International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 / https://doi.org/10.14445/22315373/IJMTT-V68I5P502
Volume 68 Issue 5, 6-10, May 2022

Original Article

Common Fixed Point Theorems Satisfying Property (E.A.) in Complex Valued Metric Spaces

Rajesh Pandya[1], Aklesh Pariya[2], Sandeep Kumar Tiwari[3]

[1,3] School of studies in Mathematics, Vikram University (M.P.), India.
[2] Department of Mathematics, S.V.P. Govt. College, Kukshi, (M.P.), India.

Received: 10 March 2022 Revised: 02 May 2022 Accepted: 16 May 2022 Published: 27 May 2022

Abstract *- In this paper, we prove some common fixed point theorems in complex valued metric spaces satisfying (E.A.) property using weakly compatible mappings. Our result generalizes the result of Verma and Pathak [8] and other existing results in complex valued metric spaces.*

Keyword *- Complex-valued metric space, Weak compatible mappings, Common fixed point, property (E.A).*

2010 MSC: *47H10; 54H25*

1. Introduction.

Mathematical results on fixed points of contraction type mappings are well known for their utility in determining the existence and uniqueness of solutions to various mathematical models. Azam et al. [3] proposed the notion of complex valued metric spaces and came up with some fixed point solutions for a pair of mappings that meet a rational expression for contraction criteria. Verma and Pathak[8] recently developed the concept of property (E.A.) on a complex valued metric space in order to derive some common fixed-point findings for two pairs of weakly compatible mappings that meet a max type contractive condition. Ahmad et al. [2] establish some common fixed results for mappings fulfilling rational expressions on a closed ball in complex valued metric spaces, while Rafiq et al. [6] prove some common fixed point theorems of weakly compatible mappings in complex valued metric spaces.

2.1. Basic Definitions and Preliminaries

We remember some notations and definitions that will be used in the conversation that follows. Azam et al. [3] recently proposed the following definition:

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$.

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(iii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2$, $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 2.1[3]. Let X be a nonempty set, whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d: X \times X \to \mathbb{C}$, satisfies the following conditions:

(d_1) $0 \precsim d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d_3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Example 2.1[6]. Let $X = \mathbb{C}$. Define the mapping $d: X \times X \to \mathbb{C}$ by

$$d(x, y) = e^{ik} |x - y| \text{ where } k \in \mathbb{R} \text{ and for all } x, y \in X.$$

Then (X, d) is a complex valued metric space.

Definition 2.2[3]. Let (X, d) be a complex valued metric space and $B \subseteq X$.

(i) $b \in B$ is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that $N(b, r) \subseteq B$, where $N(b, r) = \{y \in X: d(b, y) \prec r\}$

(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$, $N(x, r) \cap (B \backslash \{X\}) \neq \phi$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A whereas a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$F = \{N(x, r): x \in X, 0 \prec r$ is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition2.3[3]. Let (X, d) be a complex valued metric space and $\{x_n\}_{n\geq1}$ be a sequence in X and $x \in X$. We say that

(i) the sequence $\{x_n\}_{n\geq1}$ converges to x if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n$ or $x_n \to x$, as $n \to \infty$,

(ii) the sequence $\{x_n\}_{n\geq1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$,

(iii) the metric space (X, d) is a complete complex valued metric space if every Cauchy sequence is convergent.

Aamri and Moutawakil [1] developed several common fixed point theorems for mappings having the property (E.A.) on a metric space under rigorous contractive conditions. In complex valued metric space, Verma and Pathak [8] defined property (E.A.) as follows:

Definition 2.4 [8]. Let $S, T: X \to X$ be two self maps of a complex valued metric space (X, d). The pair (S, T) is said to satisfy property $(E.A.)$, if there exist a sequence$\{x_n\}$ in X such that $\lim_{n\to\infty} S x_n = \lim_{n\to\infty} T x_n = t$ for some $t \in X$.

Definition 2.5 [4] Let f and g be two self-maps defined on a set X, then f and g are said to be weakly compatible if they commute at coincidence point.

Definition 2.6 [8]. The 'max' function for the partial order $\precsim$ is defined as follows:

(1) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.

(2) $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.

(3) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$

2.2. Main Results

Azam et al. [3] introduced the notion of complex-valued metric spaces and derived certain common fixed point findings in the setting of complex-valued metric spaces. This study proves the existence of common fixed points for weak compatible mappings on complex-valued metric spaces. Similar findings in the literature are reconciled, extended, and complemented by our findings. For four mappings employing weak compatibility and property (E.A.) in ordinary metric space and complex valued metric space, our result expands numerous known results, including Azam et al [3], Sintunawarat and Kumam [7], Verma and Pathak [8].

