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*Original Article*

# Local Existence and Uniqueness of Solutions to Yang-Mills Heat Flow Problem

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***Abstract -*** *In this paper, we consider the following nonlinear equation*

$$\frac{\partial u}{\partial t} = \partial_r^2 u + \frac{d+1}{r}\partial_r u - 3(d-2)u^2 - (d-2)r^2u^3$$

*where* $u:(r,t)\in\mathbb{R}_+^2 \mapsto \mathbb{R}, d\in\mathbb{N}^*$. *This equation has been investigated by Grotowski in 2001 in studying the Yang-Mills heat flow connections on Riemann manifolds. In the paper, we prove the local Cauchy problem for above equation that is well-posed in* $L^\infty_{1+r^2}(\mathbb{R}_+)$. *More precisely, for any initial data* $u_0 \in L^\infty_{1+r^2}(\mathbb{R}_+)$, *there exists* $T(u_0)>0$ *such that the above equation has a unique solution* $u(t)\in L^\infty_{1+r^2}(\mathbb{R}_+)$ *for all* $t\in[0,T(u_0)]$.

***Keywords -*** *Local Cauchy problem, Local existence and uniqueness problem, Yang-Mills heat flows, Yang-Mills connections, Geometric flows.*

## I. INTRODUCTION

In this paper, we are interested in the following nonlinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \partial_r^2 u + \dfrac{d+1}{r}\partial_r u - 3(d-2)u^2 - (d-2)r^2u^3, \ (r,t)\in(0,+\infty)\times[0,T), \\ u(0) = u_0 \in L^\infty(\mathbb{R}_+), \end{cases} \tag{1}$$

where $u:(r,t)\in\mathbb{R}_+^2 \mapsto \mathbb{R}, d\in\mathbb{N}^*$. Equation (1) has a strong connection to the problem of Yang Mills connections in $\mathbb{R}^d \times SO(d)$ which is an important part in Yang-Mills theory. We would like to mention that the theory was used to study the weak nuclear forces, governing the nuclear decay of some particles which is considered as a non-commutative version of Maxwell's electromagnetism, see more details in [1]-[6]. In particular, the Yang-Mills heat flow has received a lot of attention from both mathematics and physics communities. Results on existence and uniqueness of weak solutions in other functional spaces were obtained in [7] for $d=2,3$, and [8] and [9] for $d=4$; global existence has proved in [10]-[12]; the regularity was established in [13] in for higher dimensions; stability were proved in [14] and [15]; singularity formations have been studied in [16]-[24], the local well posedness was studied in [25] and [26] in Sobolev spaces . In particular, the study of Yang-Mills connections usually is considered in abstract spaces such as Riemann manifolds which get lot of inconveniences. For that reason, the author in [17] reduced the study of Yang-Mills connections to the problem introduced in (1).

The main goal of this paper is to study the local Cauchy problem to (1) since strong connections to the Yang-Mills problem. Recently, the authors in [24] solved the local Cauchy problem for (1) in $C_0^\infty(\mathbb{R}_+)$ defined as the set of smooth functions with compact supports. However, it remains open if the local Cauchy problem in can be solved $L^\infty(\mathbb{R}_+)$. In this work, we aim to prove the Cauchy problem in $L^\infty_{1+r^2}(\mathbb{R}_+)\subset L^\infty(\mathbb{R}_+)$ which is strictly larger than $C_0^\infty(\mathbb{R}_+)$.

For more convenience, we mention below some important notations which will be used in the proof of the paper. Let $\Omega$ be a Lebesgue measurable set in $\mathbb{R}^n, n \in \mathbb{N}^*$, we denote $L^\infty(\Omega)$ as the set of all Lebesgue measurable functions $f$ on $\Omega$ satisfying

$$\inf\left\{B \text{ such that } \mu\left(\left\{x \in \Omega : \left|f(x)\right| > B\right\}\right) = 0\right\} < +\infty$$

which is a Banach space with the following norm

$$\|f\|_{L^\infty(\Omega)} = \inf\left\{B \text{ such that } \mu\left(\left\{x \in \Omega : \left|f(x)\right| > B\right\}\right) = 0\right\}.$$

