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Original Article

Generalization of Homeomorphism in Topological Space

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Abstract - *A new-class of homeomorphisms called gg- homeomorphism and gg*-homeomorphism are introduced in topological spaces. Interesting properties on these homeomorphisms are investigated. Also, their relation with few homeomorphisms in topological space is presented.*

Keywords - *Homeomorphism, gg-homeomorphism and gg*-homeomorphism.*

1. Introduction

Many researchers and mathematicians contributed their work in the field of general topology, particularly homeomorphisms in topological spaces, as it plays a very important role in topology. in 1991, Maki et all [11], introduced and studied generalized homeomorphism in topological space. Later, N. Nagaveni [2], investigated rwg homeomorphism in topological space. in 2002, M Sheik John [16] investigated ω homeomorphism in topological space. Recently, different types of homeomorphisms are introduced by the authors Basavaraj M. Ittanagi [1-6], R.S. Wali [24-26], Vivekananda Dembre [20-23] and T.Shyla Isac Mary. the aim of this paper is to introduce and study gg-homeomorphism, gg*-homeomorphism and their relationship with few homeomorphisms in topological space. Also, interesting properties on these homeomorphisms are investigated.

2. Prerequisites

the following is the list of definitions and notations used in present paper. Here, $M\ or\ (M,\tau)$ and $N\ or\ (N,\sigma)$ denote topological spaces. for $B \subseteq (M,\tau), cl(B), int(B)$ and $M-B\ or\ B^c$ stands for closure, interior and complement of B respectively. Also, $C-(M,\tau)$ denotes collection of closed sets.

2.1 Definitions Let $\boldsymbol{B} \subseteq (\boldsymbol{M},\tau)$. Then $\boldsymbol{B}$ is said to be
a) Generalized closed set (g-closed) [10] if $cl(B) \subseteq S$, whenever $B \subseteq S$ and S is open in (M,τ).
b) $r^\wedge g$-closed set [15] if $gcl(B) \subseteq S$, whenever $B \subseteq S$ and S is regular open in (M,τ).
c) βwg**-closed set [18] if $\beta wg * cl(B) \subseteq S$ whenever $B \subseteq S$ and S is regular open in (M,τ).
d) gg-closed set [7] if $gcl(B) \subseteq S$, whenever $B \subseteq S$ and S is regular semi open in (M,τ).
the complement of closed sets listed above are their open sets respectively and vice versa.

2.2 Definitions the bijective map $h: (M,\tau) \to (N,\sigma)$ is called
a) homeomorphism if h is both open and continuous
b) generalized homeomorphism (g-homeomorphism) if h is both g-continuous and g-open.
c) ω-homeomorphism [16] if h is both ω-continuous and ω-open.
d) swg*-homeomorphism [12] if it is both swg*-continuous and swg*-open.
e) rwg-homeomorphism [13] if it is both rwg-continuous and rwg-open.
f) r^g-homeomorphism if it is both r^g-continuous and r^g-open.
g) βwg $**$-homeomorphism if h is both βwg**-continuous and βwg**-open.
h) gc-homeomorphism if h and h^{-1} are gc-irresolute
i) ω*-homeomorphism [16] if h and h^{-1} are ω-irresolute.

3. Results and Discussion

3.1 Definition the bijective map $h:(M,\tau)\to(N,\sigma)$ is known as gg-homeomorphism if h is both gg-continuous and gg-open

3.2 Example Let $M=N=\{1,2,3\},\tau=\{\varphi,\{1\},\{2\},\{1,2\},\{1,3\},M\}$ and $\sigma=\{\varphi,\{1\},\{2,3\},N\}$. Here, the bijective map h, defined by $h(1)=2,h(2)=3,h(3)=1$ is gg-homeomorphism.

3.3Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is homeomorphism is always a gg-homeomorphism and converse is untrue.
Proof. Consider a homeomorphism, $h:(M,\tau)\to(N,\sigma)$. Then, h is bijective, continuous, and open map. Since every continuous is gg-continuous and every open map is gg-open, f is homeomorphism.

3.4 Example in example 3.2, h is a gg-homeomorphism but not a homeomorphism as $h^{-1}\{2,3\}=\{1,2\}\notin C(M,\tau)$ for $\{2,3\}\in C(N,\sigma)$ that is, h is not a gg-continuous.

3.5 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is g-homeomorphism is always a gg-homeomorphism and converse is untrue.
Proof. the proof is based on the results that every generalized continuous is gg-continuous and every generalized-open map is gg-open map.

