International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V68I6P506
Volume 68 Issue 6, 54-59, June 2022

Original Article

Exponential Transform of Some Special Functions

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Received: 14 April 2022 Revised: 30 May 2022 Accepted: 03 June 2022 Published: 22 June 2022

Abstract *- In the recent paper the concept of Exponential transform is introduced in [1]. In this paper we have obtained the exponential transform of Error function, Unit step function, Periodic function, Dirac-Delta function, Bessel function and the exponential transform of Leguerre polynomial.*

Keywords *- Error Function, Unit Step Function, Periodic Function, Dirac-Delta Function, Bessel Function, Leguerre Polynomial.*

1. Introduction

Most of the functions encountered in introductory analysis belong to the class of elementary functions. This class is composed of polynomials, rational functions, transcendental functions (trigonometric, exponential, logarithmic, and so on), and functions constructed by combining two or more of these functions through addition, subtraction, multiplication, division, or composition. Beyond these functions lies a class of special functions which are important in a variety of engineering and physics applications. the use of integral transforms is heavily interlaced with special functions like the gamma function, error function, Bessel functions, and so forth Also, functions such as the Heaviside unit function and the impulse function, which are employed in a variety of engineering applications.

Recently N. S. Ambarkhane, H. A. Dhirbasi, K.L. Bondar [1] introduced an integral transform “Exponential transform” and proved its some properties like Linearity, Shifting, Second Shifting, Change of Scale. Moreover, exponential transform of some basic functions is derived. Ambarkhane Nagnath S., Bondar Kirankumar L. [2] discussed convolution theorem and the exponential transform of derivatives and integrations of a function f(t). Nagnath S. Ambarkhane [3] discussed some results and application of the inverse exponential transform. Several authors [4-16] discussed the applications of different integral transformations along with its results.

The aim of this paper is to obtain the exponential transform of Error function, Unit step function, Periodic function, Dirac-Delta function, Bessel function and the exponential transform of Leguerre polynomial.

2. Preliminaries

2.1. Exponetial Transform

Definition 2.1[1]: Let $f(t)$ be function defined for all positive values of t, then

$$\bar{f}(s) = \int_0^\infty a^{-st} f(t)dt, \quad a > 1.$$

Provided the integral exists is called exponential transform of $f(t)$. It is denoted by $A[f(t)]$. Thus,

$$A[f(t)] = \bar{f}(s) = \int_0^\infty a^{-st} f(t)dt, \quad a > 1.$$

Here A is called exponential transformation operator, the parameter s is real or complex number.

2.2. Exponential Transform of some functions [1]:

i) $A[1] = \frac{1}{(s \log a)}, a > 1, Re(s) > 0.$

ii) $A[t^n] = \frac{n!}{(s \log a)^{n+1}}, a > 1, n \geq 0, Re(s) > 0.$

iii) $A[t^n] = \frac{\Gamma(n+1)}{(s\,log\,a)^{n+1}}, n > -1, a > 1, Re(s) > 0.$

iv) $A[e^{kt}] = \frac{1}{(s\,log\,a-k)}, a > 1, (s\,log\,a) > k, Re(s) > 0.$

vi) $A[cosh\,k\,t] = \frac{(s\,log\,a)}{(s\,log\,a)^2-k^2}, a > 1, (s\,log\,a)^2 > k^2, Re(s) > 0.$

vii) $A[sinh\,k\,t] = \frac{k}{(s\,log\,a)^2-k^2}, a > 1, (s\,log\,a)^2 > k^2, Re(s) > 0.$

viii) $A[sin\,k\,t] = \frac{k}{(s\,log\,a)^2+k^2}, a > 1, Re(s) > 0.$

ix) $A[cos\,k\,t] = \frac{(s\,log\,a)}{(s\,log\,a)^2+k^2}, a > 1, Re(s) > 0.$

3. Main Results

Theorem 3.1 : *the exponetial transform of unit step function is,*

$$A[u(t-k)] = \frac{a^{-ks}}{(s\,log\,a)}, (s\,log\,a) > 0.$$

Proof: we have,

$$A[u(t-k)] = \int_0^\infty a^{-st}u(t-k)dt$$

$$= \int_0^k a^{-st}.0dt + \int_k^\infty a^{-st}.1dt \quad \{\because u(t-k) = \{ \begin{smallmatrix} 0, t<0 \\ 1, t\geq k \end{smallmatrix}$$

