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Original Article

Finite Difference Analysis of Pressure Surge at the Valve of a Closed Pipeline

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Abstract - *Water hammer is very essential for risk analysis and operation of pipeline system. This work studies the effects of steady friction factor, wall constraint, pipe diameter and liquid density for the analysis of water hammer in a pipeline system with special emphasis on the finite difference method (FDM) due to its accurate and reliable evaluation of the water hammer problems. We model water hammer for pipe made from polymeric material by two partial differential equations (PDEs). The PDEs are then discretized along the spatial dimension to give a set of ordinary differential equations, ODEs. For a given set of parameters, we then solve the resulting ODEs numerically and plot the pressure dynamics at the valve to determine the water hammer characteristics. It was observed that the model with steady friction factor is appropriate for the simulation of water hammer in pressurized pipes made from polymers such as Polyvinyl Chloride. The maximum pressure amplitude simulated by the model is estimated accurately. The pipe wall constraint, liquid density, pipe diameter has a significant influence on obtained results. The lower the values of Pipe diameter* A_d*, effective modulus of system* D_s *and cross-sectional area , the lower the amplitudes as presented graphically. Also, during the stimulation, the experimental pressure runs shows the pressure increases after the valve closure linearly. Numerical validation of the FDM demonstrates that the FDM performs well and is as good as the method of characteristics in terms of computational speed and accuracy.*

Keywords - *Darcy-Weisbach equation, Finite difference method, Partial differential equation, Reynold number, Water hammer.*

1. Introduction

The study of water hammer stated in the 1st century B.C., when Marcus Vitruvius Pollio studied its effect in lead and tube pipes in the Roman water supply system [1]. According to Ismaier [2], a water hammer entails a pressure waves at the end of a pipeline system whenever there is a sudden closure at the valve of a pipe. The forced closure results to a sudden change of water molecules which generates a rapid pressure waves hit at the valve of the pipe. This pressure hit is also called hydraulic shock [3]. According to [4], whenever the valve closed rapidly there is a rapid change of direction or velocity of the water such that the pressure energy is transferred to the pipe wall and valve. This pressure waves (otherwise shock waves) travels to and fro on encountering a solid obstacle, and it equals the speed of the sound until it is annulled by friction losses. This is a major effect of water hammer particularly in older houses.

The internal variation or fluctuation within the water system caused by the movement of slow mass oscillation of water is called the surge. The surge is a less severe form of water hammer which can be identified as slower pressure wave developing within the water system. Both the surge and water hammer are referred to as transient pressure. Both are very dangerous because neither the water or pipe will absorb the shock waves by compressing it, and if not controlled will yield the same outputs: fittings, damage to pipe, leakage, pipeline system destruction, etc.

However, it was not until the 19th century that the theory of water hammer gained a rapid recognition as a municipal water supply system to civil engineers. This period witnessed the installation of water supply systems. For instance, Whitehurst [5] built a worktable water system for homes in Cheshire, England. Also, Joseph (1740 – 1810) developed a water hammer for his paper industry in Voiron. Soon enough, the theory of water hammer captivated the interest of physiologist for the study of the circulatory systems [6].

Water hammer is a key component in the study of pipeline systems because of the danger is posses. A critical analysis of water hammer allows us to choose relevant parameters of systems like liquid density, viscosity, pipe diameter, wall thickness, roughness, length, cross-sectional area, etc. The theory of water hammer is generally accredited to the German physiologist Johannes Von Kries in 1883, who was investigating a scheduled analysis of pulse in blood vessels ([6] - [8]). However, many researches were solely based on classic water hammer model. That is, the main assumptions of these models were linear elastics characterization of pipe wall [9]. For instance [10] carried out an explicit analysis of water hammer along the ring line for Khobar-Daminan water transmission. Also, [11] discussed water hammer effect in spiral cases in other to analyze the effect of sudden flow variation in an hydraulic turbine.

Since the advent of digital computers, researchers have shifted ground to water hammer models for pipe made from polymeric material involving quality resistant characteristics such as polyvinyl Chloride.
Thus, in this paper, we study the effects of steady friction factor, wall constraint, pipe diameter and liquid density for the analysis of water hammer in a pipeline system with special emphasis on the finite difference method (FDM) due to its accurate and reliable evaluation of the water hammer problems. For a given set of parameters, the FDM solves the resulting ODEs numerically and plots the pressure dynamics at the valve for numerical validation..

2. Materials and Methods

2.1 Mathematical formulation of the problem

For clarity of purpose, we assumed the liquid flow to be laminar, one dimensional, viscous, fully developed, incompressible and Newtonian flow through a pipe. The constricted wall of this pipe is assumed to thick and rough. The figure below shows showing water hammer effect on a closed pipe.

