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Original Article

Nano Compactness with Respect to Nano Ideal in Nano Topological Space

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Abstract - *The aim of this paper is to study nano compactness with respect to nano ideal, called nano $\mathcal{J}$-compact space and discuss some of their properties. Some of the result in compact spaces have been generalized in terms of nano $\mathcal{J}$-compact spaces.*

Keywords - *Nano ideal, Nano $\mathcal{J}$-compact, f (g)-nano continuous, Nano-ideal topology.*

AMS Subject Classification: *54A05, 54A10, 54B05.*

I. INTRODUCTION

An ideal [3] $\mathcal{J}$ on a non-empty set X is a collection of subsets of X which satisfies,

(i) $A \in \mathcal{J}$, and $B \subset A$ implies $B \in \mathcal{J}$
(ii) $A, B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$

Given topological space (X, τ) with an nano ideal $\mathcal{J}$ on X and if $\mathcal{P}(X)$ is set of all subsets of X, set operator $(\cdot)^* : \mathcal{P}(X) \to \mathcal{P}(X)$ is called local function [3] of A with respect to τ and $\mathcal{J}$ defined as follows:
For $A \subset X$, $A^*(\mathcal{J}, \tau) = \{x \in X \,/\, U \cup A \notin \mathcal{J}$, for every $U \in \tau(x)\}$,
where $\tau(x) = \{U \in \tau, x \in U\}$.

A Kuratowski's closure operator $cl^*(\cdot)$ for a topology $\tau^*(\mathcal{J}, \tau)$ called $*$-topology finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{J}, \tau)$ [?]. When there is no change for confusion, we will simply write A^* for $A^*(\mathcal{J}, \tau)$ and τ^* for $\tau^*(\mathcal{J}, \tau)$. If $A \subset X$, then $cl(A)$ and $int(A)$, denote closure and interior of A in (X, τ) respectively. The interior and closure of A in (X, τ^*) is denoted by $int^*(A)$ and $cl^*(A)$ respectively.

The notation of a nano ideal topological space was introduced by Parimala et al. [8]. They studied its properties and various characterization. Also some new notions in the context of nano ideal topological spaces and investigate their basic properties [8]. Also Krishna Prakash et al. [4] introduce nano compactness, nano connectedness and study their properties.

II. PRELIMINARIES

Now we recall the following definitions and some important properties.

Definition 2.1. [5]
Let U be a non-empty finite set of objects called the universe and R be an equivalence classes. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$, then

(i) The lower approximation of X with respect to R is $L_R(X) = \cup\{R(X) : R(X) \subseteq X, x \in U\}$ where $R(X)$ denotes the equivalence class determined by $x \in U$

(ii) The upper approximation of X with respect to R is $U_R(X) = \cup\{R(X) : R(X) \cap X \neq \emptyset, x \in U\}$

(iii) The boundary region of X with respect to R is $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2.

Let U be an universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \emptyset, U_R(X), B_R(X), U_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms.

(i) U and $\emptyset$ are in $\tau_R(X)$

(ii) The union of the elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ is topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as a nano topological space. The elements of $\tau_R(X)$ are called nano open sets (briefly *n*-open sets). Also any non-empty sets satisfies properties of (i), (ii) in $(U, \tau_R(X))$ is called nano ideal.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano interior and nano-closure of subsets of U are denoted by *n-int*(A) and *n-cl*(A) respectively.

Definition 2.3.

A collection $\{A_i / i \in \mathcal{J}\}$ of nano-open sets in a nano topological space $\{U, \tau_R(X)\}$ is called a nano open cover of a subset B of U if $B \subset \bigcup_{i=1}^{n}\{A_i\}$.

III. NANO-COMPACTNESS WITH RESPECT TO NANO IDEAL

Definition 3.1.

A subset A of a space $(U, \mathcal{N})$ is said to be nano $\mathcal{J}$-compact or nano-compact modulo an ideal if for every open cover $\{U_\alpha / \alpha \in A\}$ of A by nano open sets of X, there exist a finite subsets Λ_0 of Λ, such that $A - \cup \{A_\alpha / \alpha \in \Lambda_0\}$ \in $\mathcal{J}$. The space $(U, \mathcal{N}, \mathcal{J})$ is said to be nano $\mathcal{J}$-compact if U is nano $\mathcal{J}$-compact.

