International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V68I6P502

Volume 68 Issue 6, 8-16, June 2022

Original Article

Time-Fractional Fornberg-Whitham Equation Solved by Fractional Homotopy Perturbation Transform Method

Subodh Pratap Singh[1], Amardeep Singh[1]

[1,*]*Department of Mathematics, Glocal University, Saharanpur, U.P., India.*

Received: 07 April 2022 Revised: 25 May 2022 Accepted: 04 June 2022 Published: 14 June 2022

Abstract - *The paper presents the solution of time fractional fornberg-whitham equation by a fractional homotopy perturbation method (FHPTM). The traditional Adomian Decomposition method (ADM) for solving time-fractional fornberg-whitham equation gives good approximation in the neighborhood of initial conditions only. Here, we have presented the solution with the method FHPTM. The analysis suggests that FHPTM shows more accuracy as compared to Homotopy Analysis Method (HAM) and Adomian Decomposition Method (ADM). Obtained numerical results shows the efficacy of the method.*

Keywords - *Fractional homotopy perturbation transform method, Fornberg-whitham equation, Adomian Decomposition method, Homotopy analysis method, Partial differential equation, Perturbation methods.*

1. Introduction

Previous decades, there has been tremendous variety of application presented of fractional calculus. Fractional calculus includes theory of derivatives and integrals of fractional non-integer order. Much interest has been devoted in the fractional calculus and several applications have been presented in the areas of science and engineering. In modeling a real world phenomenon, fractional differential equations are using extensively in the areas like, control system using dynamical systems, random walk models neural network, signal processing, system recognition, reaction diffusion processes, [1,2]. Fractional calculus gives us an important tool to characterizes of memory and hereditary properties of several processes, [3]. Despite of unavailability of exact solution, some approximate analytical and numerical methods is used. Many important methods have been developed for solving fractional differential equation. The methods includes finite difference methods, Homotopy perturbation method, the Adams- Bashforth-Moulten method, generalized differential equation, Adomian decomposition method (ADM), variation iteration method (VIM) [4-17]. However, there is no simple method neither perturbation nor nonperturbation exist which adjust and control convergence region and rate.

Liao proposed the method homotopy analysis method (HAM). This method gives us an effective way to approximate analytical solutions of wide variety of differential equations. HAM is independent of small or large parameter based on homotopy of topology, [18-22]. HAM is applied without limitation of perturbation techniques and solution of many nonlinear problems in science and engineering, [23–28].

Assuming linear and nonlinear fractional PDE of the form

$$D_t^{\alpha} u(x,t) = f\left(u, u_{x,} u_t, u_{xx}, u_{xxx}, u_{xxt}\right), \qquad n-1, \alpha, \leq n, t > 0, \tag{1}$$

subject to the initial conditions

$$u^{(k)}(x,0) = g_k(x),\ k = 0, 1, 2, \ldots\ldots\ n-1, \tag{2}$$

The operator form of the nonlinear fractional partial differential equations (1)

$$D_t^{\alpha} u(x,t) = A(u, u_{xxt}) + B\left(u, u_{x,} u_{xx}, u_{xxx}\right) + C(x,t), \qquad n-1, \alpha, \leq n,\ t > 0, \tag{3}$$

Subject to the initial conditions

$$u^{(k)}(x,0) = g_k(x),\ k = 0, 1, 2, \ldots\ldots\ n-1, \tag{4}$$

where A is a linear operator which might include other fractional derivatives B is a nonlinear operator a and C is a known analytic function.

In the present paper, we shall apply FHPTM to find the approximate analytical solution of a special case of (3) and (4) the so-called time-fractional Fornberg–Whitham equation and compare it with the exact solution, Homotopy analysis Method and Adomian Decomposition Method.

2. Preliminaries

Definition 2.1 *Consider a real function h (χ), χ > 0. It is called in space* $C_\varsigma, \varsigma \in R$ *if* ∍ *a real no. b (> ζ), s.t. h (χ) =* $\chi^b h_1(\chi)$, $h_1 \in C[0,\infty]$. *It is clear that* $C_\varsigma \subset C_\gamma$ *if* $\gamma \le \zeta$.

