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Original Article

From the Embodied to the Axiomatic World of Mathematics: Students' Perceptions on the Concept 'Limit'

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***Abstract* -** *This study aims to explore whether and if so, at what extent the students of the Mathematics department have developed the axiomatic/typical understanding of the concept of 'limit', after having firstly explored and examined students' current perceptions upon this concept. The sample of the study consisted of 37 students. The sample students were given a questionnaire constructed according to the theory of D. Tall about the three worlds of Mathematics (Embodied, Proceptual, and Axiomatic). The data were analyzed using a mixed approach; including a content analysis method and the use of two statistical programs (SPSS and CHIC). Findings of the study revealed students' perceptions regarding the concept 'limit' seem deviant from the typical concept of 'limit' that axiomatic world presents and accepts. As such they strongly suggest that the teaching of the concept limit shall follow a more holistic approach emphasizing the transition from the institutional comprehension of the 'limit' to its typical apprehension and reverse. Towards, this direction the use of a plethora of mathematical tools is deemed the way forward.*

***Keywords* -** *Concept limit, University students, Three worlds of Mathematics, Teaching, Cognitive difficulties, Calculus.*

1. Introduction

1.1. The Three Worlds of Mathematics and the Concept 'Limit'

This Research conducted in the field of 'Mathematics Education' refers consistently to the fundamental nature of the concept 'limit' ([1], [2], [13]. [14], [15], [16], [18], [19], [20], [21], [22]) as well to the extreme difficulties students meet in order the latter obtain a reliable and robust knowledge on the concept 'limit'. A review in the literature shows that although there are many research studies that explore and define the ways to overcome students' difficulties, there is little academic research verifying that these methods or ways are actually successful to address them ([3]). The studies of Bezuidenhout [2] and of Przenioslo [4], distinctively demonstrate the aforementioned gap as they verify that an integrated conception of the 'limit' amongst the students is extremely rare. According to these scholars, the difficulty of understanding this concept partially lies on their weakness to pass from an untypical/dynamic conceptualization of 'limit' (i.e. the concept of 'limit' hems the concept of movement as it is demonstrated in one of the procedures of approaching) to a more typical/formal understanding. Accordingly, most research studies in Greece while they explore the perceptions and misconceptions of students about the concept of 'limit' and the ways these misunderstandings can be addressed; they mainly refer to secondary level students ([5], [17], [24]). Thus, the current study aims to explore the existing perceptions of students of the Department of Mathematics and whether these students have developed the typical understanding of the concept. In order for the purpose of the study to be accomplished, we based on the theoretical framework of the three worlds of Mathematics of David Tall. Tall's theory [6] embraces all the kind of representations and processes applied in the field of Mathematics and states that the mathematical development is taking place through these three worlds:

- Embodied world: a version of the perceptual/physical world enriched with conceptual categorizations. This world has a Platonic dimension and presents an increasing complexity where physical experiences are transformed into imaginary ideas. For instance, at first a student perceives a straight line through his/her senses and then, with the use of language he/she conceptualizes it as an abstract shape without thickness, beginning or end.
- Proceptual world: a world with objects represented by a symbol, which renders both a process and a concept (procept). By maintaining the symbol common for both the process and its result (e.g. the $\lim_{x \to a} f(x)$ represents both the process of calculating the respective limit and the result of its calculation), a world of symbolic concepts is constructed, which, although arises from the perceptual/physical world, is not bonded to the latter.

- Axiomatic world: a world where Mathematics constitute an integrated theoretical edifice basing on certain axioms. Typical examples are the various topologies and the axioms of real numbers that is the basis of Calculus.