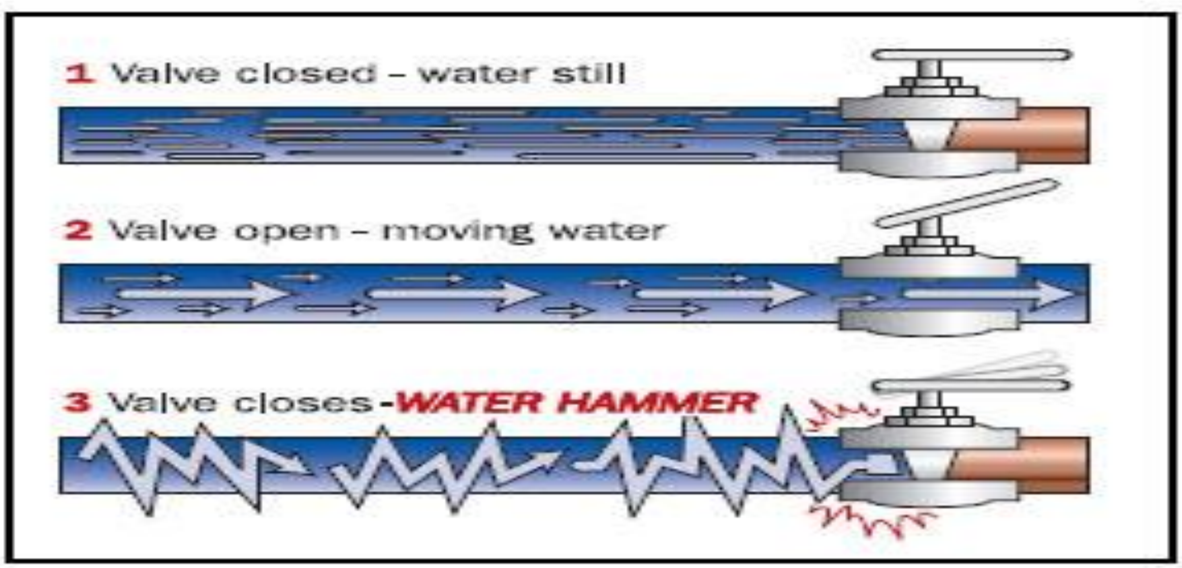

Fig. 1 Schematic diagram of water hammer

Source: https://www.theprocesspiping.com/water-hammer/

The water hammer model is modeled by two partial differential equations given as:

$$\begin{cases} \frac{\partial W(r,t)}{\partial t} + \frac{1}{\rho}\frac{\partial P(r,t)}{\partial r} + \frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) = 0, \\ D_s^2 \frac{\partial W(r,t)}{\partial r} + \frac{\partial P(r,t)}{\partial t} = 0 \end{cases}$$

where, $W(r,t) =$ mean velocity flow at position r and time t, $P(r,t) =$ pressure at the position r and time t, friction = friction factor at $W(r,t)$, or the Darcu-Weisbach friction factor, $\rho =$ fluid density, $\mathrm{A_d} =$ the pipe diameter, $D_s =$ effective bulk modulus of the system, $\mathrm{r} =$ the axial distance along the streamline, $\mathrm{z} =$ elevation at any point of the streamline. Note that equation (1) contains only one spatial dimension (r). For fluid flowing in a conduit, the mean flow velocity, $W(r,t)$ is assumed to be uniform through the pipe cross-sectional area, A. Thus, this give the relationship between the volumetric flow (V) and the cross-sectional area of the pipe as

$$V = AW(r,t) \tag{2}$$

We note the following partial derivatives

$$\frac{dW(r,t)}{dr} = \frac{\partial W(r,t)}{\partial r}\frac{dr}{dt} + \frac{\partial W(r,t)}{\partial t} = \frac{\partial W(r,t)}{\partial r}W(r,t) + \frac{\partial W(r,t)}{\partial t}\ ,$$

$$\frac{dP(r,t)}{dt} = \frac{\partial P(r,t)}{\partial r}\frac{dr}{dt} + \frac{\partial P(r,t)}{\partial t} = \frac{\partial P(r,t)}{\partial r}W(r,t) + \frac{\partial P(r,t)}{\partial t}\ . \tag{3}$$

Substituting (3) into (1), we have:

$$\frac{\partial W(r,t)}{\partial r}W(r,t) + \frac{\partial W(r,t)}{\partial t} + \frac{1}{\rho}\frac{\partial P(r,t)}{\partial r} + g\frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) = 0,$$

$$D_s^2\frac{\partial W(r,t)}{\partial r} + \frac{\partial P(r,t)}{\partial r}W(r,t) + \frac{\partial P(r,t)}{\partial t} = 0\ . \tag{4}$$

Simplifying (4) by putting the convective acceleration term to zero, we have:

$$\frac{\partial W(r,t)}{\partial t} + \frac{1}{\rho}\frac{\partial P(r,t)}{\partial r} + g\frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) = 0,$$

$$D_s^2\frac{\partial W(r,t)}{\partial r} + \frac{\partial P(r,t)}{\partial t} = 0\ . \tag{5}$$

3. Review of Solution Methods

3.1. Analytic Solution

Let the equation

$$L + \frac{W^2(r,t)}{2g} = C, \tag{6}$$

suggest a constant total head along a streamline [14],

where,

$$L = \frac{P}{\beta} + z, \text{ is the piezometric head,} \tag{7}$$

$z =$ elevation head, which does not change with time.

$\frac{W^2(r,t)}{2g} =$ velocity head,

$\frac{P}{\beta} =$ pressure head,

$\beta = Pg$ is the specific heat weight of fluid.

Here, ` $\frac{W^2(r,t)}{2g}$ corresponds to `$\frac{\partial W(r,t)}{\partial r} W(r,t)$, and is negliable in many cases when compared with the frictional term. It is more conventional to express equation (5) in terms of L instead of `P. Hence, from equation (7), we define

$\frac{\partial L}{\partial r} = \frac{1}{\beta}\frac{\partial P}{\partial r} + \frac{\partial z}{\partial r} \implies \frac{\partial P}{\partial r} = \beta\left(\frac{\partial L}{\partial r} - \frac{\partial z}{\partial r}\right)$.

But $\beta = Pg$, which implies that

$$\frac{\partial P}{\partial r} = Pg\left(\frac{\partial L}{\partial r} - \frac{\partial z}{\partial r}\right), \text{ and } \frac{\partial P}{\partial t} = Pg\left(\frac{\partial L}{\partial t} - \frac{\partial z}{\partial t}\right) \quad (8)$$

Since the elevation head, z does not change with time, we have that $\frac{\partial z}{\partial t}$ is zero, and $\frac{\partial z}{\partial r}$ is expressed as $\sin \emptyset$, where $\emptyset$ is the angle of the slope.