Definition 2.2 *Consider a function h (χ), χ > 0. It is called in space* $C_\varsigma^m, m \in N \cup \{0\}$, *if* $h^{(m)} \in C_\varsigma$

Definition 2.3 *The (left sided) Riemann–Liouville integral of fractional order* $\nu > 0$ *of a function h,* $h \in C_\varsigma, \varsigma \ge -1$ *is defined as*

$$I^\nu h(t) = \frac{1}{\Gamma\nu}\int_0^t \frac{h(\tau)}{(t-\tau)^{1-\nu}}\, d\tau = \frac{1}{\Gamma(\nu+1)}\int_0^t h(\tau)\,(d\tau)^\nu$$

$$I^0 h(t) = h(t).$$

Definition 2.4 *The Caputo fractional derivative (left sided) of h,* $h \in C_{-1}^m, m \in N \cup \{0\}$,

$$D_t^\beta h(t) = \begin{cases} [I^{m-\beta} h^{(m)}(t)], m-1 < \beta < m, m \in N, \\ \dfrac{d^m}{dt^m} h(t), \beta = m \end{cases}$$

a. $I_t^\varsigma h(x,t) = \frac{1}{\Gamma\varsigma}\int_0^t (t-s)^{\varsigma-1} h(x,s)\, ds;\ \varsigma, t > 0.$

b. $D_\tau^\nu V(x,\tau) = I_\tau^{m-\nu}\frac{\partial^m V(x,\tau)}{\partial t^m}, m-1 < \nu \le m.$

c. $I_t^\varsigma D_t^\varsigma h(t) = h(t) - \sum_0^{m-1} h^k(0+)\frac{t^k}{\lfloor k}$

d. $I^\nu t^\varsigma = \frac{\Gamma(\varsigma+1)}{\Gamma(\nu+\varsigma+1)} t^{\nu+\varsigma}.$

Definition 2.5 *Mittag-Leffler function is demarcated by the given series representation, valid in entire complex plane:*

$$E_\varsigma(z) = \sum_{m=0}^\infty \frac{z^m}{\Gamma(1+\varsigma m)}, \varsigma > 0, z \in C.$$

Definition 2.6 Laplace transform of a (piecewise) continuous function g (t) in [0,∞) is given as:

G (p) = L [g (t)] = $\int_0^\infty e^{-pt} g(t)dt$

Definition 2.7

a. Laplace transform of (Riemann–Liouville) fractional integral is given as:

$$L[I^\alpha f(t)] = p^{-\alpha} F(p)$$

b. Laplace transform of (Caputo) fractional derivative is given as:

$$L[D^{\alpha} g(t)] = p^{\alpha} F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1}\, g^{(k)}(0), n-1 < \alpha \leq n$$

3. Basic Plan of FHPTM for nonlinear time-fractional differential equations

To illustrate the process of solution of the FHPTM, we ponder over the system of nonlinear time-fractional PDEs :

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + S(u,v) + Q(u,v) = g(x,t), \tag{5}$$

with initial values u(x,0) = h(x), (6)

Here, $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is (Caputo) fractional derivative of order α, S and Q are operators, linear & nonlinear respectively; g_1 and g_2 are the source terms. Also, $0 < \alpha \leq 1$.

The method comprises of taking Laplace transform on both sides of Eq. (2) and Eq. (3), as

$$L[D_t^{\alpha} u(x,t)] + L[S(u,v)] + L[Q(u,v)] = L[g(x,t)] \tag{7}$$

By differentiation property of Laplace transform,

$$L[u(x,t)] = p^{-1} h(x) - p^{-\alpha} L[g(x,t)] + p^{-\alpha} L[S(u,v) + Q(u,v)] \tag{8}$$

Taking inverse transform in Eqs. (5), we get

$$u(x,t) = G(x,t) + L^{-1}[p^{-\alpha} L\{S(u,v) + Q(u,v)\}] \tag{9}$$

Here G(x, t) are the terms coming from the source term and initial values.