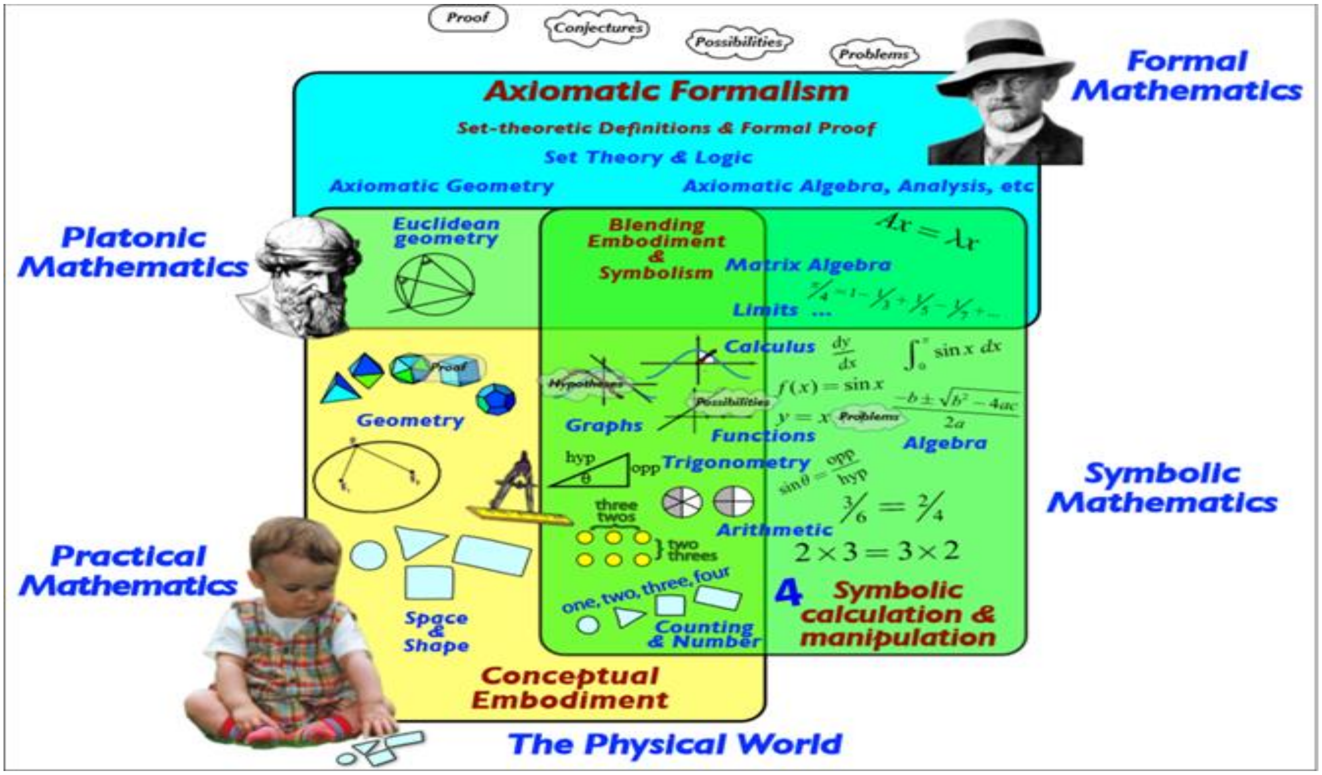


Fig. 1 Three Worlds of Mathematics, adopted from Tall (2009)

The first two worlds, the Embodied and the Proceptual, are the worlds that respectively dominate in the teaching of Mathematics in the primary and secondary education, whereas the axiomatic way of thinking begins to be cultivated in the upper-secondary education and is completed in the tertiary one ([5]). The three worlds of Tall interact with each other and are developed in a circular instead of linear way. More specifically, one needs to function efficiently in each of these worlds separately and simultaneously to be able to move with flexibility and convenience from one world to another. In other words, in order to develop a more sophisticated perception on a mathematical concept, he/she has to execute the ‘complete cognitive circle’ (i.e. to travel from the general to the abstract, from the embodiment and the symbolization to formalism and reverse) ([7]). Consequently, it is not expected from a university student of Mathematics to just ‘be’ in the Axiomatic world but rather to be able to travel through and in-and-out these three worlds with convenience and if possible to combine them depending on the tasks and methods the student has to apply. The key components to this procedure of ‘transmission’ are the external representations/semiotics (symbols, graphic representations, verbal phrases etc) ([8]).

