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Original Article

Analysis of a Feedback Bi-Tandem Queue Network in Fuzzy Environment

Vandana Saini[1], Deepak Gupta[2], A.K. Tripathi[3]

[1,2,3]*Department of Mathematics, MMDU, Mullana, Ambala Haryana, India.*

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Abstract *- The paper's objective is to analyse a feedback queue network in a fuzzy environment. For this purpose, a basic model is considered, a bi-tandem queue network linked to a common server. All queue performance measures are obtained in a stochastic environment at the initial stage. After that, work is extended to a fuzzy environment because practically all system characteristics are not exact; they are uncertain in nature. In a fuzzy environment, all fuzzy queue characteristics are obtained by using the α-cut approach, triangular fuzzy membership function, and fuzzy arithmetic operations. Yager's formula is used to defuzzify the fuzzy values. A numerical illustration is given to validate the results. This model applies to shopping complexes, administrative offices, production management, banks, and many fields.*

Keywords *- Bi-tandem queue, Feedback, Fuzzy environment, Stochastic environment, Triangular fuzzy α-cut.*

1. Introduction

Fuzzy queue models are the most absolute than crisp queue models. These are highly absolute in real-world situations. Lie, and Lee introduced the fuzzy queue models in 1989. After that, many researchers did a lot of work on fuzzy queues. T.P.Singh(2009), T.P.Singh & Kusum (2012),T.P.Singh & Arti(2013,2014), Meenu Mittal(2015), analyze queue models in fuzzy environment. Seema, Deepak Gupta, and Sameer Sharma(2013) introduced a model in which servers were linked to a common server and analysed the model in a fuzzy environment. Reeta Bhardwaj, Vijay Kumar, and T.P.Singh (2015) introduced a feedback bi-tandem queue network linked with a common channel in a steady-state environment. In this paper, feedback was allowed from the common server to each of the two channels, and the probability of revisit was considered the same as that of the first visit.

The present work is an extension of above said papers. Our paper combines both concepts: the analysis of a bi-tandem queue network with feedback in a fuzzy environment. Feedback is allowed from each channel to another channel and only for one time. This model applies to shop malls, administrative offices, and many other areas.

2. Fuzzy set theory

These sets are somewhat like sets whose elements have a degree of membership. Fuzzy sets were introduced independently by L.A.Zadeh and Dieter Klaua in 1965 as an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in binary terms, i.e., an element either belongs or does not belong to the set. On the other hand, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0,1].

Definition: In-universe of discourse X, a fuzzy subset $\tilde{A}$ on X is defined by the membership function $\mu_{\tilde{A}}(X)$, which maps each element x into X to a real number in the interval [0,1], here $\mu_{\tilde{A}}(X)$ denotes the grade or degree of membership, and it is usually denoted as

$\mu_{\tilde{A}}(X): X \rightarrow [0,1]$X is a non-member in A if $\mu_{\tilde{A}}(X)=0$ and x is a member in A if $\mu_{\tilde{A}}(X)=1$

2.1. Triangular fuzzy number

A fuzzy number $\tilde{A}$ is said to be a triangular fuzzy number only if there exist real numbers a<b<c such that,

$$\tilde{A} = \begin{cases} \frac{x-a}{b-a}, for(a \le x \le b) \\ \frac{x-a}{b-a}, for(b < x \le c) \\ 0, otherwise \end{cases}$$

2.2. Fuzzy Arithmetic Operations

Let $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$ be two fuzzy triangular numbers, then the basic arithmetic operations on these fuzzy triangular numbers are as follows:

(i) Sum of two fuzzy numbers = $\tilde{A} + \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

(ii) Difference of two fuzzy numbers = $\tilde{A} - \tilde{B} = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$ if $DP(\tilde{A}) \geq DP(\tilde{B})$, where

$DP(\tilde{A}) = \frac{c_1 - a_1}{2}$ and $DP(\tilde{B}) = \frac{c_2 - a_2}{2}$ here DP denotes the different points of triangular fuzzy numbers, otherwise $\tilde{A} - \tilde{B} = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$

(iii) Multiplication = $\tilde{A} \times \tilde{B} = (a_1 b_2 + b_1 a_2 - b_1 b_2, b_1 b_2, c_1 b_2 + b_1 c_2 - b_1 b_2)$

(iv) Division = $\tilde{A} / \tilde{B} = (\frac{2a_1}{a_2 + c_2}, \frac{b_1}{b_2}, \frac{2c_1}{a_2 + c_2})$

2.3. Defuzzification of triangular fuzzy numbers

Let $\tilde{A} = (a_1, b_1, c_1)$ be the triangular fuzzy number. Then by Yager's [1981] formula, crisp $A = \left(\frac{a_1 + 2b_1 + c_1}{4} \right)$

Table 1. Notations

n = number of arriving customers	
λ = arrival rate	$\tilde{\lambda}$ = fuzzy arrival rate
μ = service rate	$\tilde{\mu}$ = fuzzy service rate
p_{ij} = probability of moving first time from one server to another in the state (i.j)	$\tilde{p}_{ij}$ =fuzzy probability of moving first time from one server to another in the state (i.j)
q_{ij} = probability of moving a second time from one server to another in the state (i.j)	$\tilde{q}_{ij}$ = fuzzy probability of moving a second time from one server to another in the state (i.j)
L = average queue length of the system	$\tilde{L}$ = fuzzy average queue length of the system
L_i =partial queue length of the server, where i=1,2,3	$\tilde{L}_i$ =fuzzy partial queue length of the server, where i=1,2,3
E(W)= expected waiting time for customers	$E(\tilde{W})$ =fuzzy expected waiting time for customers