Applying FHPTM, it is assumed that the result may be articulated as a power series,

$$u_n(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{10}$$

Here, p is reflected as a small parameter $(p \in [0,1])$ called homotopy parameter.
The non-linear term is decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \tag{11}$$

where H_n is He's polynomials of $u_0, u_1, u_2, u_{3,}......, u_n$ respectively. They are calculated by the given formulae:

$$H_n(u_0, u_1, u_2, ...) = \frac{1}{\lfloor n} \frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i \right)\right]_{p=0}, n = 0,1,2,3,... \tag{12}$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G_1(x,t) + pL^{-1}[p^{-\alpha} L\{S_1(u,v) + Q_1(u,v)\}] \tag{13}$$

This is a pairing of FHPTM and transform of Laplace using He's polynomials.

Equating coefficients of the identical powers on both the sides, we get,

$$p^0 : u_0(x,t) = G(x,t)$$

Continuing in this way, the enduring components can completely be achieved also. Thus the series solution is fully calculated. At last, the analytical solution is approximated by the series,

$$u(x,t) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n(x,t)$$

The above solutions in series converge very rapidly, in general. Cherruault and Abbaoui proved the convergence of this above kind of series.

4. Implementation of FHPTM

Ex. 1. To illustrate the process of solution of the FHPTM, we ponder over the system of nonlinear time-fractional PDEs :

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} = u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, t > 0, 0 < \alpha \le 1, , \quad (14)$$

with initial values $\quad u(x,0) = \frac{4}{3}\, e^{\frac{1}{2}x}.$ (15)

The method comprises of taking Laplace transform on both sides of Eq. (15), as

$$L\left[\frac{\partial^\alpha u}{\partial t^\alpha}\right] = L\left[u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial u}{\partial x}\right], t > 0, 0 < \alpha \le 1, \quad (16)$$

By differentiation property of Laplace transform,

$$L[u(x,t)] = p^{-1}\left(\frac{4}{3}e^{\frac{1}{2}x}\right) - p^{-\alpha}L[0] + p^{-\alpha}L\left[u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial u}{\partial x}\right] \quad (17)$$

Taking inverse transform in Eqs. (17), we get

$$u(x,t) = \left(\frac{4}{3}e^{\frac{1}{2}x}\right) + L^{-1}\left[p^{-\alpha}L\left[u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial u}{\partial x}\right]\right] \quad (18)$$

Applying FHPTM, it is assumed that the result may be articulated as a power series,

$$u_n(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (19)$$

Here, p is reflected as a small parameter $(p \in [0,1])$ called homotopy parameter.
The non-linear term is decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \tag{20}$$

where H_n is He's polynomials of $u_0, u_1, u_2, u_{3,}......, u_n$ respectively. They are calculated by the given formulae:

$$H_n(u_0, u_1, u_2, ...) = \frac{1}{\lfloor n} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, n = 0,1,2,3,... \tag{21}$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) == \left(\frac{4}{3} e^{\frac{1}{2}x} \right) + L^{-1}\left[p^{-\alpha} L\left[u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial u}{\partial x} \right]\right] \tag{22}$$

This is a pairing of FHPTM and transform of Laplace using He's polynomials.

Equating coefficients of the identical powers on both the sides, we get,

$$p^0 : u_0(x,t) = G(x,t) \tag{23}$$

we start with an initial approximation $u_0 = u(x,0)$ given by Eq. (4) by the above Eq. (23), we can obtain the other components as

$u_0(x,t) = \frac{4}{3} e^{\frac{1}{2}x}$,

$u_1(x,t) = \frac{2}{3} e^{\frac{1}{2}x} (2 - t)$,

$u_2(x,t) = \frac{1}{6} e^{\frac{1}{2}x} (8 - 9t + t^2 + \frac{4\, t^{2-\alpha}}{\Gamma\, (3-\alpha)})$,

$u_3(x,t) = \frac{1}{6} e^{\frac{1}{2}x} (8 - 9t + t^2 + \frac{4\, t^{2-\alpha}}{\Gamma\, (3-\alpha)})$,

and so on, in the same manner the remaining components can be obtained from Mathematica.