2. Research Methodology

2.1. Research Questions

The purpose of the research is to explore students’ perceptions about the concept of limit, as well as the degree to which they have developed a formal understanding of it. Thus, the research questions are as follows:

- Which is the dominant students' perception on the concept of limit?
- At what extent students use their abilities/experiences acquired from all the three worlds in order to understand the concept of limit?
- At what extent students combine these obtained abilities/experiences in order to be able to easily move amongst the three worlds?

2.2. Research Sample

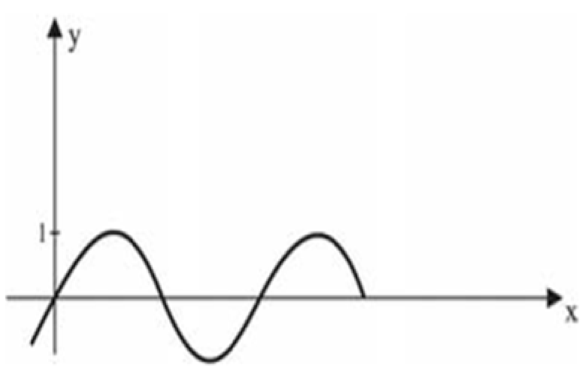
The research sample consists of thirty seven (37) university students of a department of Mathematics. The students were selected under the criterion of whether they had already attended the course of 'CALCULUS' taught at the first semester and of the criterion of approachability. The majority of the selected students had successfully completed the above course.

2.3. Measurement Tool

For the data collection we constructed and distributed a questionnaire to the students. The questionnaire consisted of three different sections of open and closed questions. Each section aimed to answer each of the research questions respectively. The first section consisted of four (04) sub-themes: in the first two (02), students had to describe with their own words how in general they perceive the concepts of function limit and limit of sequence of real numbers, while in the last two (02) themes they had to specifically describe how they perceive and conceptualize the given facts that (a) the limit of one function is the number L while x converges to a point (Conv) and (b) the function limit does not exist (Nonconv). In these last two sub-themes, students had also to graphically represent the functions in both conditions (Conv, Nonconv). The second section consisted of three sub-themes, corresponding to the three words. In the first theme, students were given a graph of a function interrupted in several points and were asked to calculate nine (09) limits, which either existed or not and justify their answers. In the second theme, students were asked to calculate four (04) limits by applying a certain calculation process/algorithm (in the case students believe these limits exist). In the third theme, they asked to calculate a limit by using the typical ε-δ definition .

Lastly, the third section consisted of three themes (Pr2Emb, Pr2Ax, Ax2Emb). In the thirst theme and with the intention to explore students' ability to translate a symbolic representation (Proceptual world) to a visual one (embodied), students were asked to fill in the graph of four different functions (if they think it is possible) in order the fact $\lim_{x\to+\infty} f(x)$=1 to be valid (Figure 2). Additionally, students were given three different graphs of functions and were asked to justify whether or not the fact $\lim_{x\to 1} f(x)$ =2 can be valid in some of these given graphs (Figure 2).

Moreover, in order to explore the students 'transition abilities' from the Proceptual to the Axiomatic world, we asked them to extensively report all the theorems and properties of 'CALCULUS' they had used at the second theme of the second section. Finally, we gave to students the proof of the theorem of uniqueness of the sequence of the limit and we presented them a case were one of a fellow student struggled to understand it. Thus, the sample students were asked to describe how they explained the proof by using any shape, symbolization or/a visual representation so as to ensure that the fellow student completely understand and perceived the proof. The utter purpose of this question was to examine the extent students were able to imprint schematically and visual-spatially an abstract/axiomatic phrase, which is quite far from their intuitions and subsequently to explore students' ability to move from Axiomatic to Embodied world. Students had one (01) hour to complete the questionnaire strictly.