3. Mathematical Modelling

The proposed model is comprised of three servers C_1, C_2 C_3. Servers C_1, C_2 are biserial servers and C_3 are commonly connected to both the bi-tandem servers, as shown in the figure below

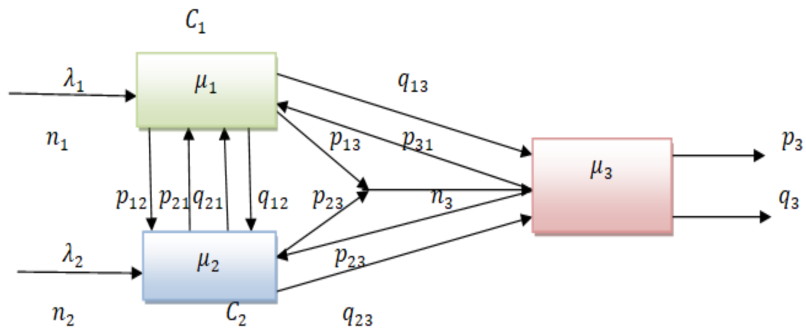

Fig. 1 Bi-tandem feedback queue model

The customers will arrive at servers C_1, C_2 with the Poisson arrival rate λ_1 λ_2. The customers, taking service after, C_1 can go to any of the servers C_2 or C_3 similar condition holds for the customers at the server C_2. The customers will get all the same possibilities when they revisit any server.

On 1st visit customer, after taking service, C_1 will leave this server with probability a and move either of the servers C_2 or C_3 with probabilities p_{12}, p_{13}. When the customers revisit the system, then he will leave C_1 with probability b and will move the service channels C_2, C_3 with probabilities q_{12}, q_{13} in such a way that $p_{12} + p_{13} = 1$ $q_{12} + q_{13} = 1$ and $ap_{12} + ap_{13} + bq_{12} + bq_{13} = 1$. Similarly, the customers will leave the server C_2 the first time with probability c and the second time with probability d and can move to C_1, C_3 with probabilities p_{21}, p_{23} in the first visit and with probabilities q_{21}, q_{23} in the second visit satisfying the conditions $p_{21} + p_{23} = 1$ $q_{21} + q_{23} = 1$ $cp_{21} + cp_{23} + dq_{21} + dq_{23} = 1$. After getting the service from the server, C_3 the customers will leave this server with probability e on 1st visit and probability f in revisit. After getting the service from C_3 the customers, either leave the system with probability p_3 or can revisit the servers C_1, C_2 and then finally exit the system with probabilities p_{31}, p_{32} and q_3 respectively. In this case, the conditions will be, $p_{31} + p_{32} + p_3 = 1$ and $ep_{31} + ep_{32} + ep_3 + fq_3 = 1$.

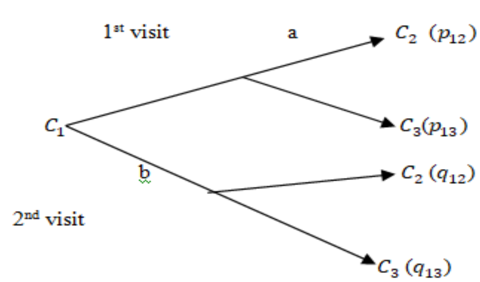

Fig. 2 Possible states of leaving the server C_1

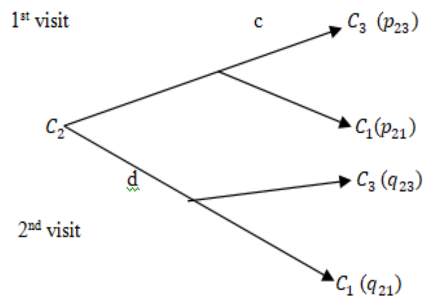

Fig. 3 Possible states of leaving the server C_2

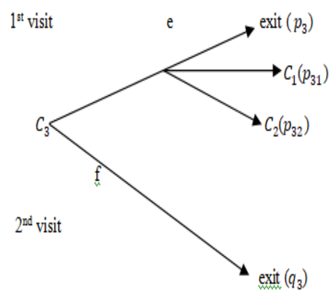

Fig. 4 Possible states of leaving the server C_3

4. Steady-state analysis

Suppose $P_{n_1,n_2,n_3}(t)$ denotes the probability of n_1, n_2, n_3 customers in front of the servers C_1, C_2, C_3 respectively, where $n_1, n_2, n_3 \geq 0$. The equations for the model in steady-state are as follows:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3)P_{n_1,n_2,n_3} = \lambda_1 P_{n_1-1,n_2,n_3} + \lambda_2 P_{n_1,n_2-1,n_3} + \mu_1(ap_{12}+bq_{12})P_{n_1+1,n_2-1,n_3} + \mu_1(ap_{13}+bq_{13})P_{n_1+1,n_2,n_3-1}$$
$$+\mu_2(cp_{21}+dq_{21})P_{n_1-1,n_2+1,n_3} + \mu_2(cp_{23}+dq_{23})P_{n_1,n_2+1,n_3-1} + \mu_3(ep_{31})P_{n_1-1,n_2,n_3+1} + \mu_3(ep_{32})P_{n_1,n_2-1,n_3+1} + \mu_3(ep_3+fq_3)P_{n_1,n_2,n_3+1} \ldots\ldots\ldots(1)$$