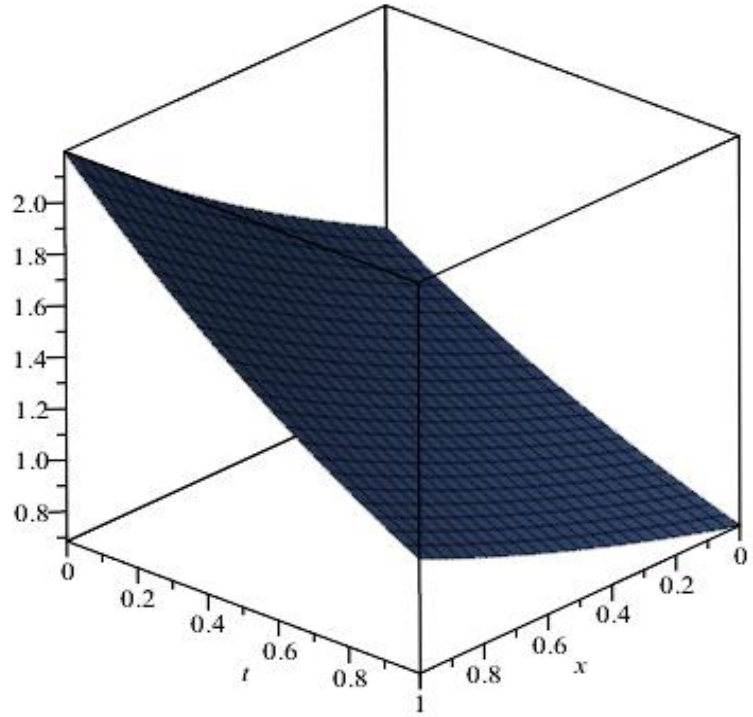

Fig 1. Shows Exact solution with respect to u, x and t.

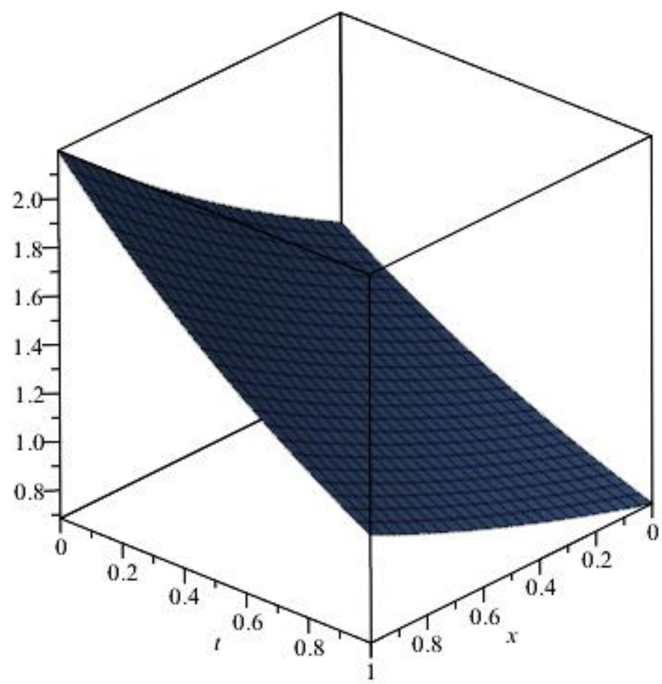

Fig 2. Shows approximate solution with respect to u, x and t.