Similarly, we also define $L^\infty_{1+|x|^2}(\Omega)$

$$L^\infty_{1+|x|^2}(\Omega) = \left\{f \in L^\infty(\Omega) \text{ such that } \left\|\left(1+|x|^2\right) f(.)\right\|_{L^\infty(\Omega)} < +\infty\right\}$$

and it is also a Banach space with the norm

$$\|f\|_{L^\infty_{1+|x|^2}(\Omega)} = \left\|\left(1+|x|^2\right) f(.)\right\|_{L^\infty(\Omega)}.$$

We call $f$ a radially symmetric function on $\Omega$ if and only if for all orthogonal matrices $A$ and $x \in \Omega$, then it satisfies that $Ax \in \Omega$ and $f(Ax) = f(x)$. Hence, we introduce

$$L^\infty_{rad}(\Omega) = \left\{f \in L^\infty(\Omega) \text{ such that } f \text{ radially symmetric}\right\},$$

which is a Banach space with $\|.\|_{L^\infty(\Omega)}$ norm. In particular, we also define the abstract space of Banach-valued functions $X^T, T > 0$ by

$$X^T = L^\infty\left([0,T], L^\infty_{rad}(\Omega) \cap L^\infty_{1+|x|^2}(\Omega)\right)$$

which is also a Banach space with the norm

$$\|z\|_{X^T} = \|Z(t)\|_{L^\infty([0,T])} \text{ where } Z(t) = \|z(t)\|_{L^\infty_{1+|x|^2}(\Omega)}.$$

Let $\Delta$ be Laplace operator in Euclide space $\mathbb{R}^{d+2}$ defined by $\Delta = \sum_{j=1}^{d+2} \partial^2_{x_j x_j}$. By taking $\Omega = \mathbb{R}^{d+2}$, we recall the definition of semi-group $\left\{e^{\Delta t}\right\}_{t>0}$ as follows

$$e^{\Delta t} : L^\infty\left(\mathbb{R}^{d+2}\right) \to L^\infty\left(\mathbb{R}^{d+2}\right)$$

and

$$e^{\Delta t} f(x) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} f(y) e^{-\frac{|x-y|^2}{4t}} dy. \tag{2}$$

We would like to mention [27, Proposition 48.4] the following fundamental estimate

$$\left\|e^{\Delta t} f\right\|_{L^\infty\left(\mathbb{R}^{d+2}\right)} \le \|f\|_{L^\infty\left(\mathbb{R}^{d+2}\right)}. \tag{3}$$

## II. MAIN RESULTS

In this section, we aim to prove the local Cauchy problem for equation (1) in $L^\infty_{1+|x|^2}(\mathbb{R}_+)$. However, it is so hard to give a direct proof to equation (1) due to complexity of the linear operator

$$\partial_r^2 + \frac{d+1}{r}\partial_r.$$

To overcome this challenge, we used the idea investigated in [28] where the authors successfully handled the Cauchy problem for harmonic heat flows. Let $f$ be a function defined in $\mathbb{R}_+$, then we denote $\tilde{f}$ as $f$ 's extension on $\mathbb{R}^{d+2}$ given by

$$\tilde{f}(x) = f(|x|). \tag{4}$$

We can see that the extension is always a radially symmetric function on $\mathbb{R}^{d+2}$. In particular, once $f \in C^2(\mathbb{R}_+)$, then we have $\tilde{f} \in C^2\left(\mathbb{R}^{d+2}\right)$ and the following identity holds

$$\Delta\tilde{f}(x) = \partial_r^2 f + \frac{d+1}{r}\partial_r f(|x|).$$

Let us consider $u$ is a $C^2(\mathbb{R}_+)$-solution to (1) (so-called the classical solution), then the extension $\tilde{u}$ belongs to $C^2(\mathbb{R}^{d+2})$ and $\tilde{u}$ satisfies the following equation

$$\begin{cases} \partial_t\tilde{u} = \Delta\tilde{u} - 3(d-2)\tilde{u}^2 - (d-2)|x|^2\tilde{u}^3, (x,t) \in \mathbb{R}^{d+2}\times(0,T), \\ \tilde{u}(0) = \tilde{u}_0 \in L^\infty_{rad}(\mathbb{R}^{d+2}). \end{cases} \tag{5}$$