$n_1 = 0, n_2, n_3 > 0$

$$(\lambda_1+\lambda_2+\mu_2+\mu_3)P_{0,n_2,n_3} = \lambda_2 P_{0,n_2-1,n_3} + \mu_1(ap_{12}+bq_{12})P_{1,n_2-1,n_3} + \mu_1(ap_{13}+bq_{13})P_{1,n_2,n_3-1}$$
$$+\mu_2(cp_{23}+dq_{23})P_{0,n_2+1,n_3-1} + \mu_3(ep_{32})P_{0,n_2-1,n_3+1} + \mu_3(ep_3+fq_3)P_{0,n_2,n_3+1} \ldots\ldots\ldots(2)$$

$n_2 = 0, n_1, n_3 > 0$

$$(\lambda_1+\lambda_2+\mu_1+\mu_3)P_{n_1,0,n_3} = \lambda_1 P_{n_1-1,0,n_3} + \mu_1(ap_{13}+bq_{13})P_{n_1+1,0,n_3-1} + \mu_2(cp_{21}+dq_{21})P_{n_1-1,1,n_3}$$
$$+\mu_2(cp_{23}+dq_{23})P_{n_1,1,n_3-1} + \mu_3(ep_{31})P_{n_1-1,0,n_3+1} + \mu_3(ep_3+fq_3)P_{n_1,0,n_3+1} \ldots\ldots\ldots(3)$$

$n_3 = 0, n_1, n_2 > 0$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2)P_{n_1,n_2,0} = \lambda_1 P_{n_1-1,n_2,0} + \lambda_2 P_{n_1,n_2-1,0} + \mu_1(ap_{12}+bq_{12})P_{n_1+1,n_2-1,0}$$
$$+\mu_2(cp_{21}+dq_{21})P_{n_1-1,n_2+1,0} + \mu_3(ep_{31})P_{n_1-1,n_2,1} + \mu_3(ep_{32})P_{n_1,n_2-1,1} + \mu_3(ep_3+fq_3)P_{n_1,n_2,1} \ldots\ldots\ldots(4)$$

$n_1 = 0, n_2 = 0, n_3 > 0$

$$(\lambda_1+\lambda_2+\mu_3)P_{0,n_2,n_3} = \mu_1(ap_{13}+bq_{13})P_{1,0,n_3-1} + \mu_2(cp_{23}+dq_{23})P_{0,1,n_3-1} + \mu_3(ep_3+fq_3)P_{0,0,n_3+1} \ldots\ldots(5)$$

$n_1 = 0, n_3 = 0, n_2 > 0$

$$(\lambda_1+\lambda_2+\mu_2)P_{0,n_2,0}=\lambda_2 P_{0,n_2-1,0}+\mu_1(ap_{12}+bq_{12})P_{1,n_2-1,0}+\mu_3(ep_{32})P_{0,n_2-1,1}+\mu_3(ep_3+fq_3)P_{0,n_2,1}\ldots\ldots(6)$$

$$n_2=0, n_3=0, n_1>0$$

$$(\lambda_1+\lambda_2+\mu_1)P_{n_1,0,0}=\lambda_1 P_{n_1-1,0,0}+\mu_2(cp_{21}+dq_{21})P_{n_1-1,1,0}+\mu_3(ep_{31})P_{n_1-1,0,1}+\mu_3(ep_3+fq_3)P_{n_1,0,1}\ldots\ldots(7)$$

$$n_1=n_2=n_3=0$$

$$(\lambda_1+\lambda_2)P_{0,0,0}=\mu_3(ep_3+fq_3)P_{0,0,1}\ldots\ldots(8)$$

5. Solution of equations

The generating function technique to solve the above equations is defined as:

$$G(x,y,z)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}P_{n_1,n_2,n_3}x^{n_1}y^{n_2}z^{n_3}\ldots(9)$$

where $|x|\le 1, |y|\le 1, |z|\le 1$

And the partial generating functions are:

$$G_{n_1,n_2}(x)=\sum_{n_1=0}^{\infty}P_{n_1,n_2,n_3}x^{n_1}\ldots\ldots(10)$$

$$G_{n_3}(x,y)=\sum_{n_2=0}^{\infty}G_{n_1,n_2}(x)y^{n_2}\ldots\ldots(11)$$

$$G(x,y,z)=\sum_{n_3=0}^{\infty}G_{n_3}(x,y)z^{n_3}\ldots\ldots(12)$$

Solving the above equations by using the generating function, we get the solution,

$$G(x,y,z)=\frac{A_1\mu_1\left(1-\frac{Ay}{x}-\frac{Bz}{x}\right)+A_2\mu_2\left(1-\frac{Cx}{y}-\frac{Dz}{y}\right)+A_3\mu_3\left(1-\frac{Ex}{z}-\frac{Fy}{z}-\frac{G}{z}\right)}{\lambda_1(1-x)+\lambda_2(1-y)+\mu_1\left(1-\frac{Ay}{x}-\frac{Bz}{x}\right)+\mu_2\left(1-\frac{Cx}{y}-\frac{Dz}{y}\right)+\mu_3\left(1-\frac{Ex}{z}-\frac{Fy}{z}-\frac{G}{z}\right)}\ldots(13)$$

