number a because

$$(a,0)+(c,0)=(a+c,0)=a+c,$$
$$(a,0).(c,0)=(ac,0)=ac.$$

Now we introduce another definition of complex numbers as linear polynomials, here, vector (a,b) in the plane can be interpreted by the linear polynomial $a+bX$, where X is a free variable. Therefore, the addition and product of vectors can be defined as addition and product of the corresponding polynomials:

$$\begin{aligned}(a+bX)+(c+dX)&=(a+c)+(b+d)X,\\(a+bX).(c+dX)&=ac+(ad+bc)X+bdX^2.\end{aligned}\tag{1}$$

However, there is indeed a possibility to remain in the plane: one only has only to identify the square X^2 with a real number, for instance, $X^2=-1$. From this we can reduce (1.1) to a linear polynomial

$$(a+bX).(c+dX)=(ac-bd)+(ad+bc)X.$$

Introduce an equivalence relation for polynomials

We known that, if we using $X^2=-1$ then the polynomials

$$P(X)=...+cX^k+...,(k\geq 2)$$

can be reduced to

$$P(X)=...+cX^{k-2}+...$$

Two polynomials $P(X)$ and $Q(X)$ are said to be equivalence if the difference is a polynomial such that each term contains the factor (X^2+1).

Consequently, complex number are equivalence classes of polynomials where two polynomials are said to be equivalent if their difference contains the factor X^2+1 (see[2,3,4,5]).

Remark 1.1. *If we identify the square X^2 with a real number 1, the one comes to that Guochu We called a hyperbolic number $a+bj$, where j is the hyperbolic unit, $j^2=1$ (see [26]).*

If we identify the square X^2 with a real number $-\beta X+\alpha$, then one comes to elliptic complex numbers which were introduced by I. M. Yaglom [27].

In the following we consider the similar problem in higher dimensions $\mathbb{R}^{n+1}, n\geq 2$.

II. CLIFFORD ALGEBRAS DEFINED BY EQUIVALENCE CLASSES

We now consider the Euclidean space $\mathbb{R}^{n+1}$ with coordinate $x_1,x_2,...,x_n$ and basic vector $e_1,e_2,...,e_n$. The point $(a_1,a_2,...,a_n)$ can be interpreted as linear polynomials

$$a_0+a_1X_1+...+a_nX_n$$

If the unit vector $e_0=(1,0,0,...,0)$ is identified with the real number one, while the unit vector $e_0=(0,...,j,...,0)$ is denoted by X_j. The addition and product of the polynomials can be written as

- $\left(a_0+a_1X_1+....+a_nX_n\right)+\left(b_0+b_1X_1+....+b_nX_n\right)=\left(a_0+b_0\right)+\left(a_1+b_1\right)X_1+...+\left(a_n+b_n\right)X_n$
- $\left(a_0+a_1X_1+....+a_nX_n\right).\left(b_0+b_1X_1+....+b_nX_n\right)=a_0b_0+a_0b_1X_1+...+a_nb_nX_n^2$

Now we consider the Clifford algebras as equivalence classes of polynomials. Let $R[X_1,X_2,...,X_n]$ be the ring of polynomials in the variables $X_1,X_2,...,X_n$. Then the points $\left(a_0,a_1,...,a_n\right)$ in $\mathbb{R}^{n+1}$ can be interpreted as linear polynomials

$$a_0 + a_1 X_1 + ... + a_n X_n .$$

In order to multiply points in $\mathbb{R}^{n+1}$ for higher-order polynomials, we need following properties:

Property 1. *If we have* $X_j^2 = -1,$ *then the degree of the polynomials can be reduced.*

Consider the Cauchy-Riemann operator and it adjoint

$$D = \partial_0 + \sum_{j=1}^{n} X_j \partial_j \, \overline{D} = \partial_0 - \sum_{j=1}^{n} X_j \partial_j ,$$

we have

$$\begin{aligned}\overline{D}D &= \partial_0^2 - \sum_{j=1}^{n} X_j^2 \partial_j^2 - \sum_{j<k} X_k X_j \partial_k \partial_j - \sum_{j>k} X_k X_j \partial_k \partial_j \\ &= \partial_0^2 + \sum_{j=1}^{n} \partial_j^2 - \sum_{j<k} \left(X_k X_j \partial_k \partial_j + X_j X_k \partial_j \partial_k \right) \\ &= \Delta_{n+1} - \sum_{j<k} \left(X_k X_j + X_j X_k \right) \partial_j \partial_k ,\end{aligned}$$

where Δ_{n+1} is Laplace operator in $\mathbb{R}^{n+1}$.

