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Original Article

Domination Polynomials for Inverse Graphs

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Abstract *- Let $G = (V, E)$ be a simple undirected graph with order n. The inverse, or complement, of a graph G is denoted by $\bar{G}$ and is obtained by filling in all the remaining edges needed to form a complete graph while removing all previously existing edges. Equivalently, $V(G) = V(\bar{G})$ and $E(\bar{G}) = E(Kn) - E(G)$ and also $|E(\bar{G})| = \binom{n}{2} - |E(G)|$. A domination polynomial of an inverse graph of G is given as $D(\bar{G}, x) = \sum_{i=\gamma(\bar{G})}^{n} d(\bar{G}, x)x^i$, where $d(\bar{G}, x)$ is the number of all dominating sets of $\bar{G}$ with size i and $\gamma(\bar{G})$ is the minimum domination number of $\bar{G}$. In this paper, we introduce a generating function called a domination polynomial for inverse dominating sets of cycle, star, wheel graphs, and triangular book graphs. Also, the bounds of the minimum dominating number $\gamma(G)$ discussed with their corresponding inverse of the minimum dominating number $\gamma(\bar{G})$.*

Keywords *- Dominating set, Domination polynomials, Inverse of graph.*

1. Introduction and preliminary result

All graphs considered in this paper are finite and simple. In a graph $G = (V, E)$, let $V(G)$ denote the set of all vertices of G and let $E(G)$ denote the set of all edges of G. The collected works on the subject of domination parameters in graphs have been discussed in these two books [5, 6]. Our objective in this paper is to study the domination polynomial of the inverse or compliment of particular graphs. The domination polynomial $D(G, x)$ is a generating function for the dominating sets of graphs with respect to their cardinalities. We also discussed the bounds of the minimum domination number of an inverse graph, $\gamma(\bar{G})$ and its corresponding minimum domination number, of γ(G).

Consider $G = (V, E)$ to be a simple undirected graph with a $|V(G)| = n$ order. The open neighborhood of any vertex $v \in V(G)$ is the set $N(v) = \{u \in V(G) | (u, v) \in E(G)$ and the closed neighborhood of v is the set $N[v] = N(v) \cup S\{v\}$. For a set $S \subseteq V$ (G), the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. If $N[S] = V$ or, equivalently, every vertex in $V(G) - S$ is adjacent to at least one vertex in S , then a subset $S \subseteq V(G)$ is said to be a dominating set.[1, 2, 5]. We have an invariant polynomial for graphs in graph theory. Polynomials in graph theory were presented for the first time by J.J. Sylvester in 1878 [11], and were additional considered by J.Petersen [9]. The idea of domination polynomial was first introduced by L.A. Jorge, and B. Llano (2000) [4] and it is widely studied in the literature. In a domination polynomial of a simple graph G we can determine the cardinality of a minimum dominating set in the graph G. The size of such a minimum set of the graph is called the minimum domination number and denoted by $\gamma(G) = min\{d(G, i) > 0\}$. If $D(G, i)$ is the family of dominating sets and $d(G, i) = |D(G, i)|$ is the number of dominating sets in G with cardinality i, then the domination polynomial is defined as [2, 4] $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, x)x^i$.

Currently, among several graph polynomials that have been published in the literature, Saied Alikhani is the one who has presented the domination polynomials for numerous types of graphs in his academic research work since 2008. Moreover, our current research work [13, 14] as well as some other interrelated research work, inspired us to evaluate graph polynomials for the inverse of dominating sets for specific graphs. The focus of this paper is to drive a generating function called dominating polynomials of an inverse graphs of cycle, star, wheel, triangular book graph, and the inverse of a union of cycle C_n with the complete graph K_1. The inverse domination polynomial of a simple graph G is given as $D(\bar{G},x) = \sum_{i=\gamma(\bar{G})}^{n} d(\bar{G},i)x^i$, where $\gamma(\bar{G}) = min\{d(\bar{G},i) > 0\}$, $D(\bar{G},i)$ is the family of inverse dominating sets and $d(\bar{G},i) = |D(\bar{G},i)|$ is the number of inverse dominating sets with cardinality i. In the following section, for the graphs such as $\bar{C}_n$, $\bar{W}_n$, $\bar{B}_n$and $(\overline{C_n \cup K_1})$ their domination polynomial is driven and also, the bounds of the inverse of a minimum dominating number are discussed with their corresponding graphs.