If $(U, \mathcal{N})$ is nano topological space with ideal $\mathcal{J} = \{\emptyset\}$, (U, τ) is nano topological compact $\Rightarrow$ (U, τ) is nano $\mathcal{J}$-compact.

From this, compact $\Rightarrow$ compact modulus an ideal $\Rightarrow$ Nano compactness $\Rightarrow$ nano compact modulo an ideal.

Theorem 3.1.

Every nano-closed subset of nano-compact a $\mathcal{J}$-nano compact.

Proof.

Let A be a nano-closed subset of $(U, \mathcal{N}, \mathcal{J})$. Then A^c is nano open in $(U, \mathcal{N}, \mathcal{J})$. Let $\{B_\alpha / \alpha \in A\}$ be an $\mathcal{J}$-nano open cover of A by nano open subsets in $(U, \mathcal{N})$. Then $\{B_\alpha / \alpha \in \Lambda\} \cup A^c$ is a nano $\mathcal{J}$-open cover of $(U, \mathcal{N})$. Since $(U, \mathcal{N})$ is nano $\mathcal{J}$-compact, there exist a finite subcover $\{B_\alpha / \alpha \in \Lambda_0\} \cup A^c$ such that

$$
\begin{aligned}
& U - \{(B_\alpha / \alpha \in \Lambda_0) \cup A^c\} \in \mathcal{J} \\
\Rightarrow\ & U - \{((B_\alpha / \alpha \in \Lambda_0) \cup A^c) \cap A\} \subseteq U - (B_\alpha / \alpha \in \Lambda_0) \\
\Rightarrow\ & U - \{(B_\alpha / \alpha \in \Lambda_0) \cup A^c \cap A\} \in \mathcal{J} \\
\Rightarrow\ & A - \{B_\alpha / \alpha \in \Lambda_0\} \in \mathcal{J}.
\end{aligned}
$$

Hence A is nano $\mathcal{J}$-compact. □

Note 3.1.

Every nano *g*-closed subset of nano compact is nano $\mathcal{J}$-compact.

Note 3.2.

If F is nano-closed and K is nano $\mathcal{J}$-compact subset of U, then $F \cap K$ is nano $\mathcal{J}$-compact.

Theorem 3.2.

Continuous image of nano $\mathcal{J}$-compact space is nano $f(\mathcal{J})$-compact.

Proof.

Let $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be a continuous map, where $(U, \mathcal{N}, \mathcal{J})$ is nano $\mathcal{J}$-compact. Let $\{B_\alpha / \alpha \in \Lambda\}$ be an nano open covering of the set $f(X)$ by sets of nano open in Y. Since f is continuous, the collection $\{f^{-1}(B_\alpha): \alpha \in \Lambda\}$ is an open covering of X. Given that X is nano $\mathcal{J}$-compact, such that finite subcover Λ_0 such that

$$
\begin{aligned}
& X - \cup\{f^{-1}(B_\alpha) : \alpha \in \Lambda_0\} \in \mathcal{J} \\
\Rightarrow\ & f(X - \cup\{f^{-1}(B_\alpha) : \alpha \in \Lambda_0\}) \in f(\mathcal{J}) \\
\Rightarrow\ & f(X) - f(\cup\{f^{-1}(B_\alpha) : \alpha \in \Lambda_0\}) \in f(\mathcal{J}) \\
\Rightarrow\ & f(X) - \cup\{B_\alpha : \alpha \in \Lambda_0\} \subset f(X) - f(\cup f^{-1}(B_\alpha) : \alpha \in \Lambda_0) \\
\Rightarrow\ & f(X) - \cup\{B_\alpha : \alpha \in \Lambda_0\} \in f(\mathcal{J}) \\
\Rightarrow\ & f(X) \text{ is nano } f(\mathcal{J})\text{-compact}
\end{aligned}
$$

Hence continuous image of nano $\mathcal{J}$-compact space is nano $f(\mathcal{J})$-compact. □

Theorem 3.3.

If A and B are nano $\mathcal{J}$-compact in ideal space, then $A \cup B$ is nano $\mathcal{J}$-compact in $(U, \tau_R(X))$.

Proof.