3.6 Example in example 3.2, h is a gg-homeomorphism but not a g-homeomorphism as $h^{-1}\{2,3\}=\{1,2\}\notin C(M,\tau)$ for $\{2,3\}\in C(N,\sigma)$ that is,h is not a gg-continuous and hence h is not a g-continuous.

3.7 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is ω-homeomorphism is always a gg-homeomorphism and converse is untrue.
Proof. Based on facts that, every ω-continuous is gg-continuous and every ω -open map is gg-open map.

3.8 Example in example 3.2, h is a gg-homeomorphism but not a ω-homeomorphism as $h^{-1}\{2,3\}=\{1,2\}\notin\omega-C(M,\tau)$ for $\{2,3\}\in C(N,\sigma)$ that is,h is not a gg-continuous and hence h is not a ω-continuous.

3.9 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is $\omega*$-homeomorphism is always a gg-homeomorphism and converse is untrue.
Proof. Based on results that, every ω*-homeomorphism is ω -homeomorphism and Theorem 3.7.

3.10 Example in example 3.2, h is a gg-homeomorphism but not a ω*-homeomorphism as $h^{-1}\{2,3\}=\{1,2\}\notin\omega-C(M,\tau)$ for $\{2,3\}\in\omega-C(N,\sigma)$ that is, h is not a gg-irresolute.

3.11 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is $swg*$-homeomorphism is always a gg-homeomorphism and converse is untrue.
Proof. Based on facts that, every swg*-continuous is gg-continuous and every swg*-open map is gg-open map.

3.12 Example in example 3.2, h is a gg-homeomorphism but not a swg*-homeomorphism as $h^{-1}\{2,3\}=\{1,2\}\notin swg*-C(M,\tau)$ for $\{2,3\}\in C(N,\sigma)$ that is, h is not a swg*-continuous

3.13 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is gc-homeomorphism is always a gg-homeomorphism and converse is untrue.
Proof. Based on facts that, every gc-homeomorphism is swg*-homeomorphism and Theorem 3.11

3.14 Example. in example 3.2, h is a gg-homeomorphism but not a gc-homeomorphism as $h^{-1}\{2,3\}=\{1,2\}\notin g-C(M,\tau)$ for $\{2,3\}\in g-C(N,\sigma)$ that is, h is not a g-irresolute.

3.15 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$ which is gg-homeomorphism is always a rwg-homeomorphism and converse is untrue.

Proof. Consider a gg-homeomorphism $h:(M,\tau)\to(N,\sigma)$. Then, h is both gg-continuous and gg-open. That is, h is both rwg continuous and rwg open. Therefore, h is rwg-homeomorphism.

3.16 Example Let $M=N=\{1,2,3,4\}$, $\tau=\{\varphi,\{1\},\{2\},\{1,2\},\{1,2,3\},M\}$ and σ={φ, {1, 2}, {3, 4}, N}. the bijective map h, defined by $h(1)=1,h(2)=3,h(3)=2,h(4)=4$ is a rwg-homeomorphism but not a gg-homeomorphism as $h^{-1}\{1,2\}=\{1,3\}\notin gg-C(M,\tau)$ for $\{1,2\}\in C(N,\sigma)$ that is, h is not a gg-continuous.

3.17 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is gg-homeomorphism is always a r^g-homeomorphism and converse is untrue.
Proof. Based on results that, every gg-continuous is rwg-continuous and every gg-open map is r^g-open map.

3.18 Example in example 3.16, h is a r^g-homeomorphism but not a gg-homeomorphism as $h^{-1}\{1,2\}=\{1,3\}\notin gg-C(M,\tau)$ for $\{1,2\}\in C(N,\sigma)$ that is, h is not a gg-continuous.

3.19 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is gg-homeomorphism is always a βwg**-homeomorphism and converse is untrue.
Proof. Follows from results that, every gg-continuous map is βwg**-continuous and every gg-open map is βwg**-open map.

3.20 Example in example 3.16, h is a βwg**-homeomorphism but not gg-homeomorphism as $h^{-1}\{1,2\}=\{1,3\}\notin gg-C(M,\tau)$ for $\{1,2\}\in C(N,\sigma)$ that is, h is not a gg-continuous.

3.21 Remark Following two examples show that gg-homeomorphism is independent with few homeomorphisms in topological spaces.