Property 2. *If*

$$X_k X_j + X_j X_k = 0, \forall k \neq j$$

is satisfied, then $\overline{D}D = \Delta_{n+1}$ *is true.*

Definition 2.1. *Two polynomials* $P(X_1, X_2, ..., X_n)$ *and* $Q(X_1, X_2, ..., X_n)$ *are called equivalent if their difference* $(P-Q)$ *is a polynomial such that each term contains at least one of the factors*

$$(X_j^2 + 1) \;\; or \;\; (X_j X_k + X_k X_j), \, j \neq k.$$

Equivalent polynomials P and Q are denoted by $P \sim Q$. The set of all equivalence classes with respect to relation " $\sim$ " is called the Clifford algebra generated by $X_1, X_2, ..., X_n$ and is denoted by $\mathcal{A}_n$. Note that, $X_k X_j + X_j X_k = 0$ implies that the product of polynomials is not commutative.

From Property 1 and Property 2, each equivalence class of polynomials can be written in the form

$$\sum_{\mu_1 < ... < \mu_m} a_{\mu_1 ... \mu_m} X_{\mu_1} ... X_{\mu_m} ,$$

where $a_{\mu_1 ... \mu_m}$ are real numbers. Indeed, if we have, for instance $X_{\mu_1} ... X_{\mu_m}$ then we have some following situations:

- If $X_{\mu_j} X_{\mu_k} = X_{\mu_1}^2$ then this can be replacsd by -1 ,
- If $X_{\mu_j} X_{\mu_k}, \mu_j \neq \mu_k$ then $(\mu_j > \mu_k)$

$$X_{\mu_j} X_{\mu_k} = -X_{\mu_k} X_{\mu_j} .$$

Therefore each term can be reduced to

$$X_1 = e_1, ..., X_n = e_n .$$

Introduce the abbreviations

$$X_1X_2 = e_{12}, ..., X_1...X_n = e_{1...n}.$$

Then the elements of $\mathcal{A}_n$ can be written in the form

$$\sum_A a_A e_A ,$$

where a_A are real numbers and A is a permutation of m element $\mu_1, ..., \mu_m$ with

$$1 \le \mu_1 < ... < \mu_m \le n.$$

Since the number of ways of selecting m objects out of n objects equals to $\binom{n}{m}$ the set $\mathcal{A}_n$ turns out to be a linear space of dimension

$$1 + n + \binom{n}{2} + ... + \binom{n}{n-1} + 1 = (1+1)^n = 2^n.$$

III. CLIFFORD ALGEBRAS $\mathcal{A}_n(p|k_j, \alpha_j(p), \gamma_{ij}(p))$

Let p be any parameter running in a subset of $\mathbb{R}$, and let $\alpha(p), \gamma_{ij}(p) = \gamma_{ji}(p)$ be real-valued function depending on p $(i, j = 1, ..., n$ and $i \ne j)$, $k_j \ge 2$, $j = 1, ..., n$ be natural number.

Let $\Re^*[X_1, ..., X_n]$ be the ring of polynomials in n free variables $X_1, ..., X_n$ with real coefficients. Then the vector of R^{n+1} can be identified with linear polynomials, where this identification preserves the linear structure. Again two terms $X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_m}$ differing in the order of the factors are to be distinguished. By that way one gets an infinite-dimensional extension of $\mathbb{R}^{n+1}$ in which the multiplication of vectors is possible. In order to get an only finite-dimensional extension, we consider equivalence classes of polynomials with respect to the finitely many structure polynomial

$$X_j^{k_j} + \alpha_j(p) \qquad \text{and} \qquad X_iX_j + X_jX_i - 2\gamma_{ij}(p), \tag{2}$$

where $i, j = 1, 2, ..., n$ and $i \ne j$.

Using these structure polynomials, each term of polynomial in $X_1, X_2, ..., X_n$ can be written in the form

$$cX_1^{\nu_1}.X_2^{\nu_2}...X_n^{\nu_n},$$

where c is a real constant and $0 \le \nu_j \le k_j - 1, j = 1, ..., n.$

The Clifford algebras generated by the structure polynomials (2) will be denoted by

$$\mathcal{A}_n(p|k_j, \alpha_j(p), \gamma_{ij}(p)) \text{ if } n \ge 2 \text{ and } \mathcal{A}_1(p|k, \alpha(p)) \text{ if } n = 1.$$

In case coefficients α_j, γ_{ij} do not depend on the parameter p, we write $\mathcal{A}_n(k_j, \alpha_j, \gamma_{ij})$ and $\mathcal{A}_1(k, \alpha)$ resp, and specially, the usual Clifford algebra is $\mathcal{A}_n(2,1,0)$.