$$= \int_k^\infty a^{-st}dt$$

$$= \left[\frac{a^{-st}}{-s\,log\,a}\right]_k^\infty$$

$$= \frac{1}{-(s\,log\,a)}[0 - a^{-ks}]$$

$$= \frac{a^{-ks}}{(s\,log\,a)}$$

$$\therefore A[u(t-k)] = \frac{a^{-ks}}{(s\,log\,a)}, (s\,log\,a) > 0.$$

Theorem 3.2: [*Exponential transform of periodic function*]

If $f(t)$be a periodic function with period T, then

$$A[f(t)] = \frac{\int_0^T a^{-st}f(t)}{1-a^{-sT}}dt, (1 - a^{-sT}) > 0.$$

Proof: We have,

$$A[f(t)] = \int_0^\infty a^{-st}f(t)dt$$

$$A[f(t)] = \int_0^T a^{-st}f(t)dt + \int_T^{2T} a^{-st}f(t)dt + \int_{2T}^{3T} a^{-st}f(t)dt + \ldots\ldots\ldots\ldots\ldots$$

Substituting $t = u + T$in second integral and $t = u + 2T$, in third integral and so on, we get

$$\therefore A[f(t)] = \int_0^T a^{-st}f(t)dt + \int_0^T a^{-s(u+T)}f(u+T)du + \int_0^T a^{-s(u+2T)}f(u+2T)du + \ldots\ldots\ldots$$

$$= \int_0^T a^{-st} f(t)dt + a^{-ST} \int_0^T a^{-su} f(u)du + a^{-2ST} \int_0^T a^{-su} f(u)du + \dots\dots\dots$$

$$[\because f(u) = f(u+T) = f(u+2T) = f(u+3T) = \dots.]$$

$$= \int_0^T a^{-st} f(t)dt + a^{-ST} \int_0^T a^{-st} f(t)dt + a^{-2ST} \int_0^T a^{-st} f(t)dt + \dots\dots\dots$$

$$= [1 + a^{-ST} + a^{-2ST} + a^{-3ST} + \dots] \int_0^T a^{-st} f(t)dt$$

$$\therefore A[f(t)] = \frac{\int_0^T a^{-st} f(t)}{1 - a^{-sT}} dt, (1 - a^{-sT}) > 0. \left\{\because 1 + a + a^2 + a^3 + \dots\dots = \frac{1}{1-a}\right.$$

Theorem 3.3.: *If Dirac -Delta function given by*

$$\delta(t-k) = \lim_{\epsilon \to 0} \delta_\epsilon(t-k) \text{ where } \delta_\epsilon(t-k) = \frac{1}{\epsilon}, \text{for } k \le t \le k+\epsilon$$

$$= 0 \text{ otherwise,}$$

then $A[\delta(t-k)] = a^{-ks}$.

Proof: we have,

$$A[\delta(t-k)] = \int_0^\infty a^{-st} \delta(t-k)dt$$

$$= \lim_{\epsilon \to 0} \left[\int_0^\infty a^{-st} \delta_\epsilon(t-k)dt\right]$$

$$= \lim_{\epsilon \to 0} \left[\int_0^k a^{-st}.0dt + \int_k^{k+\epsilon} a^{-st} \frac{1}{\epsilon} dt + \int_{k+\epsilon}^\infty a^{-st}.0dt\right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{\epsilon} \int_k^{k+\epsilon} a^{-st} dt\right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{\epsilon} \int_k^{k+\epsilon} e^{-st \log a} dt\right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{\epsilon} \left(\frac{e^{-st \log a}}{-s \log a}\right)_k^{k+\epsilon}\right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{-\epsilon s \log a} \left(e^{-s \log a (k+\epsilon)} - e^{-sk \log a}\right)\right]$$

