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Original Article

Star Coloring of Book Graph and Ladder Graph

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Abstract *- In recent years, graph coloring includes star coloring, prime coloring, face coloring etc., plays a vital role among researchers. In this disquisition, we examine the star chromatic number χ_s of the line graph of ladder graph $L(L_n)$ and book graph $L(B_n)$.*

Keywords *- Line Graph, Proper Coloring, Star Coloring, Star Chromatic Index.*

AMS Subject Classification 2010: *05C78, 05C15.*

I. INTRODUCTION

All Graphs considered in this paper are loopless, simple and undirected graphs without multiple edges. A Graph has various special patterns like star, path, cycle, regular graph, bipartite graph etc., [4] Ladder and Book graph are described for graph cartesian product where ladder graph is defined as $L_n = P_n \times P_2$ where P_n is path graph and book Graph is the combination of star graph and path graph on two vertices and it is defined by $B_n = S_{n+1} \times P_2$ where S_{n+1} is star graph and P_2 is path graph [8]. Hosoya and Harary (1993) used ladder graph for graph cartesian product $K_2 \times C_n$ where K_2 is the complete graph on two vertices and C_n is the cycle graph on n vertices. A Line Graph notion was introduced by Whitney in 1932 [6]. In 1973, Branko Grunbaum introduced the star chromatic number. Here, we discuss the star chromatic number of line graph of ladder graph and book graph The Star Chromatic Index of a graph G [2] is the minimum number of colors needed to properly color the vertices and edges of the graph so that no path or cycle of length 4 is bi-colored.

II. PRELIMINARIES

Definition 2.1: Line Graph

The Line Graph L(G) [1], [4] of a graph G is defined as

- Each edge of G represents an vertex of L(G)
- Two vertices of L(G) are adjacent iff their corresponding edges share common endvertices in G.

Definition 2.2: Proper Coloring

A Proper vertex coloring or proper edge coloring of a graph G is the assignment of colors such that no two adjacent vertices or edges receive the same color [5].

Definition 2.3: Star Coloring

A Star Coloring of a graph G is a proper coloring of graph G such that no path of length 3 in G is bicolored [3], [7].

Definition 2.4: Star Chromatic Number

The Star Chromatic Number of an undirected graph G, denoted by χ_s, is the smallest integer k for which G admits a star coloring with k colors [9], [10].

III. STAR COLORING OF LINE GRAPH OF BOOK GRAPH

Theorem 3.1

The Star Chromatic Number of line graph of book graph for $n \geq 3$ is $\chi_s(L(B_n)) = 4$

Proof

Let $V(B_n) = \{v_i : 1 \leq i \leq 2n+6\}$ and $E(B_n) = \{e_i : 1 \leq i \leq 3n+1\}$ where e_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$. By line graph definition $V[L(B_n)] = E(B_n) = \{v'_i : 1 \leq i \leq 3n+1\}$, where v'_i represents the vertices of line graph of book graph. $L(B_n)$ is n-rhombus at one midpoint.

Assign the coloring as follows:

Let σ be the proper star coloring

$$\sigma(v'_i) = \begin{cases} c_1, & i = 1 \\ c_2, & i = 3n - 1 \\ c_3, & i = 3n \\ c_4, & \text{otherwise} \end{cases}$$

To Prove: $\chi_s(L(B_n)) = 4$

Suppose $\chi_s(L(B_n)) < 4$, say 3. Here, n – rhombus $(L(B_n))$ is $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, \ldots\}$ color with $\{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$. Obviously, midvertex v'_i is colored with c_i . We conclude that $\{ v_{3n}, v_{3n-1}, v_{3n+1}\}$ needs atleast 3 colors. Our assumption is contradiction to proper star coloring. So, four colors is adequate to color the $L(B_n)$.

Therefore, $\chi_s(L(B_n)) = 4$

Theorem 3.2

Let B_n be a book graph with (3n+1) edges, then $\chi'_s(L(B_n)) = 2n$ for all $n \geq 3$

Proof

Let $E(B_n) = \{e_i : 1 \leq i \leq 3n+1\}$, $V(B_n) = \{v_i : 1 \leq i \leq 2n+6\}$ and $E[L(B_n)] = \{e'_i : 1 \leq i \leq 4n\}$, $V[L(B_n)] = \{v'_i : 1 \leq i \leq 3n+1\}$ here $E(B_n) = V[L(B_n)] = \{v'_i : 1 \leq i \leq 3n+1\}$ where v'_i and e'_i represents the vertices and edges of line graph of book graph respectively. $L(B_n)$ is n – rhombus at one midpoint.