From the symmetricity of equation (5), the solution $\tilde{u}$ remains radially symmetric as well as the solution exists. In the following, we aim to prove the local Cauchy problem to equation (5) in $L^\infty_{rad}(\mathbb{R}^{d+2}) \cap L^\infty_{1+|x|^2}(\mathbb{R}^{d+2})$. Firstly, we have the following Lemma:

**Lemma 2.1.** Let us consider $\alpha$ be a positive number and $f \in L^\infty(\mathbb{R}^{d+2})$ satisfying

$$\left\|\left(1+|x|^\alpha\right) f(.)\right\|_{L^\infty(\mathbb{R}^{d+2})} \le A. \tag{6}$$

Then, it holds that

$$\left\|\left(1+|x|^\alpha\right) e^{t\Delta} f\right\|_{L^\infty(\mathbb{R}^{d+2})} \le C(\alpha) A, t>0,$$

where the semi-group $e^{t\Delta}$ defined as in (2).

*Proof:* By the explicit formula in (2), we write as follows

$$e^{\Delta t} f(x) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

Let us consider $R>0$ large enough and fixed later. For all $|x| \le R$, we use (6) to derive

$$\left|(1+|x|^\alpha)e^{\Delta t} f(x)\right| \le \frac{A(1+R^\alpha)}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} \frac{e^{-\frac{|x-y|^2}{4t}}}{1+|y|^\alpha} dy \le C(R,\alpha)A.$$

It remains to the case $|x| \ge R$. Let us define $K_1 = \left\{y : |y| \le \frac{|x|}{2}\right\}$ and $K_1 = \left\{y : |y| > \frac{|x|}{2}\right\}$. It is obvious that $K_1 \cup K_2 = \mathbb{R}^{d+2}$. Then, we have the decomposition

$$\left(1+|x|^\alpha\right)e^{\Delta t} f(x) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} \frac{1+|x|^\alpha}{1+|y|^\alpha} e^{-\frac{|x-y|^2}{4t}} \left(1+|y|^\alpha\right) f(y)dy = I_1 + I_2,$$

where

$$I_1 = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{K_1} \frac{1+|x|^\alpha}{1+|y|^\alpha} e^{-\frac{|x-y|^2}{4t}} \left(1+|y|^\alpha\right) f(y)dy,$$

and

$$I_2 = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{K_2} \frac{1+|x|^\alpha}{1+|y|^\alpha} e^{-\frac{|x-y|^2}{4t}} \left(1+|y|^\alpha\right) f(y)dy.$$

+ For $I_1$: We use the facts that $t \in (0,1)$ and $y \in K_1, |x-y| \ge |x| - |y| \ge \frac{|x|}{2}$, then it follows

$$|I_1| \le \frac{A}{(4\pi t)^{\frac{d+2}{2}}} \int_{K_1} \left(1+|x|^\alpha\right) e^{-\frac{|x-y|^2}{4t}} dy.$$

Using the changing variable $z=\frac{x-y}{2\sqrt{t}}$ , we get the following

$$\begin{aligned}
&\frac{1}{(4\pi t)^{\frac{d+2}{2}}}\int_{K_1}\left(1+|x|^{\alpha}\right)e^{-\frac{|x-y|^2}{4t}}dy\\
&=\frac{\left(1+|x|^{\alpha}\right)}{(\pi)^{\frac{d+2}{2}}}\int_{y\in K_1,z=\sqrt{t}y-x}e^{-z^2}dz\\
&\le\frac{\left(1+|x|^{\alpha}\right)}{(\pi)^{\frac{d+2}{2}}}\int_{\frac{|x|}{2}\le|z|\le\frac{3|x|}{2}}e^{-z^2}dz\le C\left(1+|x|\right)^{d+2+\alpha}e^{-\frac{|x|^2}{4}}\\
&\le C(R,\alpha),
\end{aligned}$$

which implies $|I_1|\le C(R,\alpha)A$.