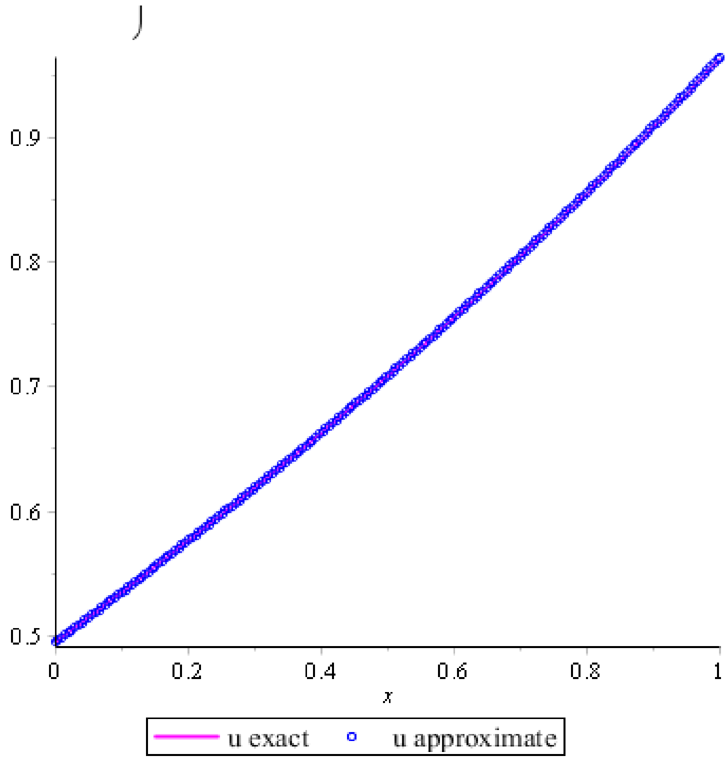

Fig. 3 Shows Exact and Approxiamte solution with respect to u and t

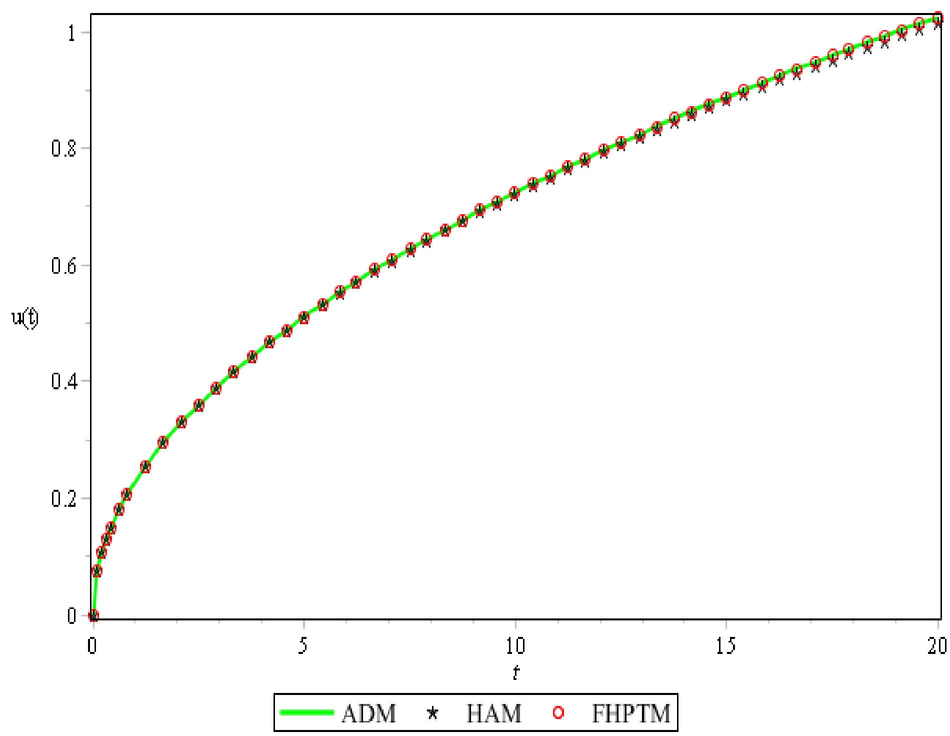

Fig. 4 Comparison between ADM, HAM and FHPTM

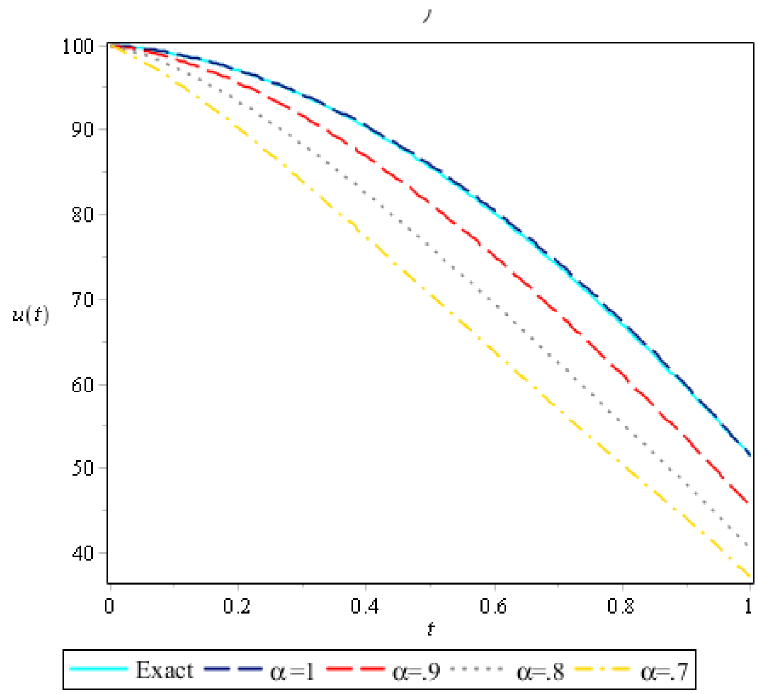

Fig. 5 Comparison between exact solution and different values of α

Table 1. Comparison between Exact Solution and Approxiamte solution

(FHPTM, HAM and ADM)

x	***Exact Solution*** $u(x)$	***FHPTM***	***HAM***	***ADM***
-4	0.0024787521	0.0024787400	0.0015123400	0.0056037265
2	0.0067379469	0.0067368000	0.0041109600	0.0152325081
0	0.0183156388	0.0183450901	0.0111748000	0.0414062500
2	0.0497870683	0.0497800050	0.0303761000	0.1125538569
4	0.1353352832	0.1353343227	0.0825709000	0.3059531040

5. Conclusion

In this paper, FHPTM is successfully applied to obtain a rapidly convergent approximate numerical solution of Riccati differential equations. Results obtained are compared with those from MVIM and VIM. It is seen that FHPTM is capable of reducing size of calculations and easy to use for both small and large parameters in nonlinear fractional problems. FHPTM is applied to solve easily, effectively and perfectly, a big class of non-linear problems with the numerical approximations which quickly congregate to exact solution. In this work, FHPTM is efficaciously used to obtain the analytical