Now, substituting (8) into (5), we arrive at

$$\frac{\partial W(r,t)}{\partial t} + \frac{1}{\rho}\left(Pg\left(\frac{\partial L}{\partial r} - \frac{\partial z}{\partial r}\right)\right) + g\frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) = 0,$$

$$D_s^2\frac{\partial W(r,t)}{\partial r} + Pg\frac{\partial L}{\partial t} = 0\ .$$

$$\implies \frac{\partial W(r,t)}{\partial t} + g\frac{\partial L}{\partial r} + \frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) = 0 \quad (9)$$

$$D_s^2\frac{\partial W(r,t)}{\partial r} + Pg\frac{\partial L}{\partial t} = 0\ .$$

$|W(r,t)|)\ \ W(r,t)|W(r,t)|)$ is dependent on the sign of the velocity and is non-linear. Thus, this makes the analytic method difficult and complicated even though the boundary conditions are simple. To proceed, we take partial derivatives of (9) with respect to t and r such that it is a frictionless case.

$$\frac{\partial^2 W(r,t)}{\partial t^2} + g\frac{\partial^2 L}{\partial t \partial r} = 0, \frac{\partial^2 W(r,t)}{\partial t \partial r} + \frac{Pg}{D_s^2}\frac{\partial^2 L}{\partial t^2} = 0 \quad (10)$$

and

$$D_s^2\frac{\partial^2 W(r,t)}{\partial r^2} + Pg\frac{\partial^2 L}{\partial t \partial r} = 0, \frac{\partial^2 W(r,t)}{\partial t \partial r} + g\frac{\partial^2 L}{\partial r^2} = 0 \quad (11)$$

Now, substituting (11) from (10), we have

$$\frac{\partial^2 W(r,t)}{\partial t^2} - D_s^2\frac{\partial^2 W(r,t)}{\partial r^2} = 0,\ \frac{\partial^2 L}{\partial t^2} - D_s^2\frac{\partial^2 L}{\partial r^2} = 0. \quad (12)$$

The equations (12) are called the one-dimensional wave equations which can be solved individually for flow velocity or piezometric head. Basically, the analytic solution to (12) is due to d' Alembert (IEEE, 2008). It is given as

$$L(r,t) = \frac{f(r-D_s)+f(r+D_s)}{2} - \frac{1}{2D_s}\int_{r-D_s}^{r+D_s} g(s)ds, \quad (13)$$

with conditions:

- $L(r,0)$ defines the initial pressure across the conduit,
- $g(r) = \frac{\partial L(r,0)}{\partial t}$ defines the change of pressure with time.

The equation (13) is vital for understanding the phenomenon of water hammer. It is applied in a single dominant conduit with very low friction levels.

3.2 *Methods of Characteristics*

The method of characteristics transforms the two hyperbolic partial differential equations in (5) into a single ordinary differential equation (ODE). The method is described in details as presented below.

Now, introducing the Lagrange multiplier λ to equation (5), we have

$$\lambda\left[\frac{\partial W(r,t)}{\partial t} + \frac{1}{\rho}\frac{\partial P(r,t)}{\partial r} + g\frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) + D_s^2\frac{\partial W(r,t)}{\partial r} + \frac{\partial P(r,t)}{\partial t}\right] = 0 \tag{14}$$

By taking like terms together:

$$\left(\lambda\frac{\partial W(r,t)}{\partial t} + +D_s^2\frac{\partial W(r,t)}{\partial r}\right) + \left(\frac{\lambda}{\rho}\frac{\partial P(r,t)}{\partial r} + \frac{\partial P(r,t)}{\partial t}\right) + \lambda g\frac{dz}{dr} + \lambda\frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) = 0. \tag{15}$$

Hence by partial derivatives:

$$\lambda\frac{dW(r,t)}{dt} = \lambda\frac{\partial W(r,t)}{\partial t} + \lambda\frac{\partial W(r,t)}{\partial r}\frac{dr}{dt}.$$

Thus, by comparison of coefficients, we have

$$\lambda\frac{\partial W(r,t)}{\partial t} + \lambda\frac{\partial W(r,t)}{\partial r}\frac{dr}{dt} = \lambda\frac{\partial W(r,t)}{\partial t} + D_s^2\frac{\partial W(r,t)}{\partial r}.$$

$$\Rightarrow \lambda\frac{dr}{dt} = D_s^2. \tag{16}$$

Similarly,

$$\frac{dP(r,t)}{dt} = \frac{\partial P(r,t)}{\partial t} + \frac{\partial P(r,t)}{\partial r}\frac{dr}{dt},$$

$$\Rightarrow \frac{\lambda}{\rho}\frac{\partial P(r,t)}{\partial r} + \frac{\partial P(r,t)}{\partial t} = \frac{\partial P(r,t)}{\partial t} + \frac{\partial P(r,t)}{\partial r}\frac{dr}{dt},$$

$$\Rightarrow \frac{\lambda}{\rho} = \frac{dr}{dt}. \tag{17}$$

From equation (16), we arrive at,

$$\frac{dr}{dt} = \frac{D_s^2}{\lambda} \tag{18}$$

Using (18) on (17), we have:

$$\lambda^2 = P^2 D_s^2. \tag{19}$$

$\Rightarrow \lambda = \pm P D_s$,

where $P > 0$ but D_s can either be positive or negative.

Now, subjecting $\lambda = +P D_s$, the first two terms in equation become totally differentials, that is,

$$\lambda \frac{dW(r,t)}{dt} + \frac{dP(r,t)}{dt} + \lambda g \frac{dz}{dr} + \lambda \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0. \tag{20}$$

This implies that

$$PD_s \frac{dW(r,t)}{dt} + \frac{dP(r,t)}{dt} + PD_s g \frac{dz}{dr} + PD_s \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0,$$

$$\Rightarrow \frac{dW(r,t)}{dt} + \frac{1}{PD_s} \frac{dP(r,t)}{dt} + g \frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0. \tag{21}$$

Equation (21) is known as the positive characteristic equation denoted as C_+ equation.
Also, constraining $\lambda = -PD_s$, we have,

$$\Rightarrow \frac{dW(r,t)}{dt} - \frac{1}{PD_s} \frac{dP(r,t)}{dt} + g \frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0, \tag{22}$$

which is known as the negative characteristic equation denoted as C_- equation.
Let recall that $P = \rho g(L - z)$, then

$$\frac{dP(r,t)}{dt} = \rho g \left(\frac{\mathrm{dL}}{\mathrm{dt}} - \frac{\mathrm{dz}}{\mathrm{dr}} \frac{\mathrm{dr}}{\mathrm{dt}} \right). \tag{23}$$