Case – (i) For n = 3

Assign the colors as follows:

$$\sigma(e_{2k}) = \sigma(e_{2k+6}) = C_{2k}$$

$$\sigma(e_{4k-3}) = \sigma(e_{4k-1}) = C_{2k-1} \text{ for all } k = 1,2,\ldots$$

Case - (ii) For n > 3

For $n \geq 4$, color assigning order is as follows:

$$\sigma(e_{4k-3}) = \sigma(e_{4k-1}) = C_{2k-1}$$

$$\sigma(e_{4k-2}) = \sigma(e_{4k-4}) = C_{2k+2} \text{ for all } k = 1,2,\ldots$$

There remains two uncolored edges, so we assign c_2 color to $\sigma(e_4) = \sigma(e_{4k+10})$ for all $n \geq 4$.

To Prove: $\chi'_s(L(B_n)) = 2n$ for all $n \geq 3$.

Suppose $\chi'_s(L(B_n)) < 2n$, say 2n-1; graph $E(L(B_n))$ has 2n edges incident at the midvertex. By the definition of proper coloring we need 2n colors to color the $E(L(B_n))$. Our assumption is contradiction to proper coloring. So, $\chi'_s(L(B_n)) = 2n$ for all $n \geq 3$. Therefore, $\chi'_s(L(B_n)) = 2n$.

IV. STAR COLORING OF LINE GRAPH OF LADDER GRAPH

Theorem 4.1

The Star Chromatic Number of line graph of ladder graph $L(L_n)$, where n is any positive integer and $n \geq 3$ is $\chi_s(L(L_n)) = 4$.

Proof

Let $V(L_n) = \{v_i : 1 \leq i \leq 2n\}$ and $E(L_n) = \{e_i : 1 \leq i \leq 3n-2\}$ where e_i is the edge v_iv_{i+1} ($1 \leq i \leq n-1$). By the definition of line graph $V(L(L_n)) = \{v'_i : 1 \leq i \leq 3n-2\}$, $E(L(L_n)) = \{e'_i : 1 \leq i \leq 4n\}$. $E(L_n) = V(L(L_n)) = \{v'_i : 1 \leq i \leq 3n-2\}$, where v'_i represents the vertices of line graph of ladder graph.

Case – (i) n ≡ 0 mod 3

Since $V(L(L_n)) = \{v_1', v_2', \ldots, v_{2n}'\} \cup \{v_{2n+1}', v_{2n+2}', \ldots, v_{3n-2}'\}$ and σ be star coloring

$$\sigma(v_i') = \begin{cases} c_1 & i \equiv 1 \bmod 3 \\ c_2 & i \equiv 2 \bmod 3 \\ c_3 & i \equiv 0 \bmod 3 \\ c_4 & \text{otherwise} \end{cases}$$

Therefore, $\{c_1, c_2, c_3, c_1, c_2, c_3, \ldots, c_1, c_2, c_3\}$ be the colors assigned to consecutive vertices $\{v_1', v_2', \ldots, v_{2n}'\}$ and for inner cycle c_4 color is applied.

Case – (ii) n ≡ 1 mod 3

Rearranging the color sequence for n ≡ 1 mod 3 are given below:

$$\sigma(v_i') = \begin{cases} c_1 & i = 2n+1 \\ c_2 & i = 4n-2 \\ c_3 & i = 4n \\ c_4 & \text{otherwise} \end{cases}$$

By repeating the sequence of colors $\{c_2, c_1, c_3, c_1, c_2, c_1, c_3, \ldots, c_3, c_1, c_2\}$ for the vertices $\{v_1', v_2', \ldots, v_{2n}'\}$ and remaining vertices c_4 color is applied.

Case – (iii) n ≡ 2 mod 3

Let σ be the star coloring

$$\sigma(v_i') = \begin{cases} c_1 & i = 2n+1 \\ c_2 & i = 4n-2 \\ c_3 & i = 4n \\ c_4 & \text{otherwise} \end{cases}$$

To Prove: $\chi_s(L(L_n)) = 4$ for $n \geq 3$

Suppose $\chi_s(L(L_n)) < 4$, say 3; $\chi_s(L(L_n)) = 3$.