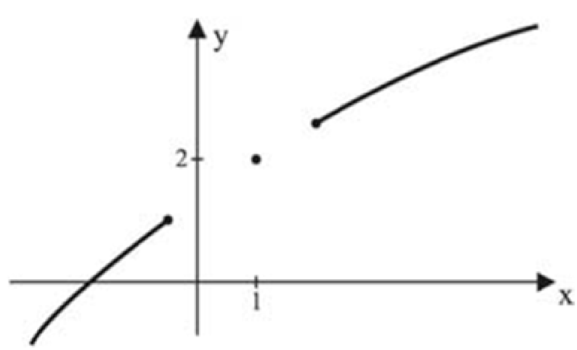


Fig. 2 An example of a question of the third section (Pr2Emb) adopted from Przenioslo (2004)

[1] « $\lim_{x\to a} f(x) = l$ ⇔For each ε>0 there exists δ>0, such that, if 0<$|x-a|$< δ then $|f(x)-l|$< ε ».

2.4. Data Analysis

Firstly, a content analysis of the data was deemed the appropriate method in order to analyze and define the perceptions of students on the concept of 'convergence'. Secondly, we used the statistical SPSS software in order to perceive and describe the general performance of the sample. Lastly, we conducted a hierarchical analysis with the use of CHIC software, which enables the detection of the inductive relationships amongst the variables ([9], [23]).

As far as the content analysis is concerned we classified the students' answers to three categories of perceptions according to Williams ([10]) and Cottrill et al. ([3]): the dynamic, the static and the mixed one. The classification was contacted under the criterion of whether each answer either matched to or explained or stated directly the categories' content. More specifically, the dynamic category contains all the answers that exclusively refer to the movement of the variable x to the number a, and to the corresponding values of the function towards the number L. The static category includes the answers that do not refer to the movement of the variables. While the mixed one includes the answers that refer both to the static and the dynamic nature of the 'limit'. The below descriptions were drawn from the data pool and represent the most distinctive examples of each category:

- Example number 1 (dynamic category):

  "The values of the function approach or tends to or converges the value L or lean towards the value L, when the variable x approaches or tends to or converges the number a",

  "As the terms of a sequence increase, it gets very close to the number to which it converges",

  "The terms of the sequence approach a number, without exceeding it, no matter how large n is."

- Example number 2 (static category):

  "The values of the function lie or live near the number L, when the variable x lies or lives near the number a",

  "The values of the function are within the space near or around L, when the variable x is within the space around or near a".

- Example number 3 (mixed category):

  "The values of the function are getting closer and closer to the number L"

  "The values of the function are in an even smaller area-radius around L",

  "The values of the function are L, when x is very close to a".

It is worth mentioned that the following answers: "the values of the function are near the number L" and "the values of the function are getting closer and closer to the number L" were categorized differently as far as the latter indicates a much more sophisticated understanding of the concept of limit ([3]). According to Cotrill et al. ([3]) the second answer is very close to the formal ε-δ definition of limit, as it is potentially, an embodiment of the expression 'for each ε>0, there is δ>0'.

3. Results

The majority of the students participating in the study perceive the concept of 'convergence' dynamically with a percentage of 37,8%. Notable is also the percentage of the students that have formed a mixed perception of the concept (35,1%), while the static perception is constrained to a percentage of up to 16,2%. Four students (10,8%) did not form a clear description and their answers have not been categorized. It is notable that misunderstandings were detected to all types of perception. For example, students with dynamic view of the convergence expressed the misconception that a limit of a sequence is unreachable (boundary limit) or that a sequence converges only if her terms increase (monotonic limit). Similarly, students with a mixed view expressed the misconception that all the values of a function must be L, as x tends to a (stationary limit). It was also observed that whilst the 86,5% of the sample students is able to successfully draw the graph of a convergent function (Conv), only the 56,8% is able to explain and graphically represent the meaning of the phrase 'the limit does not exist' (Nonconv). Those difficulties are attributed to misconceptions about the limit concept as it is demonstrated in the following answers:

"A limit does not exist as the x leans towards the point a, when the function is not defined at point a."

"A limit does not exist, means that the limit leans towards the infinity."

"A limit does not exist because the function f is not continuous."