+ For $I_2$: Using the fact that $y\in K_2, |y|\ge\frac{|x|}{2}$, then it follows $\frac{1+|x|^{\alpha}}{1+|y|^{\alpha}}\le C(\alpha)$. Hence, we estimate $I_2$ as follows

$$|I_2|\le\frac{C(\alpha)A}{(4\pi t)^{\frac{d+2}{2}}}\int_{K_2}e^{-\frac{|x-y|^2}{4t}}dy\le C(\alpha)A.$$

Thus, we conclude that for all $|x|\ge R$, we have

$$\left|(1+|x|^{\alpha})e^{\Delta t}f(x)\right|\le C(R,\alpha)A.$$

Finally, we get the conclusion of the proof.

Now, we have the following result:

**Proposition 2.2.** Let $\tilde{u}_0\in L^{\infty}_{rad}\left(\mathbb{R}^{d+2}\right)\cap L^{\infty}_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)$ arbitrarily, there exists $T=T(\tilde{u}_0)>0$ *such that problem (5) has a unique solution on* $[0,T]$ *and* $\tilde{u}(t)\in L^{\infty}_{rad}\left(\mathbb{R}^{d+2}\right)\cap L^{\infty}_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)$ *for all* $t\in[0,T]$. *In particular, the following estimate holds*

$$\|\tilde{u}\|_{X^T}\le\|\tilde{u}_0\|_{L^{\infty}_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)}+1.$$

*Proof:* The result follows from the Banach fixed point theorem (the unique fixed point of a constructive mapping). Let us consider initial data $\tilde{u}_0\in L^{\infty}_{rad}\left(\mathbb{R}^{d+2}\right)\cap L^{\infty}_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)$ and $T>0$ and define

$$B_0=\left\{\tilde{u}\in X^T \text{ such that } \|\tilde{u}-e^{t\Delta}(\tilde{u}_0)\|_{X^T}\le 1\right\}.$$

In addition, we define $\Gamma$ a mapping on $X^T$

$$\Gamma(\tilde{u})(t)=e^{t\Delta}\tilde{u}_0+\int_0^t e^{(t-s)\Delta}F\left(s,\tilde{u}(s)\right)ds, t>0 \text{ and } \Gamma(z)(0)=\tilde{u}_0, \tag{7}$$

and $F$ defined by

$$F(\tilde{u})=-3(d-2)\tilde{u}^2-(d-2)|x|^2\tilde{u}^3. \tag{8}$$

In the below, we aim to prove that once $T$ small enough, $\Gamma$ satisfies the following properties:

(H1): $\Gamma$ maps $B_0$ into itself.

(H2): $\Gamma$ is a contraction mapping on $B_0$ i.e. there exists $\lambda\in(0,1)$ such that

$$\|\Gamma(\tilde{u}_1)-\Gamma(\tilde{u}_2)\|_{X^T}\le\lambda\|\tilde{u}_1-\tilde{u}_2\|_{X^T}, \forall\tilde{u}_1,\tilde{u}_2\in B_0.$$

**-** *Proof of* (H1): Let $\tilde{u}\in B_0$ arbitrarily, we derive from (8) and the fact $\|\tilde{u}\|_{X^T}\le 1$

$$\left\|\left(1+|x|^2\right)F\left(\tilde{u}(s)\right)\right\|_{L^\infty\left(\mathbb{R}^{d+2}\right)} \le C\left(\left\|\tilde{u}(s)\right\|^2_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)} + \left\|\tilde{u}(s)\right\|^3_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)}\right) \le C\left\|\tilde{u}(s)\right\|_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)},$$

.

Applying Lemma 2.1 with $\alpha = 2$ and $t-s \in (0,1)$, we get

$$\left\|e^{(t-s)\Delta}F\left(\tilde{u}(s)\right)\right\|_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)} \le C_1\left\|\tilde{u}\right\|_{X^T},$$

which yields

$$\left\|\Gamma(\tilde{u}) - e^{t\Delta}\tilde{u}_0\right\|_{X^T} = \left\|\int_0^t e^{(t-s)\Delta}F\left(\tilde{u}(s)\right)ds\right\|_{X^T} \le C_1\sqrt{T}\left\|\tilde{u}\right\|_{X^T}, \tag{9}$$

for all $\tilde{u} \in B_0$. Taking $T \le \left(\frac{1}{C_1}\right)^2$, we get

$$\left\|\Gamma(\tilde{u}) - e^{t\Delta}\tilde{u}_0\right\|_{X^T} \le \left\|\tilde{u}\right\|_{X^T}, \forall \tilde{u} \in B_0.$$

In addition, since $\tilde{u}$ is radially symmetric, so $F(\tilde{u})$ defined as in (8) is and the convolution in (2) saves the symmetry that leads $\Gamma(\tilde{u}(t))$ is radially symmetric for all $t \in [0,T]$. Finally, we conclude (H1).