Observation 1.1 Let $\bar{G}$ be an inverse graph of G with order $|V(\bar{G})| = n$. Then

(i) If $\bar{G}$ is connected, then $d(\bar{G},n) = 1$ and $d(\bar{G},n-1) = n$

(ii) $d(\bar{G},i) = 0$ if and only if $i < \gamma(\bar{G})$ or $i > n$.

(iii) There is no constant term in D($\bar{G}$, x).

(iv) $D(\bar{G},x)$ is a strictly increasing function in $[0,\infty]$.

(v) Zero is a root of $D(\bar{G},x)$, with multiplicity $\gamma(\bar{G})$.

Theorem 1.1 [3] Let $\overline{K}_n$, be the inverse of a complete graph, K_n. Then $D(\overline{K}_n,x) = x^n$.

Observation 1.2 [12] Let G be a graph. Then the following holds

(a) $\gamma(\bar{G}) \le \delta(G) + 1$ and

(b) $\gamma(\bar{G}) \le \chi(G)$.

Theorem 1.2 [12] If G is a graph with $\gamma(G) \ge 2$, then $\gamma(\bar{G}) \le \left\lceil \frac{\delta(G)}{\gamma(G)-1} \right\rceil + 1$.

2. Domination polynomial of inverse of cycle

Definition 2.1 The cycle C_n, with $n \ge 3$ is number of vertices and the edge set is $E(C_n) = \{\{v_1,v_2\},\{v_2,v_3\},\dots,\{v_{n-1},v_n\},\{v_n,v_1\}\}$. The inverse of C_n is denoted by $\bar{C}_n$ and defined as for the vertex $V(C_n) = V(\bar{C}_n)$ and its edge set and $E(C_n) = E(K_n) - E(\bar{C}_n)$.

Theorem 2.1 Let the graph $\bar{C}_n$ be a complement of the cycle graph. Then

$$D(\bar{C}_n,x) = (1 + x)^n - (nx + 1), \quad \text{where } n \ge 5.$$

Proof: Let C_n be cycle graph with order $n \ge 3$. For $n = 3$ the invers of C_3 is an edgeless graph. i.e. $\bar{C}_3 = \overline{K}_3$. Then its domination polynomial is x^3. For vertex number $n = 4$ the inverse of C_4 is a disjoint component of two distinct paths with order 2. Let $\bar{C}_4 = [P_2] \cup [P_2^*]$. Hence, the family of dominating sets for the cardinality i, such that, $2 \le i \le 4$ are $D(\bar{C}_4,2) = D(\bar{C}_4,3) = 4$ and $D(\bar{C}_4,4) = 1$. Thus, $D(\bar{C}_4,x) = x^2(x+2)^2$ is the domination polynomial of $\bar{C}_4$ which is the same with path P_4.

For order $n \geq 5$ of the graph $\bar{C}_n$ with cardinality i, such that $2 \leq i \leq n$ the family of dominating sets of $D(\bar{C}_n, i)$ is obtained by selecting i^{th} number of cardinality from n^{th} number of the vertices, where the minimum domination number $\gamma(\bar{C}_n) = 2$. So we have $\binom{n}{i}$ possibilities to choose a dominating set. Hence, the number dominating set is $d(\bar{C}_n, i) = \binom{n}{i}$ for $2 \leq i \leq n$, $d(\bar{C}_n, n-1) = n$ and $d(\bar{C}_n, n) = 1$. Therefore, this results the domination polynomial is:

$$D(\bar{C}_n, x) = \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \cdots + \binom{n}{n}x^n$$
$$= \sum_{i=2}^{n} \binom{n}{i} x^i$$
$$= (1 + x)^n - (nx + 1), \quad \text{where } n \geq 5.$$