Substituting (23) into (21), we have:

$$\frac{dW(r,t)}{dt} + \frac{Pg}{PD_s} \left(\frac{\mathrm{dL}}{\mathrm{dt}} - \frac{\mathrm{dz}}{\mathrm{dr}} \frac{\mathrm{dr}}{\mathrm{dt}} \right) + g \frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0,$$

$$\Rightarrow \frac{dW(r,t)}{dt} + \frac{g}{D_s} \left(\frac{\mathrm{dL}}{\mathrm{dt}} \right) - \frac{g}{D_s} \frac{\mathrm{dz}}{\mathrm{dr}} \frac{\mathrm{dr}}{\mathrm{dt}} + g \frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0.$$

But $\frac{\mathrm{dr}}{\mathrm{dt}} = +D_s$,

$$\Rightarrow \frac{dW(r,t)}{dt} + \frac{g}{D_s} \left(\frac{\mathrm{dL}}{\mathrm{dt}} \right) - \frac{g}{D_s} \frac{\mathrm{dz}}{\mathrm{dr}} \frac{\mathrm{dr}}{\mathrm{dt}} + g \frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0,$$

$$\Rightarrow \frac{dW(r,t)}{dt} + \frac{g}{D_s} \left(\frac{\mathrm{dL}}{\mathrm{dt}} \right) - g \frac{\mathrm{dz}}{\mathrm{dr}} + g \frac{dz}{dr} + \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0,$$

$$\Rightarrow \frac{dW(r,t)}{dt} + \frac{g}{D_s} \left(\frac{\mathrm{dL}}{\mathrm{dt}} \right) + \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0. \tag{24}$$

Similarly, for $\frac{\mathrm{dr}}{\mathrm{dt}} = -D_s$,

$$\Rightarrow \frac{dW(r,t)}{dt} - \frac{g}{D_s} \left(\frac{\mathrm{dL}}{\mathrm{dt}} \right) + \frac{friction}{2\mathrm{A_d}} (|W(r,t)|) \; W(r,t)|W(r,t)|) = 0. \tag{25}$$

The equations (24) and (25) are the foundation for finite difference solution of equation (5).

4. The Proposed Method: Finite Difference Method

The major highlight of the method of characteristics is that it seeks relevant information about the numerical solution along the approximation to the characteristics lines. This implies that the numerical scheme for the method of characteristics offers better solution that has vivid fronts. However, major disadvantages of the method of characteristics include;

i. it is more difficult to implement than other methods such as the direct finite difference method.
ii. its approximations are unequally spaced set of points in the rt plane.

Now, let define N segments in a section of conduit such that there exist $(N + 1)$ sample points partitioned by the distance Δx.

Applying the finite difference method on (24) and (25) for the head, L_p and velocity head, W_p at the point $x,$ we have:

$$\frac{W_P-W_l}{\Delta t}+\frac{g}{D_s}[L_P-L_l]+\frac{friction}{2\mathrm{A_d}}(|W_l(r,t)|)W_l(r,t)|W_l(r,t)|=0, \quad (26)$$

$$\frac{W_P-W_r}{\Delta t}-\frac{g}{D_s}[L_P-L_r]+\frac{friction}{2\mathrm{A_d}}(|W_r(r,t)|)W_r(r,t)|W_r(r,t)|=0, \quad (27)$$

for $\Delta x=\pm D_s\Delta t.$

Note that the subscript r and l denotes sample points to the left and right of x and one Δt in the past. Moreover, the relationship between Δx and Δt subjects the solution to the characteristics lines.

Multiplying the equations (26) and (27) by Δt, we obtain

$$W_P-W_l+\frac{g}{D_s}[L_P-L_l]+\frac{friction\Delta t}{2\mathrm{A_d}}(|W_l(r,t)|)W_l(r,t)|W_l(r,t)|=0, \quad (28)$$

$$W_P-W_r-\frac{g}{D_s}[L_P-L_r]+\frac{friction\Delta t}{2\mathrm{A_d}}(|W_r(r,t)|)W_r(r,t)|W_r(r,t)|=0, \quad (29)$$

We now solve for W_p and L_p in (28) and (29) above.

4.1 Steady State Flow

Let $\frac{dW(r,t)}{dt}=0$ in (24), then

$$\frac{g}{D_s}\left(\frac{\mathrm{dL}}{\mathrm{dt}}\right)+\frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ W(r,t)|W(r,t)|)=0.$$

This implies that

$$\frac{g}{D_s}\frac{\mathrm{dL}}{\mathrm{dr}}.\frac{\mathrm{dr}}{\mathrm{dt}}+\frac{friction}{2\mathrm{gA_d}}(|W(r,t)|)\ W(r,t)|W(r,t)|)=0,$$

$$\Rightarrow\frac{\mathrm{dL}}{\mathrm{dr}}=-\frac{friction}{2\mathrm{gA_d}}(|W(r,t)|)\ W(r,t)|W(r,t)|)=0. \quad (30)$$

Thus, the slope pf piezometric head is estimated by the flow constant velocity, W_0.

Integrating (30) with respect to r, we have that

$$L(r)=-\frac{friction}{2\mathrm{gA_d}}(|W_0(r,t)|)\ W_0(r,t)|W_0(r,t)|)r+d_i=0$$

Let define $L(r=0)=L_\alpha$ to represent the head at the reservoir. Then,

$L(0)=L_\alpha=d_i$.