Here $A_1=G_0(y,z), A_2=G_0(x,z), A_3=G_0(x,y)$

$A=(ap_{12}+bq_{12}), B=(ap_{13}+bq_{13}), C=(cp_{21}+dq_{21}),$

$D=(cp_{23}+dq_{23})E=(ep_{31}), F=(ep_{32}), G=(ep_3+fq_3)$

At x=y=z=1, G(x,y,z)=1 and equation (13) reduces to indeterminate form. By applying the L'Hospital rule for the indeterminate form, we obtain,

$$-B\mu_1A_1-D\mu_2A_2+\mu_3A_3=-B\mu_1-D\mu_2+\mu_3\ldots\ldots(14)$$

$$-A\mu_1A_1+\mu_2A_2-F\mu_3A_3=-\lambda_2+\mu_2-A\mu_1-F\mu_3\ldots(15)$$

$$\mu_1A_1-C\mu_2A_2-E\mu_3A_3=-\lambda_1+\mu_1-C\mu_2-E\mu_3\ldots(16)$$

On solving the above three equations, we get,

$$A_1=1-\frac{\left(\lambda_1(1-FD)+\lambda_2(C-DE)\right)}{\mu_1\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}\ldots(17)$$

$$A_2 = 1 - \frac{\left(\lambda_1(A+BF)+\lambda_2(1-BE)\right)}{\mu_2\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]} \ldots(18)$$

$$A_3 = 1 - \frac{\left(\lambda_1(DA+B)+\lambda_2(D+BC)\right)}{\mu_3\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]} \ldots(19)$$

The solution of the steady-state differential equation is,

$$P_{n_1,n_2,n_3} = (1-G_1)^{n_1}(1-G_2)^{n_2}(1-G_3)^{n_3}G_1G_2G_3$$

$$= (1-\rho_1)^{n_1}(1-\rho_2)^{n_2}(1-\rho_3)^{n_3}\rho_1\rho_2\rho_3 \ldots\ldots\ldots(20)$$

Where,

$$\rho_1 = \frac{\left(\lambda_1(1-FD)+\lambda_2(C+DE)\right)}{\mu_1\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$= \frac{\lambda_1\{1-(ep_{32})(cp_{23}+dq_{23})\}+\lambda_2\{(cp_{21}+dq_{21})+(cp_{23}+dq_{23})(ep_{31})\}}{\mu_1\begin{bmatrix}\{1-(ap_{12}+bq_{12})(cp_{21}+dq_{21})\}\{1-(ap_{13}+bq_{13})(ep_{31})\}\\ -\{(ap_{13}+bq_{13})(cp_{21}+dq_{21})+(cp_{23}+dq_{23})\}\{(ep_{32})+(ap_{12}+bq_{12})(ep_{31})\}\end{bmatrix}}$$

$$\rho_2 = \frac{\left(\lambda_1(A+BF)+\lambda_2(1-BE)\right)}{\mu_2\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$= \frac{\lambda_1\{(ap_{12}+bq_{12})+(ap_{13}+bq_{13})(ep_{32})\}+\lambda_2\{1-(ap_{13}+bq_{13})(ep_{31})\}}{\mu_2\begin{bmatrix}\{1-(ap_{12}+bq_{12})(cp_{21}+dq_{21})\}\{1-(ap_{13}+bq_{13})(ep_{31})\}\\ -\{(ap_{13}+bq_{13})(cp_{21}+dq_{21})+(cp_{23}+dq_{23})\}\{(ep_{32})+(ap_{12}+bq_{12})(ep_{31})\}\end{bmatrix}}$$

$$\rho_3 = 1 - \frac{\left(\lambda_1(DA+B)+\lambda_2(D+BC)\right)}{\mu_3\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$= \frac{\lambda_1\{(cp_{23}+dq_{23})(ap_{12}+bq_{12})+(ap_{13}+bq_{13})\}+\lambda_2\{(cp_{23}+dq_{23})+(ap_{13}+bq_{13})(cp_{21}+dq_{21})\}}{\mu_3\begin{bmatrix}\{1-(ap_{12}+bq_{12})(cp_{21}+dq_{21})\}\{1-(ap_{13}+bq_{13})(ep_{31})\}\\ -\{(ap_{13}+bq_{13})(cp_{21}+dq_{21})+(cp_{23}+dq_{23})\}\{(ep_{32})+(ap_{12}+bq_{12})(ep_{31})\}\end{bmatrix}}$$

Also,

$$L_1 = \frac{\rho_1}{1-\rho_1}, L_2 = \frac{\rho_2}{1-\rho_2}, L_3 = \frac{\rho_3}{1-\rho_3}$$

6. Fuzzification of the model

Let us suppose that $\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3, \rho_1, \rho_2, \rho_3$ they are approximately known parameters, and we can represent them by fuzzy numbers $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3$ where $\tilde{\lambda}_j = (\lambda_j^1, \lambda_j^2, \lambda_j^3)$ $\tilde{\mu}_j = (\mu_j^1, \mu_j^2, \mu_j^3)$, $\tilde{p}_{ji} = (p_{ji}^1, p_{ji}^2, p_{ji}^3)$ for different values of j and i. by the procedure followed by Seema, Deepak Gupta, and Sameer (2013) by using α-cut, we will get,