Theorem 2.2.1. Let (X, d) be a complex valued metric space and mappings $f, g, S, T: X \to X$ satisfying

(2.2.1) $S(X) \subset g(X)$, $T(X) \subset f(X)$;

(2.2.2) $d(Sx, Ty) \precsim \alpha . max\left\{d(fx, gy), \frac{d(fx, Sx)\, d(gy, Ty)}{d(fx, Ty) + d(gy, Sx) + d(fx, gy)}\right\}$

for all $x, y \in X$ such that $x \neq y$, $d(fx, Ty) + d(gy, Sx) + d(fx, gy) \neq 0$

where α is nonnegative real with $\alpha < 1$.

(2.2.3) the pairs (S, f) and (T, g) are weakly compatible,

(2.2.4) one of the pair (S, f) or (T, g) satisfy the property $(E.A.)$.

If the range of one of the mappings f *or* g is a complete subspace of X, then mappings f, g, S and T have unique common fixed point in X.

Proof. First suppose that the pair (T, g) satisfies property$(E.A.)$, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n$ = z for some z ∈X.

Further, since $T(X) \subset f(X)$, there exists a sequence $\{y_n\}$ in X such that $Tx_n = fy_n$ for all

n ∈ N. Hence, we have $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} f\,y_n = z.$ Now we show that $\lim_{n\to\infty} S\,y_n$ = z. If not, then by using (2.2.2), with $x = y_n$ and $y = x_n$, we have

$$d(Sy_n, Tx_n) \precsim \alpha.max\left\{d(fy_n, gx_n), \frac{d(fy_n,Sy_n)\, d(gx_n,Tx_n)}{d(fy_n,Tx_n)+ d(gx_nSy_n)+ d(fy_n\, gx_n)}\right\}$$

Taking lim n → ∞, and using above conditions we get

$\lim_{n\to\infty} d(Sy_n, z) \precsim \alpha. 0$, which is a contradiction. Hence $\lim_{n\to\infty} S\,y_n$ = z

Thus $\lim_{n\to\infty} S\,y_n = \lim_{n\to\infty} Tx_n = z$.

Suppose that f(X) is complete, then $fu = z$ for some $u \in X$. Consequently, we obtain

$$\lim_{n\to\infty} S\,y_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} f\,y_n = z = fu.$$

Now we claim that Su= fu. For ,putting x=u and and $y = x_n$ in (2.2.2), we get

$$d(Su, Tx_n) \precsim \alpha.max\left\{d(fu, gx_n), \frac{d(fu,Su)\, d(gx_n,Tx_n)}{d(fu,Tx_n)+ d(gx_nSu)+ d(fu\, gx_n)}\right\}$$

letting lim n → ∞ and using results above, we get

$$d(Su, z) \precsim \alpha.max\left\{d(z, z), \frac{d(z,Su)\, d(z,z)}{d(z,z)+ d(z,Su)+ d(z,z)}\right\}$$

$\lim_{n\to\infty} d(Su, z) \precsim \alpha. 0$, which is a contradiction. Hence Su = z. Thus, Su = fu= z.

Hence u is a coincidence point of (S, f). Now the weak compatibility of the pair (S, f) implies that fSu = Sfu or fz =Sz.

On the other hand, since $S(X) \subset g(X)$, there exists v in X such that Su= gv. Thus Su = fu=gv= z.

Let us show that v is coincidence point of (T, g) i.e., Tv = gv = z. If not, then putting x=u and y =v in (2.2.2), we get

$$d(Su, Tv) \precsim \alpha.max\left\{d(fu, gv), \frac{d(fu\, , Su)\, d(gv, Tv)}{d(fu, Tv) + d(gv, Su) + d(fu, gv)}\right\}$$

$d(z, Tv) \precsim \alpha. 0$, a contradiction. Thus Tv = z. Hence Tv = gv = z and v is a coincidence point of (T, g). Further the weak compatibility of pair (T, g) implies that

Tgv =gTv , or Tz =gz. Therefore z is a common coincidence point of f, g, S, T.