Also, let $L(r=l)=L_\beta$ to represent the lower end of the conduit. Then,

$$L(l)=-\frac{friction}{2\mathrm{gA_d}}(|W_0(r,t)|)\ W_0(r,t)|W_0(r,t)|)l+L_\alpha=L_\beta. \quad (31)$$

Thus, we have,

$$|W_0(r,t)|)\ W_0(r,t)|W_0(r,t)|=\frac{2gA_d}{friction}\left(L_\beta-L_\alpha\right).$$

Hence, the steady state velocity is defined by:

$$W_0(r,t)=\sqrt[3]{\frac{2gA_d}{lfriction}}\left(L_\beta-L_\alpha\right). \quad (32)$$

Explicitly, we let $(i-1)$ and $(i+1)$ to refer to points at the left and right of p respectively. Thus, (24) and (25) can be rewritten as:

$$W_{pi} = \frac{1}{2}[W_{1-i} + W_{1+i}] + \frac{g}{D_s}[L_{1-i} - L_{1+i}] - \frac{friction\Delta t}{2\mathrm{A_d}}(|W_{1-i}(r,t)|)W_{1-i}(r,t)|W_{1-i}(r,t)| + \quad (|W_{1+i}(r,t)|)W_{1+i}(r,t)|W_{1+i}(r,t)|),$$

$$L_{pi} = \frac{1}{2}\left[\frac{D_s}{g}(W_{1-i} - W_{1+i})\right] + [L_{1-i} + L_{1+i}] - \frac{D_s}{g}\frac{friction\Delta t}{2\mathrm{A_d}}(|W_{1-i}(r,t)|)W_{1-i}(r,t)|W_{1-i}(r,t)| -$$

$$(|W_{1+i}(r,t)|)W_{1+i}(r,t)|W_{1+i}(r,t)|) = 0 \ ,$$

(33)

which is valid in the solution space $2 \leq i \leq N$ called the interior points.

4.2 Initial and Boundary Conditions Estimate

At the valve, the velocity remains at a steady state during the initial two seconds. Thereafter, there is an exponential decay of the velocity to zero as the valve closes. Suppose W_0 denotes the velocity at the steady state,

$$W_{N+1}(t) = \begin{cases} W_0 & t < 2 \\ W_0 \ -70^{(t-2)} & otherwise \end{cases} \tag{34}$$

Initial and final pressures of the pipeline is given as:

$$P_0(t) = P_0, P_{N+1}(t) = 0. \tag{35}$$

Initial pressure and velocity distribution along the pipeline is given as:

$$P_{N+1}(0) = \left[P_i(0) = P_0\left(1 - i\frac{dx}{L}\right)\right], \ i = 1 \dots N,$$

$$[W_i(0) = W_0], \ i = 1..N. \tag{36}$$

Since, there is no change in velocity at the node 0, we have that the velocities at the nodes 0 and 1 are equal, that is,

$$W_0(t) = W_1(t) \ .$$

The P_0 and W_0 can be calculated the steady state pipeline velocity from the Darcy-Weisbach equation [9] :

$$W_0 = P_0 = friction(W)\frac{L}{A_d}\frac{\rho V^2}{2}, \tag{37}$$

where, L is the length of pipe work (m), $friction(W)$ is the frictional factor, ρ is the liquid density, $\mathrm{A_d}$ is the pipe diameter, V is the cross-sectional area of the pipe.

The frictional factor , $friction(W)$ in a laminar flow in a pipe (circular) is given as [12]

$$friction(W) = \frac{1}{\left(1.8log10\left(\frac{6.9}{Rey} + \left(\frac{e}{3.7A_d}\right)^{1.11}\right)\right)^2} \ , \tag{38}$$

where *Rey* denotes Reynolds number [13], [14] given as

$$Rey = \frac{VA_d}{\mu} \ , \tag{39}$$

where μ is the kinematic viscosity.

5. Discretization of the Partial Differential Equations

Discretizing the PDEs (equation (1) and (2)) by replacing the spatial derivatives with a central difference approximation gives these equations:

$$\frac{d}{dt}W_i(t) + \frac{1}{\rho}\frac{P_{i+i}(t) - P_{i-1}(t)}{2\Delta x} + \frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) = 0\ , \tag{40}$$

$$D_s^2\frac{\partial W(r,t)}{\partial r} + \frac{\partial P(r,t)}{\partial t} = 0\ , \tag{41}$$

where $i = 1 \ldots N$.

5.1 *Discretizing the PDE into ODE*

Let the number of nodes be given as N, and the length, L of each node be defined as $dx = \frac{L}{N}$, then the Spatially discretized form of each PDE is given as

$$\frac{d}{dt}W_i(t) + \frac{1}{\rho}\frac{P_{i+i}(t) - P_{i-1}(t)}{2\Delta x} + \frac{friction}{2\mathrm{A_d}}(|W(r,t)|)\ \ W(r,t)|W(r,t)|) = 0\ , \tag{42}$$

$$D_s^2\frac{\partial W(r,t)}{\partial r} + \frac{\partial P(r,t)}{\partial t} = 0\ , \tag{43}$$

where for $i = 1..N$, generate the entire set of ODEs.

6. Solving the ODEs and Plot Pressure at Valve

In this section, we solve the entire set of ODEs generated at various nodes N, to establish a pressure plot at the valve of the closed pipe. To this effect, the MAPLE 18 software is adopted to implement all computational framework.

In line with [9], let the liquid density, $\rho = 1000$; Viscosity, $\mu = 0.001$; Pipe diameter, $\mathrm{A_d} = 0.1$; Wall thickness, thick = 0.001, Roughness, $e = 0.0001$; Length, L=25; Young's modulus, E=70 $\times 10^9$, Cross-sectional area, $A = \frac{1}{4}\pi \mathrm{A_d}^2$, Pressure at start of pipeline, psource = 0.5 $\times 10^6$; Bulk modulus, K=200 $\times 10^6$; Effective modulus of system, $D_s = \frac{1}{\left(\frac{1}{k} + \frac{\mathrm{A_d}}{E \times thick}\right)}$, we obtain the steady velocity denoted by *Wsteady* as 14.19058741 via the equations (34) – (37) through MAPLE 18 software.