Let $\{B_\lambda : \lambda \in \Lambda\}$ be an open cover of $A \cup B$ in $(U, \tau_R(X))$. Then $\{B_\alpha : \alpha \in \Lambda\}$ is an open cover of A and B. Given A and B are nano $\mathcal{J}$-compact, there exist finite subset Λ_0 and Λ_1 of Λ such that

$$
\begin{aligned}
& A - \cup \left\{B_{\alpha_i} : \alpha_i \in \Lambda_0\right\} = I_1 \ and\ B - \cup \left\{B_{\alpha_k} : \alpha_k \in \Lambda_1\right\} = I_2 \\
& \Rightarrow A = I_1 \cup \left\{\cup B_{\alpha_i} : \alpha_i \in \Lambda_0\right\} and\ B = I_2 \cup \left\{\cup B_{\alpha_k} : \alpha_k \in \Lambda_1\right\}
\end{aligned}
$$

$$
\begin{aligned}
Now\ \ A \cup B &= \cup \left\{B_{\alpha_i} : \alpha_i \in \Lambda_0\right\} \cup \left\{B_{\alpha_k} : \alpha_k \in \Lambda_1\right\} \cup I_1 \cup I_2 \\
&= \cup \left\{B_{\alpha_i} \cup B_{\alpha_k} / \alpha_i \in \Lambda_0, \alpha_k \in \Lambda_1\right\} \cup I,\ \ where\ I \in \mathcal{J} \\
\Rightarrow (A \cup B) &= \cup \left\{B_{\alpha_i} \cup B_{\alpha_k} : \alpha_i \in \Lambda_0, \alpha_k \in \Lambda_1\right\} \cup I.
\end{aligned}
$$

This implies $A \cup B$ is nano $\mathcal{J}$-compact.□

Corollary 3.1.

Finite union of nano $\mathcal{J}$-compact space X is nano $\mathcal{J}$-compact.

Theorem 3.4.

Every nano $\mathcal{J}$-compact subset of a Hausdroff ideal space is τ^*-closed.

Proof.

Let A be a nano $\mathcal{J}$ compact subset of Hausdroff space $(U, \mathcal{N}, \mathcal{J})$. Let $x \notin A$ then $x \in X - A$. For each $y \in A$, there exist a neighbourhood U_y and V_y of x and y respectively, such that $U_y \cap V_y = \varnothing$. Note that $x \notin cl(V_y)$. Now $\{V_y : y \in A\}$ is a τ-nano open cover of A which is a nano $\mathcal{J}$-compact, there exist a finite subset Λ_0 of Λ such that $A - \cup \{V_y : y \in \Lambda_0\} \in \mathcal{J}$, set $I = A - \cup \{V_{y_i} : y \in \Lambda_0\}$. Define $U = \bigcup_{i=1}^{n} U_{y_i}$. Then U is τ-nano open and $U \cap \left(\cup V_{y_i}\right) = \emptyset$. Therefore $U \cap A \subseteq I$ and hence $x \notin A^*$. Hence A is τ^*-closed. □

Theorem 3.5.

The following are equivalent for the space $(U, \mathcal{N}, \mathcal{J})$.

(i) $(U, \tau, \mathcal{J})$ is nano $\mathcal{J}$-compact.

(ii) $(U, \tau^*, \mathcal{J})$ is nano $\mathcal{J}$-compact.

(iii) For any family $\{F_\alpha : \alpha \in \Lambda\}$ of nano closed set of U such that $\cap\{F_\alpha : \alpha \in \Lambda\} = \varnothing$, there exist a subset Λ_0 of Λ such that $\cap\{F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{J}$.

Proof.

(a) ⇒ (b) Let $\{U_\lambda : \lambda \in \Lambda\}$ be a τ^*-open cover of U such that $U_\lambda = V - I$, where V_λ is nano open in X and $I_\lambda \in \mathcal{J}$. Now $\{V_\lambda : \lambda \in \Lambda\}$ is an open cover of U and hence there exist subcover Λ_0 of Λ such that

$$U - U\{V_\lambda : \lambda \in \Lambda_0\} \in \mathcal{J}.$$
$$\Rightarrow U - \cup\{U_\lambda : \lambda \in \Lambda_0\} \cup \{\cup I_\lambda : \lambda \in \Lambda_0\}.$$
$$\subset U / \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{J}.$$

∴ $(U, \tau^*, \mathcal{J})$ is compact.

(b) ⇒ (a) Proof follows from $\tau \subset \tau^*$.