Observation 2.1

i. The bounds for a minimum domination number of the cycle C_n and its inverse $\bar{C}_n$ are $\gamma(C_n) \geq \gamma(\bar{C}_n)$ for $n \geq 5$ and $\gamma(C_n) \leq \gamma(\bar{C}_n)$ for $3 \leq n \leq 4$, where $\gamma(C_n) \geq \left\lceil \frac{n}{3} \right\rceil, n \geq 3$

ii. The number of distinct real domination roots of a complement of the cycle $\bar{C}_n$ is:

$$d_r(\bar{C}_n) = \begin{cases} 2, & \text{for } n \text{ } is \text{ } odd \\ 1, & \text{for } n \text{ } is \text{ } even \end{cases}$$

3. Domination polynomial of inverse of star and wheel graph

Definition 3.1 A star graph S_n, with order $n \geq 3$ in which $n-1$ vertices have degree one and a single vertex have degree $n-1$. The vertex and edge sets are $V(S_n) = \{i \mid 1 \leq i \leq n\}$ and $E(S_n) = \{(1, i) | 1 \leq i \leq n-1\}$, respectively. The inverse of star S_n is denoted by the symbol $\bar{S}_n$ and is defined as a disjoint union of complete graphs K_{n-1} and v, where the vertex v is an isolated vertex of graph $\bar{S}_n$.

Theorem 3.1 The domination polynomial of an inverse of star graph is:

$$D(\bar{S}_n, x) = x[(1 + x)^{n-1} - 1], \text{ where } n \geq 2$$

Proof: Since the inverse of the star S_n is a disjoint union of complete graphs K_{n-1} and v. Hence, the family of dominating set $D(\bar{S}_n, i)$ with the cardinality i, such that $2 \leq i \leq n$ is obtained by selecting $(i - 1)$ vertices from the vertex set of a graph K_{n-1}.

Hence, $\binom{n-1}{i-1}$ possibilities to choose a dominating subset from $V(\bar{S}_n)$. This results that the coefficient of the polynomial $\binom{n-1}{i-1}$ for their corresponding degree of the polynomial with the size i such that $2 \leq i \leq n$. Since for $\gamma(\bar{S}_n) = 2$, the domination polynomial of graph $\bar{S}_n$ is given by $D(\bar{S}_n, x) = \sum_{i=2}^{n} d(\bar{S}_n, i) x^i$. Therefore,

$$D(\bar{S}_n, x) = \binom{n-1}{2}x^2 + \binom{n-1}{3}x^3 + \binom{n-1}{4}x^4 + \cdots + \binom{n-1}{n-1}x^n$$

$$= \sum_{i=2}^{n} \binom{n-1}{i-1} x^i$$

$$= x\left[\sum_{i=2}^{n-1} \binom{n-1}{i-1} x^{i-1} - 1\right]$$

$$= x[(1 + x)^n - 1], \quad \text{where } n \geq 2.$$

Observation 3.1

i. For star graph S_n with order $n \geq 2$ the $\gamma(S_n) < \gamma(\bar{S}_n)$.

ii. The number of distinct real domination roots of $\bar{S}_n$ is :

$$d_r(\bar{S}_n) = \begin{cases} 1, & \text{for } n \; is even \\ 2, & \text{for } n \; is \;\; odd \end{cases}$$

iii. The roots of domination polynomial is

$$Z(D(\bar{S}_n, x) = \begin{cases} 0, & \text{for all } n \\ -2, & \text{for } n \equiv 0 (mod 2) \end{cases}$$