Now, let N=30, then equation (42) and (43) generate the set of ODEs as follows:

$$\frac{d}{dt}w_1(t) + \frac{1}{5000}p_2(t) - 100 + 5friction(|w_1(t)|w_1(t)|w_1(t)|) = 0,$$

$$\frac{1}{5}w_2(t) - \frac{1}{5}w_0(t) + 6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_1(t)\right) = 0,$$

$$\frac{d}{dt}w_2(t) + \frac{1}{5000}p_3(t) - \frac{1}{5000}p_1(t) + 5friction(|w_2(t)|w_2(t)|w_2(t)|) = 0,$$

$$\frac{1}{5}w_3(t) - \frac{1}{5}w_1(t) + 6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_2(t)\right) = 0,$$

$$\frac{d}{dt}w_3(t) + \frac{1}{5000}p_4(t) - \frac{1}{5000}p_2(t) + 5friction(|w_3(t)|w_3(t)|w_3(t)|) = 0,$$

$$\frac{1}{5}w_4(t) - \frac{1}{5}w_2(t) + 6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_3(t)\right) = 0,$$

$\frac{d}{dt}w_4(t)+\frac{1}{5000}p_5(t)-\frac{1}{5000}p_3(t)+5friction(|w_4(t)|w_4(t)|w_4(t)|)=0,$

$\frac{1}{5}w_5(t)-\frac{1}{5}w_3(t)+6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_4(t)\right)=0,$

$\frac{d}{dt}w_5(t)+\frac{1}{5000}p_6(t)-\frac{1}{5000}p_4(t)+5friction(|w_5(t)|w_5(t)|w_5(t)|)=0,$

$\frac{1}{5}w_6(t)-\frac{1}{5}w_4(t)+6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_5(t)\right)=0,$

$\frac{d}{dt}w_6(t)+\frac{1}{5000}p_7(t)-\frac{1}{5000}p_5(t)+5friction(|w_6(t)|w_6(t)|w_6(t)|)=0,$

$\frac{1}{5}w_7(t)-\frac{1}{5}w_5(t)+6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_6(t)\right)=0,$

$\frac{d}{dt}w_7(t)+\frac{1}{5000}p_8(t)-\frac{1}{5000}p_6(t)+5friction(|w_7(t)|w_7(t)|w_7(t)|)=0,$

$\frac{1}{5}w_8(t)-\frac{1}{5}w_6(t)+6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_7(t)\right)=0,$

$\frac{d}{dt}w_8(t)+\frac{1}{5000}p_9(t)-\frac{1}{5000}p_7(t)+5friction(|w_8(t)|w_8(t)|w_8(t)|)=0,$

$\frac{1}{5}w_9(t)-\frac{1}{5}w_7(t)+6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_8(t)\right)=0,$

$\frac{d}{dt}w_9(t)+\frac{1}{5000}p_{10}(t)-\frac{1}{5000}p_8(t)+5friction(|w_9(t)|w_9(t)|w_9(t)|)=0,$

$\frac{1}{5}w_{10}(t)-\frac{1}{5}w_8(t)+6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_9(t)\right)=0,$

$\frac{d}{dt}w_{10}(t)+\frac{1}{5000}p_{11}(t)+5friction(|w_{10}(t)|w_{10}(t)|w_{10}(t)|)=0,$

$\frac{1}{5}w_{11}(t)-\frac{1}{5}w_9(t)+6.42857143\ \ 10^{-19}\left(\frac{d}{dt}p_{10}(t)\right)=0.$

During the initial two seconds, the velocity at the valve is at a steady state. Thus,

$$w_6=\begin{cases}14.19058741 & t<2\\ 14.19058741e^{-70t+140} & otherwise\end{cases}$$

Pressure at the start and end of the pipeline is

$p_6(0)$=0

Initial pressure and velocity distribution along the pipeline:

Initial conditons [9]:

$p_1(0)=4.5\ \ 10^5, p_2(0)=4.0\ \ 10^5,\ p_3(0)=3.5\ \ 10^5, p_4(0)=3.0\ \ 10^5$, $p_5(0)=2.5\ \ 10^5, p_6(0)=2.0\ \ 10^5,\ p_7(0)=1.5\ \ 10^5, p_8(0)=1.0\ \ 10^5, p_9(0)=0;\ w_1(0)=14.19058741, w_2(0)=14.19058741,\ w_3(0)=14.19058741, w_4(0)=14.19058741,\ w_5(0)=14.19058741, w_6(0)=14.19058741, w_7(0)=14.19058741,\ w_8(0)=14.19058741,$ $w_9(0)=14.19058741,\ w_{10}(0)=14.19058741.$