approximate solutions to the nonlinear systems of coupled time fractional PDEs. It does not require Adomian polynomials. FHPTM is applied directly without linearization or any restrictive assumptions. Hence FHPTM is more convenient and accurate than other existing methods.

References

[1] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, (1974).

[2] K. Diethelm, Analysis of Fractional Differential Equations, Springer-Verlag, Berlin, (2010).

[3] I. Podlubny, Fractional Differential Equations, Academic Press, New York, (1999).

[4] V.E. Lynch, B.A. Carreras, D. Del-Castillo-Negrete, K.M. Ferriera-Mejias, H.R. Hicks, Numerical Methods for the Solution of Partial Differential Equations of Fractional Order, J. Comput. Phys. 192 (2003) 406–421.

[5] M.M. Meerschaert, C. Tadjeran, Finite Difference Approximations for Two-Sided Space-Fractional Partial Differential Equations, Appl. Numer. Math, 56 (2006) 80–90.

[6] C. Tadjeran, M.M. Meerschaert, A Second-Order Accurate Numerical Method for the Two-Dimensional Fractional Diffusion Equation, J. Comput. Phys, 220 (2007) 813–823.

[7] K. Diethelm, N.J. Ford, A.D. Freed, Detailed Error Analysis for A Fractional Adams Method, Numer. Algorithms, 36 (2004) 31–52.

[8] Q. Wang, Homotopy Perturbation Method for Fractional Kdv Equation, Appl. Math. Comput,190 (2007) 1795–1802.