The Clifford algebras $\mathcal{A}_n(p|k_j, \alpha_j(p), \gamma_{ij}(p))$ has the basis

$$e_1^{\nu_1}e_2^{\nu_2}...e_n^{\nu_n}, 0 \le \nu_j \le k_j - 1, j = 1, 2, ..., n,$$

and so, we have

$$\dim \mathcal{A}_n(p|k_j, \alpha_j(p), \gamma_{ij}(p)) = k_1.k_2 \dots k_n.$$

For instance, $\mathcal{A}_2(k_j, 1,0)$ with $k_1 = 2$ and $k_2 = 3$ has the dimension 6, its basis is

$$1, e_1, e_2, e_1e_2, e_2^2, e_1e_2^2.$$

Similarly, $\mathcal{A}_n(3,1,0)$ has the dimension 9, its basis is

$$1, e_1, e_2, e_1^2, e_1e_2, e_2^2, e_1^2e_2, e_1e_2^2, e_1^2e_2^2.$$

We consider now some special cases for $\mathcal{A}_n(p|k_j, \alpha_j(p), \gamma_{ij}(p))$

3.1. Fourth elliptic algebras $\mathcal{A}_1(4,1)$

A Clifford algebra $\mathcal{A}_1(4,1)$ is generated by $1, e_1 = X, e_2 = X^2, e_3 = X^3$. This implies the relations

$$e_1e_1 = e_2, e_1e_2 = e_2e_1 = e_3, e_1e_3 = e_3e_1 = e_2e_2 = -1, e_2e_3 = -e_1$$

and so on.

On the other hand, if we identify X by unit element i then we can present a vector $x \in \mathbb{R}^4$ by

$$x = (x_0, x_1, x_2, x_3) = x_0 + x_1 i + x_2 i^2 + x_3 i^3,$$

where $i^4 = 1$.

It is clear that this multiplication law is associative and commutative. We will use the modulus of vector as normal Euclid modulus

$$|x| := \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

We will call our new structure is fourth elliptic complex algebras and denote it by $\mathbb{C}^{(4)}$.

Let Ω be a domain in $\mathbb{R}^2$. Let u be a function defined in Ω and valued in $\mathbb{C}^{(4)}$, $u \in C^1(\Omega)$.

$$u: \Omega \to \mathbb{C}^{(4)};$$

$$u := u_0 + u_1 i + u_2 i^2 + u_3 i^3;$$

$$u_j := u_j(x_0, x_1); u_j \in C^1(\Omega); j = 0,1,2,3.$$

We give the definition of the Cauchy-Riemann operator:

$$\partial := \partial_0 + i\partial_1 = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}.$$

So we have the following Cauchy-Riemann system:

$$\partial u = 0 \Leftrightarrow \begin{cases} \partial_0 u_0 - \partial_1 u_3 = 0 \\ \partial_0 u_1 + \partial_1 u_0 = 0 \\ \partial_0 u_2 + \partial_1 u_1 = 0 \\ \partial_0 u_3 + \partial_1 u_2 = 0. \end{cases} \tag{3}$$

Definition 3.1. *Let Ω be a domain in $\mathbb{R}^2$. A function $u \in C^1(\Omega)$ is called an analytic function in Ω if $\partial u = 0$ in Ω. The set of all* *analytic function in* Ω *denoted by* $\mathbb{A}^{(4)}(\Omega)$.

In the following, we give some results for the analytic function taking values in $\mathbb{C}^{(4)}$ (see more detail in [2]):

Lemma 3.1. *If* u *ia an analytic function in* Ω *then its components will satisfy the equation*

$$(\partial_0^4+\partial_1^4)u_j=0;\ j=\overline{0,3}.$$

(4)

For convenient, form here we will denote $z:=x_0+i^3x_1$ and $u(z)=u_0+u_1i+u_2i^2+u_3i^3, u_j=u_j(z);\ j=\overline{0,3}.$