The velocity at node 0 is equal to the velocity at node 1. Hence,

$w_0(0)=w_1(0).$

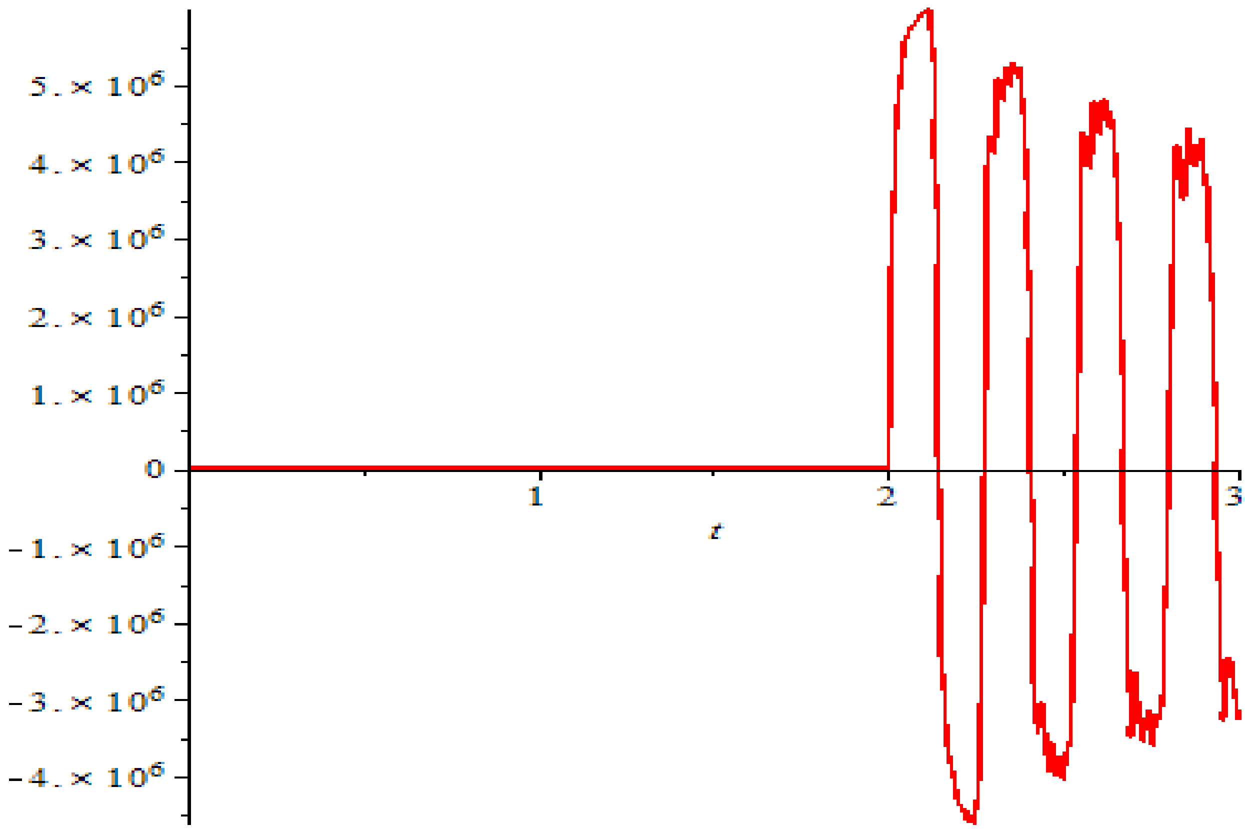

Fig. 2 A pressure $\{P_{30}(t)\ , t = 0\ \}$ plot at the valve

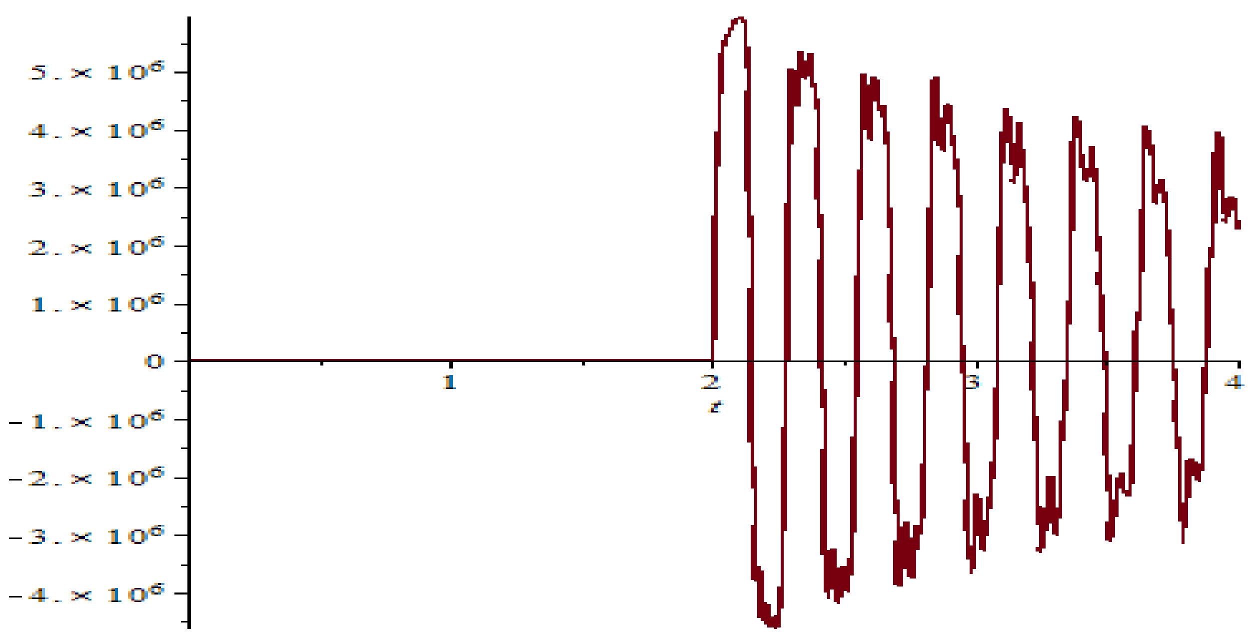

Fig. 3 A pressure $\{P_{30}(t)\ , t = 2\ \}$ plot at the valve

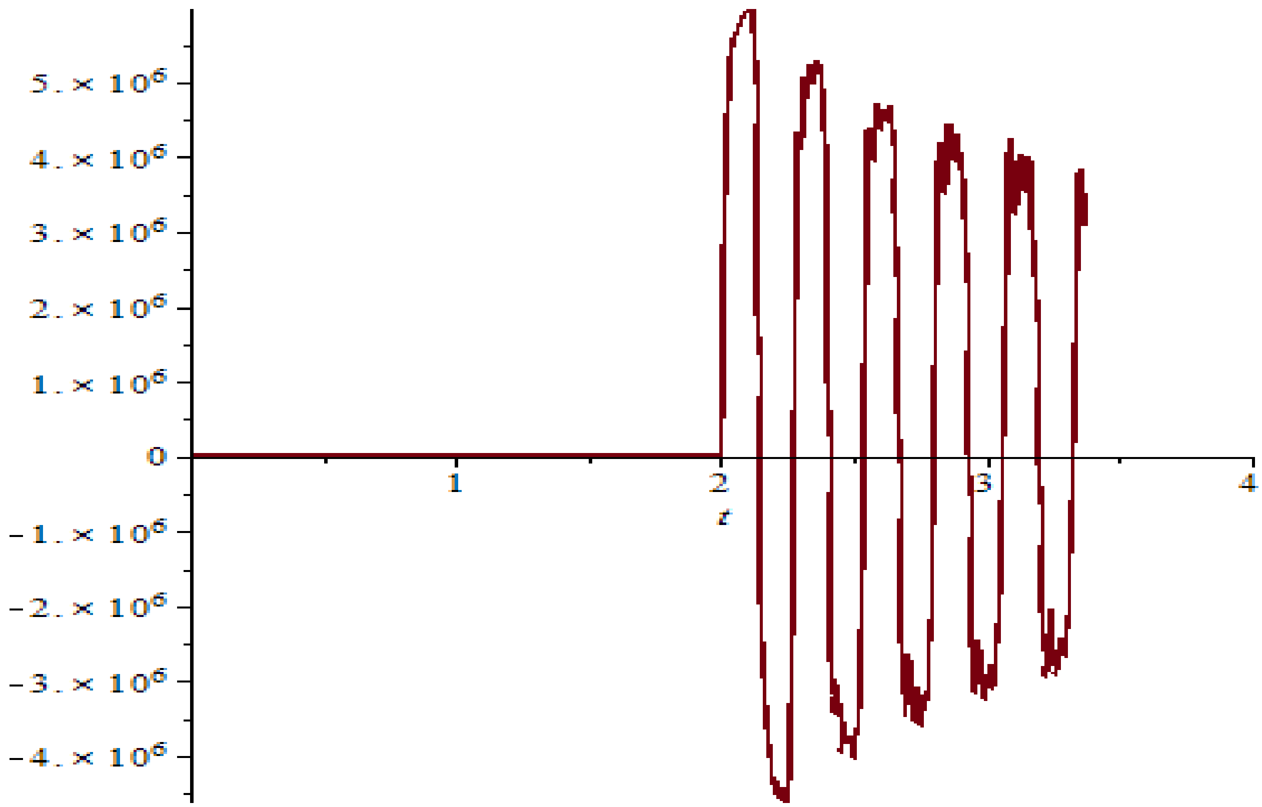

Fig 4. A pressure $\{P_{30}(t)\ , t = 0..4\ \}$ plot at the valve

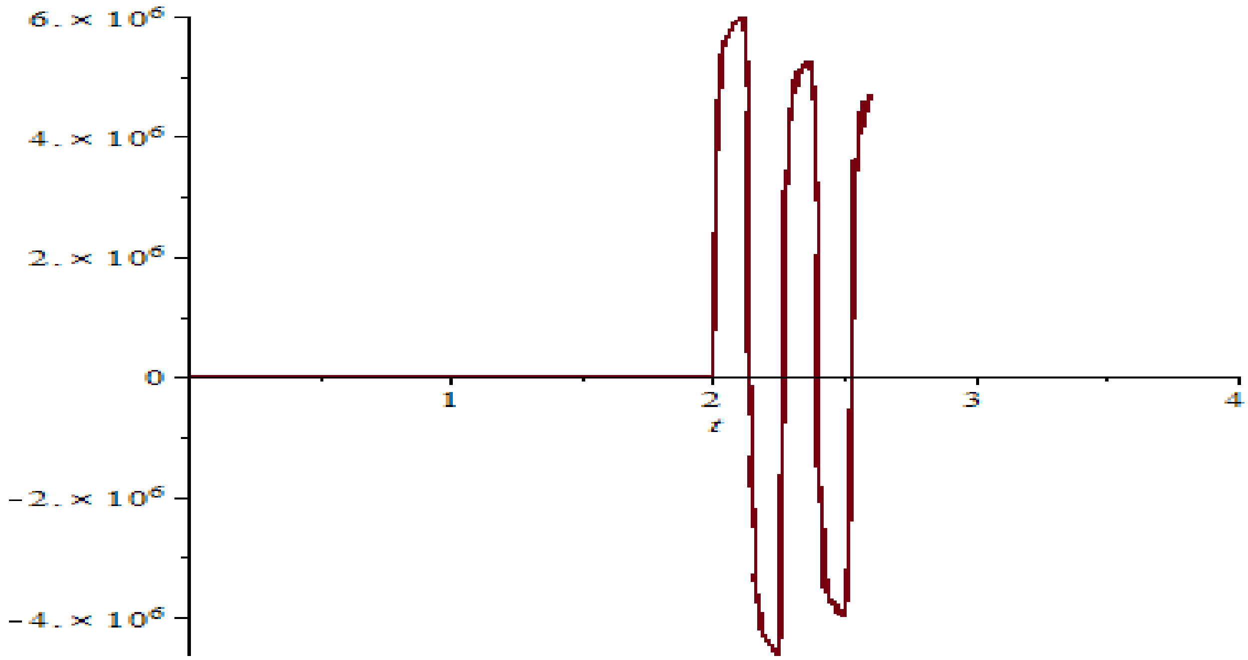

Fig. 5 A pressure $\{P_{30}(t)\ , t = 4\ \}$ plot at the valve.

7. Discussion of Results

From numerical simulations presented in Fig. 2, 3, 4 and Fig. 5, we make the following observations:

- The model (with steady friction factor) is appropriate for the simulation of water hammer in pressurized pipes made from polymers such as polyvinyl Chloride. The maximum pressure amplitude simulated by the model is estimated accurately.
- The pipe wall constraint, liquid density, pipe diameter has a significant influence on obtained results. The lower values of A_d, D_s, and A , the lower the amplitudes occurs (see Fig. 5).
- During the stimulation, the experimental pressure runs show the pressure increases after the valve closure linearly (see Fig. 2, 3 and 4).

8. Conclusion

Water hammer is very essential for risk analysis and operation of pipeline system. This work studied the effects of steady friction factor, wall constraint, pipe diameter and liquid density for the analysis of water hammer in a pipeline system with special emphasis on the finite difference method (FDM) due to its accurate and reliable evaluation of the water hammer problem. Numerical validation of the FDM demonstrates that the FDM performs well and is as good as the method of characteristics (MOC) ([9]) in terms of computational speed and accuracy.

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