Definition 3.2 *A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle. The inverse of a wheel graph is denoted by $\overline{W}_n$ and defined as $\overline{W}_n = (v_t + v_s) - e_r$, where v_t and v_s are two distinct adjacent vertices of W_n and e_r is an edge set incident from the center of wheel to the vertices of a cycle C_{n-1} and $1 \leq t, s, r \leq n-1$.*

Theorem 3.2 A domination polynomial of an inverse of wheel graph is:

$$D(\overline{W}_n, x) = x\,[((1 + x)^{n-1} - (1 + x)^2], \text{ where n} \geq 6$$

Proof: Let $\overline{W}_n$ be an inverse of a wheel graph with order $n \geq 4$. The graph $\overline{W}_4 = \overline{K}_4$, for order $n = 4$, so the minimum domination number $\gamma(\overline{W}_4) = 4$. Thus, $|D(\overline{W}_4, 4)| = 1$ and the domination polynomial $D(\overline{W}_4, x) = x^4$**.** The graph $\overline{W}_5$ is a disjoint union of three distinct complete graphs when the vertex number $n = 5$**.** Hence, for the family of dominating set $D(\overline{W}_5, i)$ with cardinality i, such that $3 \leq i \leq 5$ the number of dominating sets are $|D(\overline{W}_5, 3)| = |D(\overline{W}_5, 4)| = 4$ and $|D(\overline{W}_5, 3)| = 1$. Therefore, $D(\overline{W}_5, x) = x^3(x+2)^2$. However, orders of vertices $n \geq 6$ in $\overline{W}_n$ with cardinality i such that $3 \leq i \leq n$, the family of the dominating set $D(\overline{W}_n, i)$ is obtained by selecting $(i-1)$ size of vertices from $(n-1)$ number of the vertices of graph $\overline{W}_n$ with $\gamma(\overline{W}_n) = 3$. Hence, the number dominating of dominating set is $d(\overline{W}_n, i) = \binom{n-1}{i-1}$. Therefore, the domination polynomial of the graph $\overline{W}$ is:

$$D(\overline{W}_n, x) = \binom{n-1}{2}x^3 + \binom{n-1}{3}x^4 + \binom{n-1}{4}x^5 + \cdots + \binom{n-1}{n-2}x^{n-1} + \binom{n-1}{n-1}x^n$$

$$= \sum_{i=3}^{n} \binom{n-1}{i-1} x^{i-1}$$

$$= x\left[\sum_{i=3}^{n-1}\binom{n-1}{i-1}x^{i-1}\right]$$

$$= x[(1 + x)^{n-1} - (1+x)^2], \quad \text{where } n \geq 6.$$

Observation 3.2

i. For the vertex number n ≥ 4, Then $\gamma(W_n) < \gamma(\overline{W}_n)$.

ii. The number of distinct real domination roots of $\overline{W}_n$ is:

$$d_r(\overline{W}_n) = \begin{cases} 3, & \text{for } n\ is\ even \\ 2, & \text{for } n\ is\ odd \end{cases}$$

4. Domination polynomial for inverse of a triangular book graph

Definition 4.1 *[10]* A triangular book graph is also called a complete tripartite graph with n-number of pages is defined as n-copes of cycle $\boldsymbol{C_3}$ sharing a commune base edge and denoted by $\boldsymbol{B(3,n)}$. The inverse of a triangular book graph is denoted by $\boldsymbol{\overline{B}(3,n)}$ and defined as the disjoint union of a complete graph $\boldsymbol{K_n}$ with two distinct isolated vertices.