$\tilde{\rho}_1 = (\rho_1^1, \rho_1^2, \rho_1^3)$, where

$$\rho_1^1 = \frac{\left(\lambda_1(1-FD)+\lambda_2(C+DE)\right)}{\mu_1\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$= \frac{\lambda_1^1\{1-(e^1p_{32}^1)(c^1p_{23}^1+d^1q_{23}^1)\}+\lambda_2^1\{(c^1p_{21}^1+d^1q_{21}^1)+(c^1p_{23}^1+d^1q_{23}^1)(e^1p_{31}^1)\}}{\mu_1^3\begin{bmatrix}\{1-(a^3p_{12}^3+b^3q_{12}^3)(c^3p_{21}^3+d^3q_{21}^3)\}\{1-(a^3p_{13}^3+b^3q_{13}^3)(e^3p_{31}^3)\}\\ -\{(a^3p_{13}^3+b^3q_{13}^3)(c^3p_{21}^3+d^3q_{21}^3)+(c^3p_{23}^3+d^3q_{23}^3)\}\{(e^3p_{32}^3)+(a^3p_{12}^3+b^3q_{12}^3)(e^3p_{31}^3)\}\end{bmatrix}}$$

$$\rho_1^2=\frac{\left(\lambda_1(1-FD)+\lambda_2(C+DE)\right)}{\mu_1\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$=\frac{\lambda_1^3\left\{1-(e^3p_{32}^3)(c^3p_{23}^3+d^3q_{23}^3)\right\}+\lambda_2^3\left\{(c^3p_{21}^3+d^3q_{21}^3)+(c^3p_{23}^3+d^3q_{23}^3)(e^3p_{31}^3)\right\}}{\mu_1^1\left[\begin{array}{l}\left\{1-(a^1p_{12}^1+b^1q_{12}^1)(c^1p_{21}^1+d^1q_{21}^1)\right\}\left\{1-(a^1p_{13}^1+b^1q_{13}^1)(e^1p_{31}^1)\right\}\\-\left\{(a^1p_{13}^1+b^1q_{13}^1)(c^1p_{21}^1+d^1q_{21}^1)+(c^1p_{23}^1+d^1q_{23}^1)\right\}\left\{(e^1p_{32}^1)+(a^1p_{12}^1+b^1q_{12}^1)(e^1p_{31}^1)\right\}\end{array}\right]}$$

$$\rho_1^3=\frac{\left(\lambda_1(1-FD)+\lambda_2(C+DE)\right)}{\mu_1\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$=\frac{\lambda_1^2\left\{1-(e^2p_{32}^2)(c^2p_{23}^2+d^2q_{23}^2)\right\}+\lambda_2^2\left\{(c^2p_{21}^2+d^2q_{21}^2)+(c^2p_{23}^2+d^2q_{23}^2)(e^2p_{31}^2)\right\}}{\mu_1^2\left[\begin{array}{l}\left\{1-(a^2p_{12}^2+b^2q_{12}^2)(c^2p_{21}^2+d^2q_{21}^2)\right\}\left\{1-(a^2p_{13}^2+b^2q_{13}^2)(e^2p_{31}^2)\right\}\\-\left\{(a^2p_{13}^2+b^2q_{13}^2)(c^2p_{21}^2+d^2q_{21}^2)+(c^2p_{23}^2+d^2q_{23}^2)\right\}\left\{(e^2p_{32}^2)+(a^2p_{12}^2+b^2q_{12}^2)(e^2p_{31}^2)\right\}\end{array}\right]}$$

$\tilde{\rho}_2=(\rho_2^1,\rho_2^2,\rho_2^3)$, where

$$\rho_2^1=\frac{\left(\lambda_1(A+BF)+\lambda_2(1-BE)\right)}{\mu_2\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$=\frac{\lambda_1^1\left\{(a^1p_{12}^1+b^1q_{12}^1)+(e^1p_{32}^1)(a^1p_{13}^1+b^1q_{13}^1)\right\}+\lambda_2^1\left\{1-(a^1p_{13}^1+b^1q_{13}^1)(e^1p_{31}^1)\right\}}{\mu_2^3\left[\begin{array}{l}\left\{1-(a^3p_{12}^3+b^3q_{12}^3)(c^3p_{21}^3+d^3q_{21}^3)\right\}\left\{1-(a^3p_{13}^3+b^3q_{13}^3)(e^3p_{31}^3)\right\}\\-\left\{(a^3p_{13}^3+b^3q_{13}^3)(c^3p_{21}^3+d^3q_{21}^3)+(c^3p_{23}^3+d^3q_{23}^3)\right\}\left\{(e^3p_{32}^3)+(a^3p_{12}^3+b^3q_{12}^3)(e^3p_{31}^3)\right\}\end{array}\right]}$$

$$\rho_2^2=\frac{\left(\lambda_1(A+BF)+\lambda_2(1-BE)\right)}{\mu_2\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$=\frac{\lambda_1^3\left\{(a^3p_{12}^3+b^3q_{12}^3)+(e^3p_{32}^3)(a^3p_{13}^3+b^3q_{13}^3)\right\}+\lambda_2^3\left\{1-(a^3p_{13}^3+b^3q_{13}^3)(e^3p_{31}^3)\right\}}{\mu_2^1\left[\begin{array}{l}\left\{1-(a^1p_{12}^1+b^1q_{12}^1)(c^1p_{21}^1+d^1q_{21}^1)\right\}\left\{1-(a^1p_{13}^1+b^1q_{13}^1)(e^1p_{31}^1)\right\}\\-\left\{(a^1p_{13}^1+b^1q_{13}^1)(c^1p_{21}^1+d^1q_{21}^1)+(c^1p_{23}^1+d^1q_{23}^1)\right\}\left\{(e^1p_{32}^1)+(a^1p_{12}^1+b^1q_{12}^1)(e^1p_{31}^1)\right\}\end{array}\right]}$$