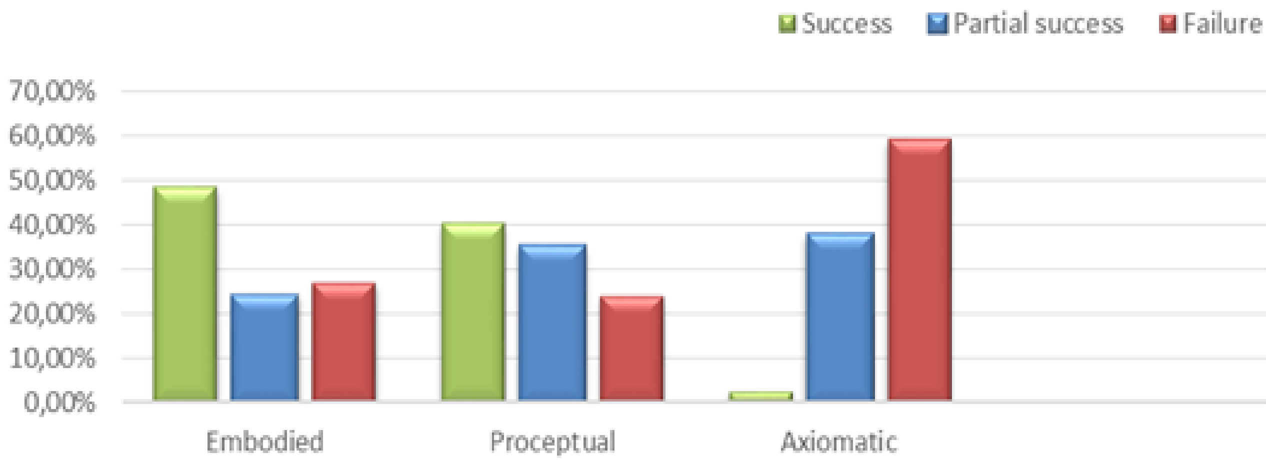


Fig. 3 Performance of sample-students in each of the categories.

Fig. 3 demonstrates the percentages of the students' performance in tasks referring to each world separately (embodied tasks, proceptual tasks, and axiomatic tasks). The sample students achieved the highest percentage at the Embodied task (48,6%), indicating that they are able to use their visuo-spatial abilities in order to calculate a limit observing the graph of a function. An almost equal percentage of the students (40,5%) managed to successfully complete the proceptual tasks and calculate the limits with the use of the algorithm. Most false answers were mainly attributed to the misapprehension that the value of the limit must be equal with the value f (xo) when the x converges to a real number xo as well as to the false connection of the concept of 'limit' with the concept of 'continuty'. Finally, all students except from one (2,7%) failed to successfully complete the axiomatic task demonstrating their overall difficulty to use the axiomatic limit definition and specifically the parameters ε and δ.