Lemma 3.2. *If* $u,v\in C^1(\Omega)$ *then we have the following formula:*

$$\partial(u,v)=\partial(u).v+u.\partial(v).$$

Theorem 3.1. *Let* Ω *be a connected domain in* $\mathbb{R}^2$. *If* $u(z)$ *is an analytic function in* $\overline{\Omega}$, *then*

$$\int_{\partial\Omega}u(z)dz=0.$$

Theorem 3.2. *Let* Ω *be a simply connected domain in* $\mathbb{R}^2$ *If* $u(z)$ *is an analytic function in* $\overline{\Omega}$ *then*

$$u(z)=\frac{1}{\beta_0}\int_{\partial\Omega}\frac{u(\zeta)d\zeta}{\zeta-z},$$

where $\beta_0=\sqrt{2}\pi i(i^2+1);\zeta=\zeta_0+i^3\zeta_1;z=x_0+i^3x_1.$

3.2. Clifford algebra $\mathcal{A}_2(4,1,0)$

We will now take our attention to the case $n=2,k_j=4,\alpha_j=1$ and $\gamma_{ij}=0.$ That is the algebra are

$$1,e_1,e_1^2,e_1^3,e_2,e_2^2,e_2^3,$$

where

$$e_1^4=e_1^2=-1,e_1e_2=e_2e_1.$$

Each element of $\mathcal{A}_2(4,1,0)$ can be represented by

$$\begin{aligned}u&=u_0+e_1u_1+e_1^2u_2+e_1^3u_3+e_2v_1+e_2^2v_2+e_2^3v_3+e_1e_2v_4+e_1^2e_2v_5+e_1^3v_6+e_1e_2^2v_7\\&\quad+e_1e_2^3v_8+e_1^2e_2^2v_9+e_1^2e_2^3v_{10}+e_1^3e_2^3v_{11}+e_1^3e_2^3v_{12}.\end{aligned}$$

Definition 3.2. *A function* $u(x,y)$ *to be called is defined in a domain* $\Omega\subset\mathbb{R}^4(x_0,x_1,y_0,y_1)$ *taking values in* $\mathcal{A}_2(4,1,0)$ *if all* $u_k,v_j,k=0,1,2,3;\ j=1,2,...,12$ *are real-valued functions defined in* Ω.*And* $u(x,y)$ *said to be a continuous are* $k-$*continuously differentiable* (C^k) *if all* u_k,v_j *are belonging to these classes, respectively.*

The generalized Cauchy-Riemann operators in $\mathcal{A}_2(4,1,0)$ are defined as

$$\begin{aligned}\partial_x&=\partial_{x_0}+e_1^2\partial_{x_1}\\ \partial_y&=\partial_{y_0}+e_1^2\partial_{y_1}.\end{aligned}$$

Their conjugate operators operators are defined by

$$\overline{\partial_x} = \partial_{x_0} - e_1^2 \partial_{x_1}$$
$$\overline{\partial_y} = \partial_{y_0} - e_2^2 \partial_{y_1}.$$

It follows that

$$\partial_x \overline{\partial_x} = \Delta_x$$
$$\partial_y \overline{\partial_y} = \Delta_y.$$

Definition 3.3. *A function $u(x,y)$ difined in Ω is said to be multi-monogenic if*

$$u(x,y) \in C^1(\Omega) \text{ and } \partial_x u = \partial_y u = 0.$$

Denote by $\mathcal{M}M(\Omega, \mathcal{A}_2(4,1,0))$ the class of all multi-monogenic functions.

Remark 3.1. *If $u \in C^1(\Omega, \mathcal{A}_2(4,1,0)) \cap \mathcal{M}M(\Omega, \mathcal{A}_2(4,1,0))$ then each element of u is harmonic on x_0, x_1, y_0, y_1.*

In the following we will give some result for theory function taking values in $\mathcal{A}_2(4,1,0)$ (see [1]).

Suppose $u \in \mathcal{M}M(\Omega, \mathcal{A}_2(4,1,0))$ we have

$$\begin{aligned}\partial_x u &= (\partial_{x_0} + e_1^2 \partial_{x_1})(u_0 + e_1 u_1 + e_1^2 u_2 + e_1^3 u_3 + e_2 v_1 + e_2^2 v_2 + e_2^3 v_3 + e_1 e_2 v_4 + e_1^2 e_2 v_5 + e_1^3 e_2 v_6 + e_1 e_2^2 v_7 \\ &\quad + e_1 e_2^3 v_8 + e_1^2 e_2^2 v_9 + e_1^2 e_2^3 v_{10} + e_1^3 e_2^2 v_{11} + e_1^2 e_2^3 v_{12} \\ &= (\partial_{x_0} u_0 - \partial_{x_1} u_2) + e_1(\partial_{x_0} u_1 - \partial_{x_1} u_3) + e_2(\partial_{x_0} v_1 - \partial_{x_1} v_5) + e_1^2(\partial_{x_0} u_2 + \partial_{x_1} u_0) + e_1^3(\partial_{x_0} u_3 + \partial_{x_1} u_1) \\ &\quad + e_2^2(\partial_{x_0} v_2 - \partial_{x_1} v_9) + e_2^3(\partial_{x_0} v_3 - \partial_{x_1} v_{10}) + e_1 e_2(\partial_{x_0} v_0 - \partial_{x_1} v_6) + e_1^2 e_2(\partial_{x_0} v_1 + \partial_{x_1} v_5) \\ &\quad + e_1^3 e_2(\partial_{x_0} v_6 + \partial_{x_1} v_4) + e_1^2 e_2^2(\partial_{x_0} v_9 + \partial_{x_1} v_2) + e_1^2 e_2^3(\partial_{x_0} v_{10} + \partial_{x_1} v_3) + e_1^3 e_2^3(\partial_{x_1} v_8 + \partial_{x_0} v_{12}).\end{aligned} \tag{5}$$