Here $\{v_1', v_2', v_3' \ldots, v_{2n}'\}$ be the vertices of outer cycle is assigned colors with $\{c_1, c_2, c_3\}$ and inner cycle $\{v_{2n+1}', v_{2n+2}', \ldots, v_{3n-2}'\}$ be uncolored vertices. By proper coloring, introduce new color c_4 to inner cycle of $V(L(L_n))$. Our assumption is contradiction to proper coloring. Hence, $\chi_s(L(L_n)) = 4$

Theorem 4.2

The line graph of ladder graph, $\chi'_s(L(L_n)) = 5$ for $n \geq 4$

Proof

Let $V(L_n) = \{v_i : 1 \leq i \leq 2n\}$ and $E(L_n) = \{e_i : 1 \leq i \leq 3n-2\}$ where e_i is the edge $v_i v_{i+1}$ $(1 \leq i \leq n-1)$. By the definition of line graph $V(L(L_n)) = \{v'_i : 1 \leq i \leq 3n-2\}$, $E(L(L_n)) = \{e'_i : 1 \leq i \leq 4n\}$ where v_i' and e_i' represents the vertices and edges of line graph of ladder graph respectively.

Let $E(L(L_n)) = \{e_1', e_2', e_3', \ldots, e_{2n}'\} \cup \{e_{2n+1}', e_{2n+2}', \ldots, e_{2n-4}'\}$. Here $\{ e_1', e_2', e_3', \ldots, e_{2n}'\}$ be the edges of outer cycle and $\{ e_{2n+1}', e_{2n+2}', \ldots, e_{2n-4}'\}$ be the edges of inner cycle of $E(L(L_n))$.

Case – (i) n ≡ 0 mod 3

Let σ be star coloring

$$\sigma(e_i') = \begin{cases} c_1 & i \equiv 1 \bmod 3 \\ c_2 & i \equiv 2 \bmod 3 \\ c_3 & i \equiv 0 \bmod 3 \\ c_4 & i = \text{odd} \\ c_5 & i = \text{even} \end{cases}$$

To Prove: $\chi'_s(L(L_n)) = 5$

Suppose $\chi'_s(L(L_n)) < 5$, say 4. The edges $\{ e_1', e_2', e_3', \ldots, e_{2n}'\}$ of the outer cycle assigned with colors $\{c_1, c_2, c_3\}$ as proper coloring. One more is not sufficient to proper color the inner cycle with edges $\{e_{2n+1}', e_{2n+2}', \ldots, e_{2n-4}'\}$. So, we introduce new color c_4 and c_5 to inner cycle. $\{c_1, c_2, c_3, c_1, c_2, c_3, \ldots, c_1, c_2, c_3\}$ be the colors assigned to consecutive edges $\{ e_1', e_2', e_3', \ldots, e_{2n}'\}$ and for inner cycle c_4 and c_5 colors are applied.

Case – (ii - a) n ≡ 1 mod 3

Rearranging the color sequence for n ≡ 1 mod 3 as follows:

$$\sigma(e_i') = \begin{cases} c_1 & i = 2n+1 \\ c_2 & i = 4n-2 \\ c_3 & i = 4n \\ c_4 & i = \text{odd} \\ c_5 & i = \text{even} \end{cases}$$

By repeating the sequence of colors $\{ c_2, c_1, c_3, c_1, c_2, c_1, c_3, \ldots, c_3, c_1, c_2 \}$ for the edges $\{ e_1', e_2', \ldots, e_{2n}'\}$ and c_4 and $c_5$colors are applied to inner cycle of $E(L(L_n))$.

Case – (ii - b) n ≡ 2 mod 3

Let σ be the star coloring and rearranging the color sequence for n ≡ 2 mod 3 as follows:

$$\sigma(e_i') = \begin{cases} c_1 & i = 3n \\ c_2 & i = 3n-1 \\ c_3 & i = 3n+1 \\ c_4 & i = \text{odd} \\ c_5 & i = \text{even} \end{cases}$$

The sequence of colors $\{ c_2, c_1, c_3, c_2, c_1, c_3, \ldots, c_2, c_1, c_3 \}$ for the edges $\{ e_2', e_3', \ldots, e_{2n}'\}$ and c_4 and c_5 colors are applied to inner cycle of $E(L(L_n))$.

Thus, there remains one uncoloured edge e_1' in outer cycle, by the definition of star coloring we use either c_4 or c_5 color for the respective edge.

Hence, $\chi'_s(L(L_n)) = 5$.

V. CONCLUSION

In this paper, we found the star chromatic number for line graph of book graph and ladder graph. In future, we can extend into some other classes of graphs by using star coloring, face coloring, map coloring, prime coloring etc,.

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