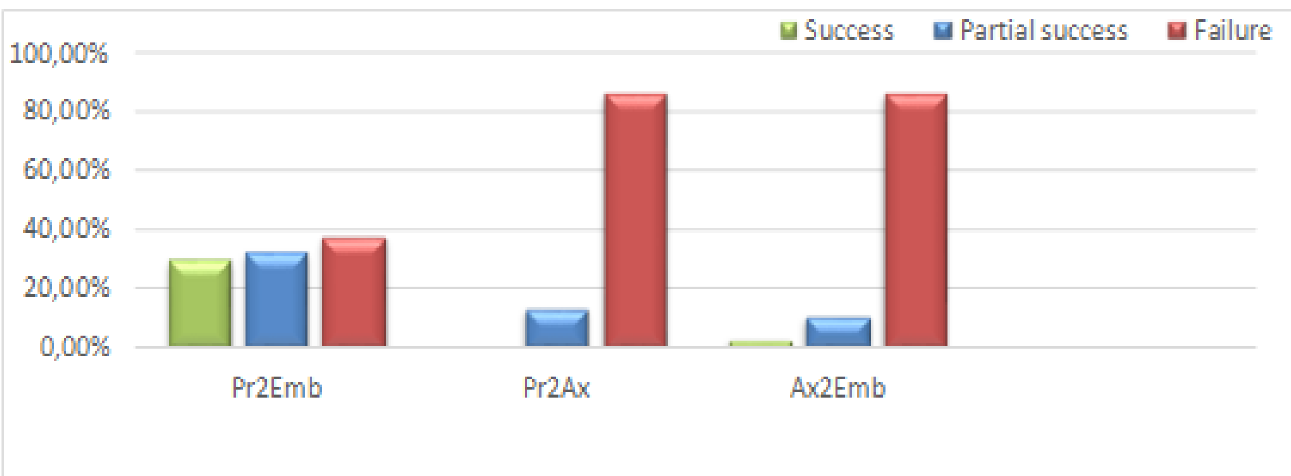


Fig. 4 Performance of the students in the transition tasks

Fig. 4 presents the percentages of the students' performance regarding the tasks of the questionnaires' third section. Students succeeded the most (29,7%) when they had to transit from the Proceptual to the Embodied world (Pr2Emb), indicating that they were able to represent a limit by transforming a symbolic representation to a graphical one. A notable conclusion is that the majority of the students experienced a high degree of difficulty to understand the relationship between the point xo and the domain of the function when the x converges to xo. Also most students struggled to understand whether this point should (or not) be the point of accumulation of the function's domain.

Students succeeded at the lowest rates when they were asked to complete tasks that demand to travel from the Proceptual to the Axiomatic world (Pr2Ax) and from the Axiomatic to the Embodied one (Ax2Emb). More specifically, none student succeeded to name the properties-theorems they had used in the proceptual task. Most students were content to write that they 'apply the algebra of limits' or that they 'apply the theorem of L' Hospital' or they just gave a non specific descriptive explanation. Notably, only one student managed to both verbally describe the proof of the uniqueness of the 'limit' and represent it visually. Most false answers revealed the overall students' difficulty to graphically represent the indicator no and to define the terms of the sequence that are near to the sequence's unique limit.