$$= e^{-ks \log a} \left[\lim_{\epsilon \to 0} \frac{1}{-\epsilon s \log a} (e^{-\epsilon s \log a} - 1)\right]$$

$$= e^{-ks \log a} \lim_{\epsilon \to 0} \frac{1}{-\epsilon s \log a} \left[\left(1 - \epsilon\, s \log a + \frac{(\epsilon s \log a)^2}{2!} - \frac{(\epsilon s \log a)^3}{3!} + \dots\dots\dots\right) - 1\right]$$

$$= e^{-ks \log a} \lim_{\epsilon \to 0} \left[\frac{1}{-\epsilon s \log a} \left(-\epsilon\, s \log a + \frac{(\epsilon s \log a)^2}{2!} - \frac{(\epsilon s \log a)^3}{3!} + \dots\dots\dots\right)\right]$$

$$= e^{-ks \log a} \lim_{\epsilon \to 0} \left[\frac{1}{-\epsilon s \log a} \left(-\epsilon\, s \log a \left[1 - \frac{(\epsilon s \log a)}{2!} + \frac{(\epsilon s \log a)^2}{3!} - \dots\dots\dots\right]\right)\right]$$

$$= e^{-ks \log a} \lim_{\epsilon \to 0} \left[1 - \frac{(\epsilon s \log a)}{2!} + \frac{(\epsilon s \log a)^2}{3!} - \dots\dots\dots\right]$$

$$= e^{-ks \log a} [1]$$

$$\therefore A[\delta(t-k)] = a^{-ks}.$$

Theorem 3.4: *Exponential transform of Error function is,*

$$A\left[erf\left(\sqrt{x}\right)\right] = \frac{1}{(s\,log\,a)\sqrt{(s\,log\,a)+1}}, (s\,log\,a) > 0.$$

Proof: We have,

$$erf\left(\sqrt{x}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} e^{-t^2}\,dt$$

$$erf\left(\sqrt{x}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} \left[1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \ldots.\right] dt$$

$$erf\left(\sqrt{x}\right) = \frac{2}{\sqrt{\pi}}\left(x^{\frac{1}{2}} - \frac{x^{3/2}}{3} + \frac{x^{\frac{5}{2}}}{5\times 2!} - \frac{x^{\frac{7}{2}}}{7\times 3!} + \ldots\right) \qquad \therefore A[erf(\sqrt{x})] = \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma\left(\frac{3}{2}\right)}{(s\,log\,a)^{\frac{3}{2}}} - \frac{\Gamma\left(\frac{5}{2}\right)}{3(s\,log\,a)^{\frac{5}{2}}} + \frac{\Gamma\left(\frac{7}{2}\right)}{5\times 2!(s\,log\,a)^{\frac{7}{2}}} - \frac{\Gamma\left(\frac{9}{2}\right)}{7\times 3!(s\,log\,a)^{\frac{9}{2}}} + \ldots\ldots\ldots.\right]$$

$$= \frac{1}{(s\,log\,a)^{\frac{3}{2}}} - \frac{1}{2}.\frac{1}{(s\,log\,a)^{\frac{5}{2}}} + \frac{1.3}{2.4}.\frac{1}{(s\,log\,a)^{\frac{7}{2}}} - \frac{1.3.5}{2.4.6}.\frac{1}{(s\,log\,a)^{\frac{9}{2}}} + \ldots..$$

$$= \frac{1}{(s\,log\,a)^{\frac{3}{2}}}\left[1 - \frac{1}{2}.\frac{1}{(s\,log\,a)} + \frac{1.3}{2.4}.\frac{1}{(s\,log\,a)^2} - \frac{1.3.5}{2.4.6}.\frac{1}{(s\,log\,a)^3} + \ldots.\right]$$

$$= \frac{1}{(s\,log\,a)^{\frac{3}{2}}}\left[1 + \frac{1}{(s\,log\,a)}\right]^{\frac{-1}{2}}$$

$$\therefore A\left[erf\left(\sqrt{x}\right)\right] = \frac{1}{(s\,log\,a)\sqrt{(s\,log\,a)+1}}, (s\,log\,a) > 0.$$