**-** *Proof of* (H2): Let $\tilde{u}_1, \tilde{u}_2 \in B_0$, and we write

$$\begin{aligned} F\left(\tilde{u}_1(s)\right) - F\left(\tilde{u}_2(s)\right) &= -3(d-2)\left(\tilde{u}_1(s) - \tilde{u}_2(s)\right)\left(\tilde{u}_1(s) + \tilde{u}_2(s)\right) \\ &\quad -(d-2)|x|^2\left(\tilde{u}_1(s) - \tilde{u}_2(s)\right)\left(\tilde{u}_1^2(s) + \tilde{u}_1(s)\tilde{u}_2(s) + \tilde{u}_2^2(s)\right). \end{aligned}$$

Since $\left\|\tilde{u}_1\right\|_{X^T} \le 1$ and $\left\|\tilde{u}_2\right\|_{X^T} \le 1$, we derive

$$\left\|F\left(\tilde{u}_1(s)\right) - F\left(\tilde{u}_2(s)\right)\right\|_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)} \le C\left\|\tilde{u}_1(s) - \tilde{u}_2(s)\right\|_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)}, \forall s \in [0,T].$$

Regarding to Lemma 2.1, we have

$$\left\|e^{(t-s)\Delta}\left(F\left(\tilde{u}_1(s)\right) - F\left(\tilde{u}_2(s)\right)\right)\right\|_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)} \le C\left\|\tilde{u}_1(s) - \tilde{u}_2(s)\right\|_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)} \le C\left\|\tilde{u}_1 - \tilde{u}_2\right\|_{X^T},$$

for all $t-s > 0, s \in (0,t)$. Thus, we derive from (6) that for all $t \in (0,T]$

$$\left\|\Gamma(\tilde{u}_1)(t) - \Gamma(\tilde{u}_2)(t)\right\|_{L^\infty_{1+|x|^2}\left(\mathbb{R}^{d+2}\right)} \le C\left\|\tilde{u}_1 - \tilde{u}_2\right\|_{X^T}\left(\int_0^t ds\right) \le C_2 T\left\|\tilde{u}_1 - \tilde{u}_2\right\|_{X^T},$$

which yields

$$\left\|\Gamma(\tilde{u}_1) - \Gamma(\tilde{u}_2)\right\|_{X^T} \le C_2 T\left\|\tilde{u}_1 - \tilde{u}_2\right\|_{X^T}, \forall \tilde{u}_1, \tilde{u}_2 \in B_0.$$

Finally, we choose $C_2\sqrt{T} \le \frac{1}{2}$ i.e. $T \le \left(\frac{1}{2C_2}\right)^2$ then

$$\left\|\Gamma(\tilde{u}_1) - \Gamma(\tilde{u}_2)\right\|_{X^T} \le \frac{1}{2}\left\|\tilde{u}_1 - \tilde{u}_2\right\|_{X^T}, \forall \tilde{u}_1, \tilde{u}_2 \in B_0,$$

.

which concludes (H2).