$$\rho_2^3=\frac{\left(\lambda_1(A+BF)+\lambda_2(1-BE)\right)}{\mu_2\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$=\frac{\lambda_1^2\left\{(a^2p_{12}^2+b^2q_{12}^2)+(e^2p_{32}^2)(a^2p_{13}^2+b^2q_{13}^2)\right\}+\lambda_2^2\left\{1-(a^2p_{13}^2+b^2q_{13}^2)(e^2p_{31}^2)\right\}}{\mu_2^2\left[\begin{array}{l}\left\{1-(a^2p_{12}^2+b^2q_{12}^2)(c^2p_{21}^2+d^2q_{21}^2)\right\}\left\{1-(a^2p_{13}^2+b^2q_{13}^2)(e^2p_{31}^2)\right\}\\-\left\{(a^2p_{13}^2+b^2q_{13}^2)(c^2p_{21}^2+d^2q_{21}^2)+(c^2p_{23}^2+d^2q_{23}^2)\right\}\left\{(e^2p_{32}^2)+(a^2p_{12}^2+b^2q_{12}^2)(e^2p_{31}^2)\right\}\end{array}\right]}$$

$\tilde{\rho}_3=(\rho_3^1,\rho_3^2,\rho_3^3)$, where

$$\rho_3^1=\frac{\left(\lambda_1(DA+B)+\lambda_2(D+BC)\right)}{\mu_3\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$=\frac{\lambda_1^1\left\{(a^1p_{13}^1+b^1q_{13}^1)+(a^1p_{12}^1+b^1q_{12}^1)(c^1p_{23}^1+d^1q_{23}^1)\right\}+\lambda_2^1\left\{(a^1p_{13}^1+b^1q_{13}^1)(c^1p_{21}^1+d^1q_{21}^1)+(c^1p_{23}^1+d^1q_{23}^1)\right\}}{\mu_1^3\left[\begin{array}{l}\left\{1-(a^3p_{12}^3+b^3q_{12}^3)(c^3p_{21}^3+d^3q_{21}^3)\right\}\left\{1-(a^3p_{13}^3+b^3q_{13}^3)(e^3p_{31}^3)\right\}\\-\left\{(a^3p_{13}^3+b^3q_{13}^3)(c^3p_{21}^3+d^3q_{21}^3)+(c^3p_{23}^3+d^3q_{23}^3)\right\}\left\{(e^3p_{32}^3)+(a^3p_{12}^3+b^3q_{12}^3)(e^3p_{31}^3)\right\}\end{array}\right]}$$

$$\rho_3^2=\frac{\left(\lambda_1(DA+B)+\lambda_2(D+BC)\right)}{\mu_3\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$=\frac{\lambda_1^3\left\{(a^3p_{13}^3+b^3q_{13}^3)+(a^3p_{12}^3+b^3q_{12}^3)(c^3p_{23}^3+d^3q_{23}^3)\right\}+\lambda_2^3\left\{(a^3p_{13}^3+b^3q_{13}^3)(c^3p_{21}^3+d^3q_{21}^3)+(c^3p_{23}^3+d^3q_{23}^3)\right\}}{\mu_3^1\begin{bmatrix}\left\{1-(a^1p_{12}^1+b^1q_{12}^1)(c^1p_{21}^1+d^1q_{21}^1)\right\}\left\{1-(a^1p_{13}^1+b^1q_{13}^1)(e^1p_{31}^1)\right\}\\-\left\{(a^1p_{13}^1+b^1q_{13}^1)(c^1p_{21}^1+d^1q_{21}^1)+(c^1p_{23}^1+d^1q_{23}^1)\right\}\left\{(e^1p_{32}^1)+(a^1p_{12}^1+b^1q_{12}^1)(e^1p_{31}^1)\right\}\end{bmatrix}}$$

$$\rho_3^3=\frac{\left(\lambda_1(DA+B)+\lambda_2(D+BC)\right)}{\mu_3\left[(1-AC)(1-BE)-(BC+D)(F+AE)\right]}$$

$$=\frac{\lambda_1^2\left\{(a^2p_{13}^2+b^2q_{13}^2)+(a^2p_{12}^2+b^2q_{12}^2)(c^2p_{23}^2+d^2q_{23}^2)\right\}+\lambda_2^2\left\{(a^2p_{13}^2+b^2q_{13}^2)(c^2p_{21}^2+d^2q_{21}^2)+(c^2p_{23}^2+d^2q_{23}^2)\right\}}{\mu_3^2\begin{bmatrix}\left\{1-(a^2p_{12}^2+b^2q_{12}^2)(c^2p_{21}^2+d^2q_{21}^2)\right\}\left\{1-(a^2p_{13}^2+b^2q_{13}^2)(e^2p_{31}^2)\right\}\\-\left\{(a^2p_{13}^2+b^2q_{13}^2)(c^2p_{21}^2+d^2q_{21}^2)+(c^2p_{23}^2+d^2q_{23}^2)\right\}\left\{(e^2p_{32}^2)+(a^2p_{12}^2+b^2q_{12}^2)(e^2p_{31}^2)\right\}\end{bmatrix}}$$