Theorem 3.5: *The exponential transform of Bessel function* $J_0(t)$ *is,*

$$A[J_0(t)] = \frac{1}{\sqrt{(s\,log\,a)^2+1}}.$$

Proof: We have the Bessel function is

$$J_0(t) = \left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \ldots..\right]$$

Taking Exponential Transform on both sides

$$A[J_0(t)] = \left[\frac{1}{(s\,log\,a)} - \frac{1}{2^2}.\frac{2!}{(s\,log\,a)^3} + \frac{1}{2^2.4^2}.\frac{4!}{(s\,log\,a)^5} - \frac{1}{2^2.4^2.6^2}.\frac{6!}{(s\,log\,a)^7} + \ldots..\right]$$

$$A[J_0(t)] = \frac{1}{(s\,log\,a)}\left[1 - \frac{1}{2}\left(\frac{1}{(s\,log\,a)^2}\right) + \frac{1.3}{2.4}.\left(\frac{1}{(s\,log\,a)^4}\right) - \frac{1.3.5}{2.4.6}.\left(\frac{1}{(s\,log\,a)^6}\right) + \ldots..\right]$$

$$A[J_0(t)] = \frac{1}{(s\,log\,a)}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{1}{(s\,log\,a)^2}\right) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}.\left(\frac{1}{(s\,log\,a)^2}\right)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}.\left(\frac{1}{(s\,log\,a)^2}\right)^3 + \ldots..\right]$$

$$= \frac{1}{(s\,log\,a)}\left[1 + \frac{1}{(s\,log\,a)^2}\right]^{\frac{-1}{2}} \qquad \text{(By Binomial Theorem)}$$

$$\therefore A[J_0(t)] = \frac{1}{\sqrt{(s\,log\,a)^2+1}}.$$

Theorem 3.6: *If Leguerre Polynomial is given by*

$$L_n(t) = \frac{e^t}{n!}.\frac{d^n}{dt^n}(e^{-t}t^n) \text{ then } A[L_n(t)] = \frac{(s\log a - 1)^n}{(s\log a)^{n+1}}, (s\log a) > 1.$$

Proof: We have,

$$A[L_n(t)] = \int_0^\infty a^{-st}.\frac{e^t}{n!}.\frac{d^n}{dt^n}(e^{-t}t^n)dt$$

$$= \int_0^\infty e^{-st\log a}.\frac{e^t}{n!}.\frac{d^n}{dt^n}(e^{-t}t^n)dt$$

$$= \int_0^\infty \frac{e^{-(s\log a-1)t}}{n!}.\frac{d^n}{dt^n}(e^{-t}t^n)dt$$

$$= \left[\frac{e^{-(s\log a-1)}}{n!}.\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)\right]_0^\infty + \int_0^\infty \frac{(s\log a-1)e^{-(s\log a-1)t}}{n!}.\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)dt$$

$$= \frac{(s\log a-1)}{n!}\int_0^\infty e^{-(s\log a-1)t}.\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)dt$$

By Continues Integrating, we get

$$= \frac{(s\log a-1)^n}{n!}\int_0^\infty e^{-(s\log a-1)t}.e^{-t}t^n dt$$

$$= \frac{(s\log a-1)^n}{n!}\int_0^\infty e^{-st\log a}t^n dt$$

$$= \frac{(s\log a-1)^n}{n!}\int_0^\infty a^{-st}t^n dt$$

$$= \frac{(s\log a-1)^n}{n!}A[t^n]$$

$$= \frac{(s\log a-1)^n}{n!} \times \frac{n!}{(s\log a)^{n+1}}$$

$$\therefore A[L_n(t)] = \frac{(s\log a-1)^n}{(s\log a)^{n+1}}, (s\log a) > 1.$$

4. Conclusion

In this work we obtained the exponential transform of Error function, Unit step function, Periodic function, Dirac-Delta function, Bessel function and the exponential transform of Leguerre polynomial.

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