Therefore $\partial_x u = 0$ if and only if the following systems are satisfy:

$$\begin{cases} \partial_{x_0} u_0 - \partial_{x_1} u_2 = 0 \\ \partial_{x_1} u_0 + \partial_{x_0} u_1 = 0, \end{cases}$$

$$\begin{cases} \partial_{x_0} u_1 - \partial_{x_1} u_3 = 0 \\ \partial_{x_0} u_3 + \partial_{x_1} u_1 = 0, \end{cases}$$

$$\begin{cases} \partial_{x_0} v_1 - \partial_{x_1} v_5 = 0 \\ \partial_{x_1} v_1 + \partial_{x_0} v_5 = 0 \end{cases}$$

$$\begin{cases} \partial_{x_0} v_2 - \partial_{x_1} v_9 = 0 \\ \partial_{x_0} v_9 + \partial_{x_1} v_2 = 0, \end{cases}$$

$$\begin{cases} \partial_{x_0} v_4 - \partial_{x_1} v_6 = 0 \\ \partial_{x_0} v_6 + \partial_{x_1} v_4 = 0, \end{cases}$$

$$\begin{cases} \partial_{x_0} v_3 - \partial_{x_1} v_{10} = 0 \\ \partial_{x_0} v_{10} + \partial_{x_1} v_3 = 0, \end{cases}$$

$$\begin{cases} \partial_{x_0} v_8 - \partial_{x_1} v_{12} = 0 \\ \partial_{x_0} v_{12} + \partial_{x_1} v_8 = 0, \end{cases}$$

$$\begin{cases} \partial_{x_0} v_7 - \partial_{x_1} v_{11} = 0 \\ \partial_{x_1} v_7 + \partial_{x_0} v_{11} = 0 \end{cases}.$$

If we set

$$\begin{aligned} &W_1 = u_0 + iu_2; W_2 = u_1 + iu_3; W_3 = v_1 + iv_5; W_4 = v_2 + iv_6; \\ &W_5 = v_4 + iv_6; W_6 = v_3 + iv_{10}; W_7 = v_8 + iv_{12}; W_8 = v_7 + iv_9; \\ &z = x_0 + ix_1, \end{aligned}$$

(6)

then we have following lemma:

Lemma 3.3. *If* $\partial_x u = 0$ *then* $W_1, W_2, ..., W_8$ *are the holomorphic functions on variable* z_1.

By the similarity calculation and setting $z_2 = y_0 + iy_1$, we have

Lemma 3.4. *If* $\partial_y u = 0$ *then* $W_1, W_2, ..., W_8$ *are the holomorphic functions on variable* z_2.

Theorem 3.3. *If* u *is a multi-monogenic function then* $W_1, W_2, ..., W_8$ *defined by (6) are the holomorphic of two Complex variables* z_1 *and* z_2. *Conversely from* 8 *given holomorphic functions* $W_1, W_2, ..., W_8$ *(of* z_1 *and* z_2*) we can get a multi-monogenic function* $u \in \mathcal{MM}(\Omega, \mathcal{A}_2(4,1,0))$.

Let $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 be a domain in $\mathbb{R}^2(x_0, x_1)$ and Ω_2 be a domain in $\mathbb{R}^2(y_0, y_1)$. Then we have:

Theorem 3.4. (Hartogs extension theorem). *Suppose that* u *is a given multi-monogenic function in* $\sum$, *where* $\sum$ *is an open neighborhood of* $\partial\Omega$, *so that* $\bar{u} \equiv u$ *in* $\sum$.

IV. CONCLUSION

The above results show that one can construct the Clifford algebras generaliza in may way, by equivalence classes of polynomials in n free variables $X_1, X_2, \ldots X_n$ one can construct more diverse structures of Clifford algebras. The richer problems for functions taking value in hypercomplex and Clifford algebras depending on parameter can be investigated.

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