And partial fuzzy queue lengths are,

$$\tilde{L}_1=\frac{\tilde{\rho}_1}{1-\tilde{\rho}_1},\ \tilde{L}_2=\frac{\tilde{\rho}_2}{1-\tilde{\rho}_2},\ \tilde{L}_3=\frac{\tilde{\rho}_3}{1-\tilde{\rho}_3}$$

Mean fuzzy queue length is, $\tilde{L}=\tilde{L}_1+\tilde{L}_2+\tilde{L}_3=(L_1,L_2,L_3)$

Also, $\tilde{\lambda}=\tilde{\lambda}_1+\tilde{\lambda}_2=(\lambda_1,\lambda_2,\lambda_3)$

Fuzzy average time of the customers = $E(\tilde{W})=\dfrac{\tilde{L}}{\tilde{\lambda}}=\dfrac{(L_1,L_2,L_3)}{(\lambda_1,\lambda_2,\lambda_3)}=(w_1,w_2,w_3)$

7. Numerical illustration

Let us take the value of each fuzzy parameter given in the table below, satisfying the conditions

$$\tilde{a}\tilde{p}_{12}+\tilde{a}\tilde{p}_{13}+\tilde{b}\tilde{p}_{12}+\tilde{b}\tilde{p}_{13}=1$$

$$\tilde{c}\tilde{p}_{21}+\tilde{c}\tilde{p}_{23}+\tilde{d}\tilde{p}_{21}+\tilde{d}\tilde{p}_{23}=1$$

$$\tilde{e}\tilde{p}_3+\tilde{e}\tilde{p}_{31}+\tilde{e}\tilde{p}_{32}+\tilde{f}\tilde{q}_3=1$$

Table 2. Fuzzy input variables of the proposed feedback queue model

No. of customers	Arrival Rate	Service Rate	Probabilities		
n_1=5	$\tilde{n}_1$=(3,4,5)	$\tilde{\mu}_1$=(14,16,18)	$\tilde{p}_{12}$=(.4,.3,.2)	$\tilde{q}_{12}$=(.3,.4,.5)	$\tilde{a}$ = (.6,.7,.8)
n_2=6	$\tilde{n}_2$=(2,3,4)	$\tilde{\mu}_2$=(15,16,17)	$\tilde{p}_{13}$ = (.6,.7,.8)	$\tilde{q}_{13}$ = (.7,.6,.5)	$\tilde{b}$ = (.4,.3,.2)
		$\tilde{\mu}_3$=(16,18,20)	$\tilde{p}_{21}$=(.3,.4,.5)	$\tilde{q}_{21}$=(.2,.4,.6)	$\tilde{c}$ = (.7,.6,.5)
			$\tilde{p}_{23}$=(.7,.6,.5)	$\tilde{q}_{23}$=(.8,.6,.4)	$\tilde{d}$ = (.3,.4,.5)
			$\tilde{p}_{31}$=(.1,.2,.3)	$\tilde{q}_3$=(1,1,1)	$\tilde{e}$ = (.4,.6,.8)
			$\tilde{p}_{32}$=(.3,.4,.5)		$\tilde{f}$ = (.6,.4,.2)
			$\tilde{p}_3$=(.6,.4,.2)		

After substituting these values in the results that we have obtained, we get

$\tilde{\rho}_1$ = (0.6006, 0.6340, 0.5445)

$\tilde{\rho}_2$ = (0.6215, 0.5335, 0.5331)

$\tilde{\rho}_3$ = (0.7314, 0.6323, 0.6075)

Also partial queue lengths are, $\tilde{L}_1 = \frac{\tilde{\rho}_1}{1-\tilde{\rho}_1} = \frac{(0.6006, 0.6340, 0.5445)}{(0.3994, 0.3960, 0.4555)}$ = (1.4051, 1.6010, 1.2738)

$\tilde{L}_2 = \frac{\tilde{\rho}_2}{1-\tilde{\rho}_2} = \frac{(0.6215, 0.5335, 0.5331)}{(0.3785, 0.4665, 0.4669)}$ = (1.4703, 1.1436, 1.2612)

$\tilde{L}_3 = \frac{\tilde{\rho}_3}{1-\tilde{\rho}_3} = \frac{(0.7314, 0.6323, 0.6075)}{(0.2686, 0.3677, 0.3925)}$ = (2.2126, 1.7196, 1.8378)

Total queue length= $\tilde{L} = \tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3$ = (5.088, 4.4642,4.3728)

And $\tilde{\lambda} = \tilde{\lambda}_1 + \tilde{\lambda}_2$ = (5, 7, 9)

Expected waiting time = $E(\tilde{W}) = \frac{\tilde{L}}{\tilde{\lambda}} = \frac{(5.088, 4.4642, 4.3728)}{(5,7,9)}$ = (0.7268, 0.6377, 0.6247)

Using Yager's formula for defuzzification, we get,
E(W) = 0.6567, E(λ) = 7

8. Result analysis

From the above calculation, it can be seen that the possible service utilization of 1st, 2nd, and 3rd servers are approximately 63%, 53%, and 63%, respectively. The mean queue length of the system will lie between 5.088 and 4.3728. The exact mean queue length of the system will be 4.4642. Similarly, the expected waiting time will be between 0.7268 and 0.6247, which is fuzzy. The possible value for the expected waiting time is 0.6377.