Theorem 4.1 The domination polynomial of an inverse of triangular book graph is:

$$D(\overline{B}(3,n),x) = x^2((1+x)^{n-2} - 1), \text{ where } n \geq 5$$

***Proof*:** Let the graph $\overline{B}(3,n)$ be the inverse of triangular book graph with order $n \geq 5$ and $D(\overline{B}(3,n),x)$ be the family of dominating set with cardinality $3 \leq i \leq n$. Since the graph G is the disjoint union of the complete graph K_m with a distinct two isolated vertices, hence $\gamma(\overline{B}(3,n)) = 3$. Let the order of G is $n = m + 2$, where m is the order of K_m. The family of dominating set $D(\overline{B}(3,n),i)$ for $n \geq 5$ is obtained by selecting a dominating sets of j vertices from a set of m vertices of complete graph K_m, where $j = \gamma(K_1), \gamma(K_2), \ldots, |V(K_m)|$, Hence, this gives the family of dominating set $|D(\overline{B}(3,n),i)| = \binom{m}{j}$, where $3 \leq i \leq n$, $i = j + 2$. Therefore, for $n \geq 5$ and $i > j$ the domination polynomial of graph G is;

$$D(\overline{B}(3,n),x) = \binom{m}{1}x^3 + \binom{m}{2}x^4 + \binom{m}{3}x^5 + \cdots + \binom{m}{m}x^{m+2}$$

$$= \sum_{j=1}^{m}\binom{m}{j}x^{j+2}$$

$$= x\left[\sum_{i=3}^{n-2}\binom{n-2}{i-1}x^{i}\right]$$

$$= x^2[(1 + x)^{n-2} - 1], \quad \text{where } n \geq 5, \text{ and } m = n - 2.$$

Observation 4.1

i. The bounds of minimum domination number of a compliment of the triangular book graph $\gamma(B(3,n)) < \gamma(\overline{B}(3,n))$, for every n ≥ 3.

ii. The roots of polynomials $\bar{B}_{(3,n)}$ is:

$$Z(D(\bar{B}(3,n),x) = \begin{cases} -2, & \text{for } n \equiv 0(mod2) \\ 0, & \text{otherwise} \end{cases}$$

Theorem 4.2 The domination polynomial of an inverse of graph $G = C_m \cup \{v_1\}$ is:

$D(\bar{G},x) = x(1+x)^{n-1}$, where C_m is a cycle, v_1 is an isolated vertex of G and $n \geq 4$

Proof: Let $G = C_m \cup \{v_1\}$ be a graph with order $n \geq 4$. The inverse of the graph G is a connected graph and one of its vertex have $(n-1)$ degree and the remaining all vertices have $(n-3)$ degree.

The family of the dominating set $D(\bar{G},i)$ for the cardinality i such that $1 \leq i \leq n$ and for the minimum dominating number $\gamma(\bar{G}) = 1$ is obtained by choosing a dominating set m from the number of vertices $n-1$. Hence, the number of dominating set $|D\,(\bar{G},i)| = \binom{n-1}{i-1}$ for overall $m = \gamma(C_3), \gamma(C_4), \dots, |V(C_m)|$ in $\bar{G}$. Therefore, the domination polynomial of the graph $\bar{G}$ is :

$$D(\bar{W}_n,x) = \binom{n-1}{0}x + \binom{n-1}{1}x^2 + \binom{n-1}{2}x^3 + \cdots + \binom{n-1}{n-1}x^n$$

$$= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i$$

$$= x(1+x)^{n-1}, \quad \text{where } n \geq 4$$

Observation: 4.2

i. For the number of vertex set $n \geq 4$ the bounds of minimum domination number $\gamma(\overline{C_m \cup \{v_0\}}) < \gamma(C_m \cup \{v_0\})$.

ii. The rotes of domination polynomial $D(\overline{C_m \cup \{v_0\}},x)$ of are 0 and −1 with multiplicity of $n-1$.

iii. For a cycle C_n and a path P_n, the $D(\overline{C_m \cup \{K_1\}},x) = D(\overline{P_m \cup \{K_1\}},x)$, where n ≥ 4

5. Conclusion

In this paper, we introduced a generating function called a dominating polynomial for the inverse of graphs, such as cycles, stars, wheels, triangular book graphs, and the union of cycles with a single isolated vertex. In this way, we identified the roots of the dominating polynomial for some graphs and the bounds of the inverse minimum dominating number of graphs with their corresponding minimum dominating number are discussed.

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