Now, we continue on the proof of the proposition, since $X^T$ is a Banach space and $\Gamma$ is a contractive map from $B_0$ to itself, so we apply Banach fixed point theorem that there uniquely exists $\tilde{u} \in B_0$ such that

$$\tilde{u}(t) = \Gamma(\tilde{u})(t), \forall t \in [0,T],$$

and we have the estimate

$$\|\tilde{u}\|_{X^T} \le \left\|e^{t\Delta}\tilde{u}_0\right\|_{X^T} + \left\|\Gamma(\tilde{u}) - e^{t\Delta}\tilde{u}_0\right\|_{X^T} \le \|\tilde{u}_0\|_{L^\infty_{1+|x|^2}(\mathbb{R}^{d+2})} + 1. \quad (10)$$

In particular, by the parabolic regularity of the semi-group $e^{t\Delta}$, we improve that $\tilde{u}(t) \in C^2(\mathbb{R}^{d+2}), \forall t \in (0,T)$ and it satisfies equation (5) for all $(x,t) \in \mathbb{R}^{d+2} \times (0,T)$ point-wise, and we derive from (10)

$$|\tilde{u}(x,t)| \le \frac{\|\tilde{u}_0\|_{L^\infty_{1+|x|^2}} + 1}{1+|x|^2}, \forall (x,t) \in \mathbb{R}^{d+2} \times (0,T].$$

Finally, we get the conclusion of the Proposition.

Consequently, **Proposition 2.2** implies the following result:

**Proposition 2.3:** *Let $d \ge 1$ be an integer number and initial choice $u_0 \in L^\infty_{1+r^2}(\mathbb{R}_+)$. Then, there exists $T = T(u_0)$ such that problem (1) has a unique solution on $[0,T]$ and $u(t) \in L^\infty_{1+|x|^2}(\mathbb{R}_+) \cap C^2(\mathbb{R}_+), \forall t \in (0,T]$. In particular, we have the following estimate*

$$|u(r,t)| \le \frac{C(u_0)}{1+r^2}, \forall (r,t) \in \mathbb{R}_+ \times (0,T).$$

*Proof:* Let $u_0 \in L^\infty_{1+r^2}(\mathbb{R}_+)$, then, it follows from the extension (4) that $\tilde{u}_0 \in L^\infty_{rad}(\mathbb{R}^{d+2}) \cap L^\infty_{1+|x|^2}(\mathbb{R}^{d+2})$. Applying Proposition 2.2, we obtain the existence and the uniqueness of the solution $\tilde{u}$ to equation (5) on $[0,T(u_0)]$ and $\tilde{u}(t) \in L^\infty_{rad}(\mathbb{R}_+) \cap L^\infty_{1+|x|^2}(\mathbb{R}_+) \cap C^2(\mathbb{R}_+), \forall t \in (0,T]$, then, the problem (1) and (8) are equivalent in the radially symmetric setting, this leads to the existence and the uniqueness of $u$ and also the conclusion the proposition completely follows.

**Remark 2.4:** We can repeat the proof of Propositions 2.2 and 2.3, to establish the local existence and uniqueness in spaces $L^\infty_{1+r^\alpha}(\mathbb{R}_+)$ where $\alpha \ge \frac{2}{3}$. Since the main difficulty is to handle the hugeness of nonlinear term $r^2u^3$ at infinity. However, once $u \in L^\infty_{1+r^\alpha}(\mathbb{R}_+), \alpha \ge \frac{2}{3}$ it follows that the nonlinearity is controlled well by

$$r^2u^3 = \frac{r^2}{(1+r^\alpha)^3}\left((1+r^\alpha)u\right)^3 \le \|u\|^3_{L^\infty_{1+r^\alpha}} (1+r^\alpha)^{\frac{2}{\alpha}-3},$$

since $\frac{2}{\alpha} - 3 \le 0$. We kindly refer the readers to check the details of the general results.

## III. CONCLUSION

As we showed in Proposition 2.3, the local Cauchy problem in $L^\infty_{1+r^2}(\mathbb{R}_+)$ is completely solved (also in $L^\infty_{1+r^\alpha}(\mathbb{R}_+), \alpha \ge \frac{2}{3}$ as in Remark 2.4). In comparison to the result proved in Donninger and Schörkhuber, 2019 where the authors considered the problem in $C_0^\infty(\mathbb{R}_+)$, our result is better. The technique of the proof replies to the idea given by Biernat and Seki, 2019 that extended the original problem to the radially symmetric one in $\mathbb{R}^{d+2}$, and then, we established a new property that the semi-group $e^{t\Delta}, t > 0$ reserves the polynomial decays showed in Lemma 2.1, then the existence and uniqueness follows by the route map based on Banach fixed point theorem.

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