9. Conclusion

In this paper, a bi-tandem feedback queue network has been studied first in a steady-state environment and then in a fuzzy environment. Researchers developed many approaches to solving fuzzy queue characteristics. But we find it suitable to use the α-cut approach and fuzzy arithmetic operations to obtain fuzzy queue characteristics. The proposed work is more suitable because it combines feedback and a fuzzy environment. Both the concepts are related to our real life. The numerical given at the end indicates the service utilization of all the servers, suggesting that the 2nd server has been made less service utilization than the other two servers. With the help of this, an analyst can easily assign extra work to the servers in their free time. Further, this work can be extended to the models with more servers.

Limitations

- The concept of balking and reneging is not considered in the analysis of the present model.
- Priority is not given to any of the customers.
- Single arrival of the customers are considered.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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References

[1] D.G. Kendell, Some problems in theory of queues, J. Roy.Statistics Soc.Series B.13(1951)151-185.
[2] G.B .O'Brien ,The solution of some queuing problems,J.Soc.Ind.Appli.Math, 2(1954) 133-142
[3] T. Suzuki,Two Queues in series ,Journal of Operational Research Society of Japan,5(1963)149-155.
[4] N.U. Parbhu, Transient behaviour of tandem queues, Management sciences,13(1967)631-639
[5] Maggu,Phase type service queues with two servers in biseries, Journal of Operational Research Society of Japan, 13(1) (1970)
[6] R.J.Li, E.S. Lee, Analysis of fuzzy queues, Computers and Mathematics with applications. 17(7)(1989)1143-1147.
[7] T.P. Singh, V. Kumar & R. Kumar,On transient behaviour of queuing network with parallel bi-series queues, JMASS,1(2)(2005).
[8] D.Gupta, T.P.Singh, R.Kumar,Analysis of a queue model comprised of biserial and parallel channel linked with a common server, Ultra Sci.19 (2M) (2007)407-418.
[9] B.V.Houdt, J.V.Velthoven, C.Blondia,QBD Markov Chains on Binomial-Like trees and its application to multilevel feedback queues, Annals of Operations Research, 16(1) (2008)3-18.
[10] T.P. Singh, Kusum& D.Gupta ,On network queue model centrally linked with a common feedback channel, Journal of Mathematics and System Sciences, 6(2) (2010)18-31.
[11] S. Singh, M.Singh, The steady-state solution of serial queuing processes with feedback and reneging, Journal of Contemporary Applied Mathematics, 2(1) (2012)32-38.
[12] Seema, Gupta ,D.&Sharma,S., Analysis of biserial servers linked to a common server in fuzzy environment, International Journal of computing science and Mathematics,68(6)(2013)26-32.
[13] S.Sharma, D.Gupta & S.Sharma ,Analysis of network of biserial queues linked with a common server, International Journal of computing science and Mathematics,5(3)(2014) 293-324.
[14] A.N.Yakubu, U.Najim,An application of queuing theory to ATM service optimization:a case study, Mathematical Theory and Modeling, 4(6) (2014),11-23.
[15] S.Sharma, D.Gupta, &Seema ,Network analysis of fuzzy bi-serial and parallel servers with a multistage flow shop model,21st International Congress on Modelling and Simulation, Gold Coast, Australia(2015)697-703.
[16] R.Bhardwaj, V. Kumar & T.P.Singh, Mathematical modelling of feedback bitandem queue network with linkage to common channel, Aryabhatta Journal of Mathematics & Informatics 7(1) 2015,187-194.
[17] M.Mittal, T.P.Singh, D.Gupta , Threshold Effect on a Fuzzy Queue Model with Batch Arrival, Aryabhatta Journal of Mathematics & Informatics, 7(1) (2015).
[18] T.P.Singh, M.Mittal, D.Gupta, Modelling of a bulk queue system in Triangular fuzzy numbers using α-cut, International Journal of IT and Engineering, 4(9)(2016)72-79.
[19] T.P.Singh, M.Mittal, D.Gupta, Priority queue model along intermediate queue under fuzzy environment with application, International Journal of physical & Applied Sciences, 3(4)(2016) 102-113.
[20] S. Agrwal, B.K. Singh,A Comprehensive study of various queue characteristics using tri-cum biserial queuing model, International Journal of Scientific Research in Mathematical and Statistical sciences,5(2) (2018) 46-56.
[21] S. Kumar and G. Taneja ,A feedback queuing model with chances of revisit at most twice to any of the three servers, International Journal of Applied Engineering Research, 13(2018),13093-13102.
[22] M.Mittal, R.Gupta R, Modelling of biserial bulk queue network linked with common server, international journal of mathematics trends and technology,56(6),2018,430-436.
[23] S.Agrawal, V.Kumar, B.Agrawal,Analysis of tri-cum biserial bulk queue model connected with a common server,J.Math.Computer science.10(6)(2020),2599-2612.
[24] V. Saini and D. Gupta ,Analysis of a complex feedback queue network, Turkish Online Journal of Qualitative Inquiry, 12(7), (2021), 2377-2383.
[25] V. Saini and D. Gupta,A steady state analysis of Tri-cum bi-series feedback queue model", proceeding of TRIBES-2021,IEEE Xplore(2022).