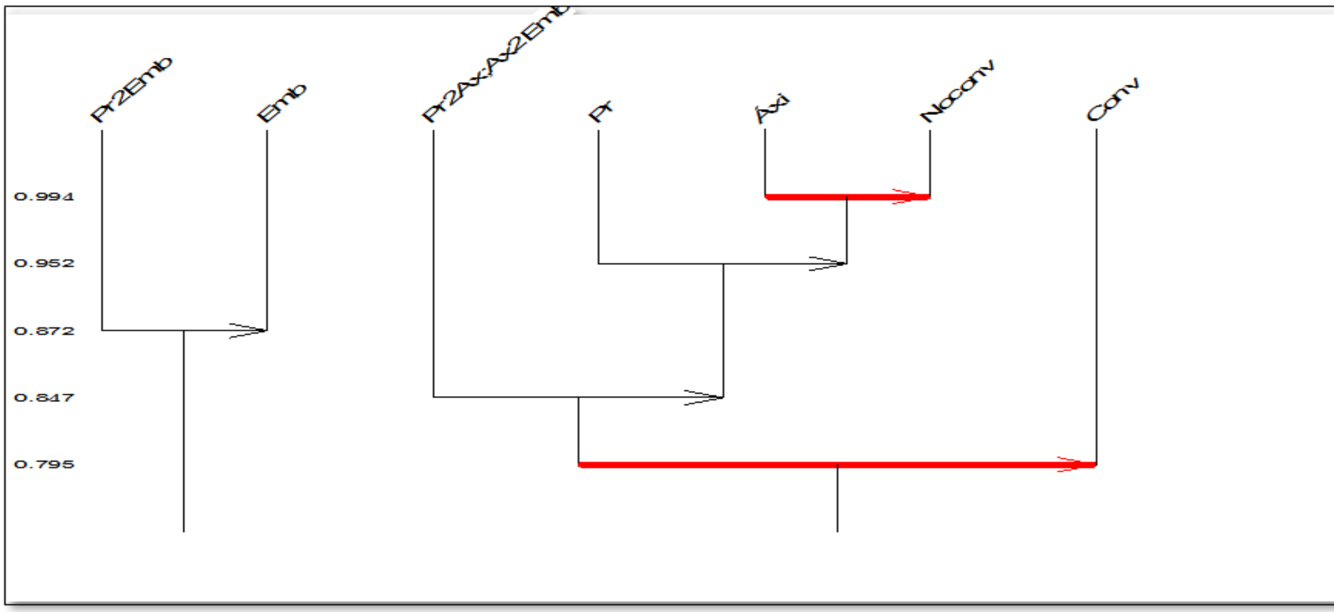


Fig. 5 Hierarchical diagram.

From the analysis of the tree-diagram of the hierarchical resemblance (Fig. 5) emerges the conclusion that the ability to move from the Proceptual to the Axiomatic world and from the Axiomatic to the Embodied one entails the ability to understand and successfully represent a function that converges (Conv). It is observed that the inductive relationship between the axiomatic tasks' success and the correct representation of a function, whose limit does not exist, is statistical important. The practical value of this finding is highly important because it indicates that a sophisticated understanding of the limit based on its axiomatic definition, leads not only to the comprehension of the concept of 'convergence' (Conv) but also of the concept of 'non-convergence' (Nonconv).

4. Discussion

Regarding the first research question, the data analysis demonstrates that the majority of students perceive the limit concept dynamically (either purely or partially) and that their perceptions diverge significantly from the typical conception of the limit. This result is in accordance with the findings of other research studies in the field, which verify the students' tendency to more dynamic interpretations of the concept of convergence even at the most advanced stages of teaching ([4]; [1]). This tendency as, Cornu advocates ([1]), is probably due to the emphasis given at the early stages of teaching, regarding the process of approaching the point via the geometrical representations. This plays a very important role as according to Przenioslo ([4]) and Cornu ([1]) the dynamic perception of 'limit' constitutes a potential conflicting factor in students' cognitive development and might yield miscomprehensions and cognitive obstacles (e.g. does a limit succeeds or not ?) ([1]).

Regarding the second and third research questions, the findings of the study show that students perform better and with higher success rate to the tasks related to the Embodied and the Proceptual world (both separately and in transition from the former to the latter) rather than the Axiomatic one. In total, we observed that the majority of students perceive the 'limit' strictly empirically or procedurally and they obtain, project and mostly apply this experiential knowledge when facing tasks with limits. Thus, students deprive from a more sophisticated, deep and typical understanding of the concept and mainly ignores the explicit properties and definitions of Calculus. It was clear from the given answers that the students strived to recall the application process of ε-δ definition and algorithmically apply it, basing on their past experiences. Therefore, either they gave a wrong answer or they reached a dead-end. The reason may lie on the contextual and complex nature of 'limit', since its calculation is non predictable, is independent from previous learning experiences and demands new cognitive capabilities that do not depend on a specific algorithm at all cases ([12]). According to our opinion, this practical dimension of limit plays a very significant role, because it might act as an obstacle for the transition to the Axiomatic world. This transition, as Tall ([7]) states, requires a circular cognitive re-construction process, which in its turn requires a much more sophisticated and deep understanding of the mathematics and a reconfiguration of the symbols and the representations through the schemata of the Axiomatic World. Plain learning of a concept definition is inadequate; students must be provided with opportunities to develop a more complete and integrated perception of the mathematic concepts by using a variety of tools (embodiments, symbols, representations, axiomatic definitions) and to lay emphasis on the relationship amongst them. This study distinctively shows that the more opportunities students are given to transit from one world to another, the greater they advance their skills in each world. As such, the teaching of the concept 'limit' shall not be constrained to a mere application of the mathematic tools; it rather should focus on developing students' ability to choose from a plethora of options and apply different tools according to the context. Towards this direction, it is deemed that teaching approaches emphasizing the transition from the untypical to the typical comprehension of the 'limit' and reverse is the way forward.

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