[9] H. Bulut, Comparing Numerical Methods for Boussinesq Equation Model Problem, Numer. Methods Partial Differ. Equ,. 25 (4) (2009) 783–796.

[10] P.K. Gupta, M. Singh, Homotopy Perturbation Method for Fractional Fornberg–Whitham Equation, Comput. Math. Appl, 61 (2011) 250–254.

[11] S. Momani, Z. Odibat, A Novel Method for Nonlinear Fractional Partial Differential Equations: Combination of Dtm and Generalized Taylor's Formula, J. Comput. Appl. Math, 220 (1–2) (2008) 85–95.

[12] J. Liu, G. Hou, Numerical Solutions of the Space- and Time-Fractional Coupled Burgers Equations By Generalized Differential Transform Method, Appl. Math. Comput, 217 (16) (2011) 7001–7008.

[13] R. Yulita Molliq, M.S.M. Noorani, I. Hashim, R.R. Ahmad, Approximate Solutions of Fractional Zakharov–Kuznetsov Equations By Vim, J. Comput. Appl. Math, 233 (2) (2009) 103–108.

[14] M.G. Sakar, F. Erdogan, A. Yildirim, Variational Iteration Method for the Time-Fractional Fornberg–Whitham Equation, Comput. Math. Appl, 63 (9) (2012) 1382–1388.

[15] M. Inc, the Approximate and Exact Solutions of the Space- and Time-Fractional Burgers Equations With Initial Conditions By Variational Iteration Method, J. Math. Anal. Appl, 345 (1) (2008) 476–484.

[16] H. Jafari, V.D. Gejji, Solving A System of Nonlinear Fractional Differential Equations Using Adomian Decomposition, Appl. Math. Comput, 196 (2006) 644– 651.

[17] S. Momani, Z. Odibat, Analytical Solution of A Time-Fractional Navier–Stokes Equation By Adomian Decomposition Method, Appl. Math. Comput, 177 (2006) 488–494.

[18] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/Crc Press, Boca Raton, (2003).

[19] S.J. Liao, An Approximate Solution Technique Which Does Not Depend Upon Small Parameters: A Special Example, Int. J. Non-Linear Mech,30 (1995) 371– 380.

[20] S.J. Liao, on the Homotopy Analysis Method for Nonlinear Problems, Appl. Math. Comput, 147 (2004) 499–513.

[21] S.J. Liao, Notes on the Homotopy Analysis Method: Some Definitions and Theorems, Commun. Nonlinear Sci. Numer. Simul, 14 (2009) 983–997.

[22] S.J. Liao, Comparison Between the Homotopy Analysis Method and the Homotopy Perturbation Method, Appl. Math. Comput, 169 (2005) 1186–1194.

[23] S.J. Liao, An Analytic Approach to Solve Multiple Solutions of A Strongly Nonlinear Problem, Appl. Math. Comput, 169 (2005) 854–865.

[24] L. Song, H. Zhang, Application of Homotopy Analysis Method to Fractional Kdv–Burgers–Kuramoto Equation, Phys. Lett, A 367 (1–2) (2007) 88–94.

[25] M. Ganjiani, Solution of Nonlinear Fractional Differential Equations Using Homotopy Analysis Method, Appl. Math. Model, 34 (6) (2010) 1634–1641.

[26] F. Abidi, K. Omrani, the Homotopy Analysis Method for Solving the Fornberg–Whitham Equation and Comparison With Adomian's Decomposition Method, Comput. Math. Appl, 59 (2010) 2743–2750.

[27] I. Podlubny, Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation, Fract. Calc. Appl. Anal, 5 (4) (2002) 367–386.

[28] G. Adomian, A Review of the Decomposition Method In Applied Mathematics, J. Math. Anal. Appl,135 (1988) 44–501.

[29] G. Adomian, Solving Frontier Problems of Physics: the Decomposition Method, Kluwer Academic Publishers, Boston, (1999).

[30] K. Abbaoui, Y. Cherruault, New Ideas for Proving Convergence of Decomposition Methods, Comput. Math. Appl, 29 (7) (1995) 103–108.