In order to show that z is a common fixed point, let us put x=u and y =z in (2.2.2), we get

$$d(Su, Tz) \precsim \alpha.max\left\{d(fu, gz), \frac{d(fu\, ,Su)\, d(gz,Tz)}{d(fu,Tz)+d(gz,Su) +d(fu,gz)}\right\} = \alpha.max\{d(z, Tz), 0\}$$

Or $|d(z,Tz)| \leq \alpha.|d(z,Tz)| < |d(z,Tz)|$, a contradiction.

Thus Tz = z. Hence Sz= fz =Tz =gz = z.

Similar argument arises if we assume that $g(X)$, is a complete sub space of X. Similarly, the property $(E.A.)$ of the pair (S,f) will give the similar result.

For uniqueness of common fixed point, let us put x=w and y =z in (2.2.2), we get

$$d(w,z) = d(Sw,Tz) \precsim \alpha.max\left\{d(fw,gz), \frac{d(fw\ ,Sw)\ d(gz,Tz)}{d(fw,Tz) + d(gz,Sw)\ + d(fw,gz)}\right\}$$

$$= \alpha.max\{d(w,z),0\}$$

Or $|d(w,z)| \leq \alpha.|d(w,z)| < |d(w,z)|$, a contradiction.

Thus w = z. Hence Sz= fz =Tz =gz = z, and z is the unique common fixed point of S, T, f, g. This completes the proof.

By setting $f = g = I$, in theorem 2.2.1, we get the following corollary:

Corollary 2.2.1 Let (X,d) be a complex valued metric space and mappings $S,T: X \to X$ satisfying

(i) $S(X) \subset T(X)$,

(ii) $d(Sx,Ty) \precsim \alpha.max\left\{d(x,y), \frac{d(x\ ,Sx)\ d(y,Ty)}{d(x,Ty)+d(y,Sx)\ +d(x,y)}\right\}$

for all $x,y \in X$ such that $x \neq y$, $d(x,Ty) + d(y,Sx) + d(x,y) \neq 0$

where α is nonnegative real with α < 1.

(iii) the pairs (S,T) is weakly compatible,

(iv) the pair (S,T) satisfy the property $(E.A.)$.

If $T(X)$ is a complete then mappings S and T have unique common fixed point in X.

By setting $f = g = S = I$, in theorem 2.2.1, we get the following corollary:

Corollary 2.2.2 Let (X,d) be any complete complex valued metric space and mapping $T: X \to X$ satisfying

$$d(Tx,Ty) \precsim \alpha.\max\left\{d(x,y), \frac{d(x,Tx)\ d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)}\right\}$$

for all $x,y \in X$ such that $x \neq y$, $d(x,Ty) + d(y,Tx) + d(x,y) \neq 0$. where α is non-negative real with α < 1. Then T has a unique fixed point in X.

3. Conclusion

The presence of common fixed points for weak compatible mappings on complex-valued metric spaces is demonstrated in this paper. Our findings reconcile, broaden, and reinforce similar findings in the literature. Our work extends multiple previous findings, including Azam et al [3], Sintunawarat and Kumam [7], Verma and Pathak [8], for four mappings in ordinary metric space and complex valued metric space that use weak compatibility and property (E.A.).

Conflicts of Interest

The authors declare that they have no Conflicts of Interest.

References

[1] M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions. J Math Anal Appl. 270 (2002) 181–188.

[2] J., Ahmad, A. Azam, and S. Saejung, Common fixed point results for contractive mappings in complex valued metric spaces, Fixed Point Theory Appl., 2014 (2014) 11.

[3] A. Azam, B. Fisher, and M. Khan, Common fixed point theorems in complex valued metric spaces, Num. Func. Anal. Opt. 32 (2011) 243- 253.

[4] G. Jungck and B.E. Rhoades, Fixed point for set valued functions without continuity, I. J. Pure Appl. Math.,29(3) (1998) 227-238.
[5] H.K. Nashine M. Imdad and M. Hasan, Common fixed point theorems under rational contractions in complex valued metric spaces, J. Nonlinear Sci. Appl. 7(2014) 42-50.
[6] A. Rafiq, F. Rouzkard, M. Imdad and Shin Min Kang, Some common fixed point theorem of weakly compatible mappings in complex valued metric spaces, Mitteilungen Klosterneuburg 65 (2015) 1 422-432.
[7] W. Sintunavarat and P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, J. Inequalities Appl. 2012(2012), 11.
[8] R.K. Verma, and H.K. Pathak, Common fixed point theorems using property (E.A) in complex valued metric spaces, Thai Journal of Mathematics, 11(2) (2013) 275-283.