(a) ⇒ (c) Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of nano closed sets of U such that $\cap\{F_\alpha : \alpha \in \Lambda\} = \emptyset$. Then $\{U - F_\alpha : \alpha \in \Lambda\}$ is a nano open covers of U. By (a) there exist a finite subset Λ_0 of Λ such that $U - \cup\{U - F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{J}$. $\Rightarrow \cap\{F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{J}$.

(c) ⇒ (a) Let $\{U_\alpha : \alpha \in \Lambda\}$ be nano-open cover of U then $\{U - U_\alpha : \alpha \in \Lambda\}$ is a collection of nano-closed sets and $\cap(U - U_\alpha : \alpha \in \Lambda) = \emptyset$. Hence thers exist finite subset Λ_0 of Λ such that $\cap(U - U_\alpha : \alpha \in \Lambda_0) \in \mathcal{J}$. $\Rightarrow U - U(U_\alpha : \alpha \in \Lambda_0) \in \mathcal{J}$. This shows that $(U, \mathcal{N}, \mathcal{J})$ is nano $\mathcal{J}$ -compact. □

Theorem 3.6.

Let $(U, \tau, \mathcal{J})$ be any ideal topological space and A be a subset of U such that for each nano open set V containing A, there is nano $\mathcal{J}$-compact set B with $A \subset B \subset V$ then A is nano $\mathcal{J}$-compact.

Proof.

Let $\{U_\alpha : \alpha \in \Lambda\}$ open cover of A, where $V_\alpha = W \cup A$ such that W is open in U. Given there exists nano $\mathcal{J}$-compact subset B of U such that $A \subset B \subset \cup W_\alpha$. Then $\{W_\alpha \cup B : \alpha \in \Lambda\}$ is τ_B nano open cover of B. As B is nano $\mathcal{J}$-compact, there exist finite subset Λ_0 of Λ such that $B - \cup\{W_\alpha \cup B : \alpha \in \Lambda_0\} \in \mathcal{J}$. Therefore

$$B - \cup(W_\alpha \cap B : \alpha \in \Lambda_0) \cup (I_1 \cap B), I_1 \in \mathcal{J}$$
$$\Rightarrow B \cap A = \cup(W_\alpha : \alpha \in \Lambda_0) \cup (I_1 \cap B \cap A)$$
$$\Rightarrow A = \cup(W_\alpha : \alpha \in \Lambda_0) \cup (I_1 \cap A)$$
$$\Rightarrow A - \cup(W_\alpha : \alpha \in \Lambda_0) = I_1 \cap A \in \mathcal{J}_A$$
$$\Rightarrow A \text{ is nano} - \text{compact.}$$ □

Theorem 3.7.

Let $\{U_\alpha : \alpha \in \Lambda\}$ be a family of nano topological spaces where each U_α is nano compact modulo to the ideal I_α in the space U_α. Let $\mathcal{J}$ be some ideal in $U = \prod_{\alpha\in\Lambda} U_\alpha$ such that $\pi_\alpha : U \to U_\alpha$ is the projection map. Then U is nano $\mathcal{J}$-compact.

Proof.

Let γ be the covering of U whose elements are numbers of standard pre basis. To prove there exist a finite number U_i of elements of γ such that $X - \bigcup_{i=1}^{n} U_i \in \mathcal{J}$. For each $\alpha_0 \in \Lambda_1$, let γ_{α_0} denote family of subsets of γ_{α_0} is covering U_{α_0} by choosing point U_α from each $U_\alpha \Rightarrow$ there exists $\beta_0 \in U$ such that γ_{β_0} is covering of U_{β_0}. Then we find $U_1^{\beta_0}, U_2^{\beta_0}, \ldots, U_n^{\beta_0}$ such that

$$U_{\beta_0} - \cup\, U_i^{\beta_0} = I_{\beta_0}$$
$$\Rightarrow \pi_{\beta_0}^{-1}(U_{\beta_0}) - \bigcup_{i=1}^{n} \beta_0^{-1}(U_i^{\beta_0}) \in \pi_{\beta_0}^{-1}(I_{\beta_0}) = \mathcal{J}$$
$$\Rightarrow U - \bigcup_{i=1}^{n} U_i \in \mathcal{J}$$

Thus U is nano $\mathcal{J}$-compact.

IV. CONCLUSION

This paper, we introduce the notion of nano compactness with respect to nano ideal and investigate some properties of nano topology and nano ideal. In future, it motivation to apply this concept in fuzzy system and graph structures.

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