3.22 Example in example 3.2, h is gg-homeomorphism but not a sg-homeomorphism [8], gs- homeomorphism, gb-homeomorphism, and gαb-homeomorphism [11] as $h^{-1}\{2,3\}=\{1,2\}$ which is not a sg-closed, gs-closed, gb-closed and gαb closed set in M, for $\{2,3\}\in C(N,\sigma)$. Thus, h is not a sg-continuous, gs-continuous, gb-continuous and gαb-continuous.

3.23 Example Let $M=N=\{1,2,3,4\}$, $\tau=\{\varphi,\{1\},\{2\},\{1,2\},\{1,2,3\},M\}$ and σ= {φ, {1, 2}, {3, 4}, N}. the bijective map h, defined by h(1)=1, h(2)=3, h(3)=2, h(4)=4 is a sg-homeomorphism, gs- homeomorphism, gb-homeomorphism and gαb-homeomorphism but not a gg-homeomorphism as $h^{-1}\{1,2\}=\{1,3\}\notin gg-C(M,\tau)$ for $\{1,2\}\in C(N,\sigma)$. Therefore h is not a gg-continuous.

3.24 Theorem the statements given below are identical for a bijective map $h:(M,\tau)\to(N,\sigma)$, which is gg-continuous. Then, h is
(i) gg-open map.
(ii) gg-homeomorphism.
(iii)gg-closed map.
Proof.
(i) ⟹ (ii)
Given, a bijective map h, which is gg-continuous and gg-open. Therefore h is a gg-homeomorphism.
(ii) ⟹ (iii)
Given, h is a gg-homeomorphism. Consider a closed set G in M and hence G^c is open set in M. Then $h(G^c)$ is open in N. But $(h(G^c))^c$. Therefore h is a gg-closed map.
(iii) ⟹ (i)
Assume that h is a gg-closed map. Consider an open set G in M and hence $G^c\in C(M)$. Then, $h(G^c)\in C(N)$. But $h(G^c)=(h(G))^c$. Therefore h is a gg-open map.

3.25 Remark Let h and g be two gg-homeomorphisms. in general, goh is need not be a gg-homeomorphism.

3.26 Example Let $M=N=P=\{1,2,3,4\},\tau=\{\varphi,\{1\},\{2\},\{1,2\},\{1,2,3\},M\}$, $\sigma=\{\varphi,N,\{1\},\{1,2\},\{1,2,3\}\}$ and η={ φ, P, {1, 2}, {3, 4}}. the bijective map h, defined by $h(1)=1,h(2)=3,h(3)=2,h(4)=4$ is a gg-homeomorphism. the identity map $g:(N,\sigma)\to(P,\eta)$ is a gg-homeomorphism. But $goh:(M,\tau)\to(P,\eta)$ is not a gg-homeomorphism as $(goh)^{-1}\{1,2\}=\{1,3\}\notin gg-C(M,\tau)$ for $\{1,2\}\in C(\mathrm{P},\eta)$. That is, h is not a gg-continuous.

3.27 Definition the bijective map $h:(M,\tau)\to(N,\sigma)$is known as gg*-homeomorphism if h and h^{-1} are gg-irresolute.
Where $gg*-(M,\tau)$ denotes collection of all gg- homeomorphisms of M on to itself.

3.28 Theorem Every bijective map $h:(M,\tau)\to(N,\sigma)$, which is gg*-homeomorphism is gg-homeomorphism and converse is untrue.
Proof. Consider a bijective map $h:(M,\tau)\to(N,\sigma)$, which is gg*-homeomorphism. Then, h and h^{-1} are gg-irresolute. This implies that h and h^{-1} are gg-continuous. From Theorem 3.24, h is a gg-homeomorphism.

3.29 Example in example 3.2, h is a gg-homeomorphism but not gg*-homeomorphism as $h^{-1}\{2\} = \{1\} \notin gg - C(M,\tau)$, for $\{2\} \in gg - C(N,\sigma)$.

3.30 Theorem Every bijective map $h: (M,\tau) \to (N,\sigma)$, which is gg*-homeomorphism is always a rwg-homeomorphism and converse is untrue.
Proof. Similar to Theorem 3.28.

3.31 Example in example 3.16, converse of Theorem 3.30 is untrue as $h^{-1}\{1,2\} = \{1,3\} \notin gg - C(M,\tau)$ for $\{1,2\} \in C(N,\sigma)$.

3.32 Theorem Every gg*-homeomorphism is r^g-homeomorphism (resp. βwg**-homeomorphism) but converse is not true.
Proof. Similar to Theorem 3.28

3.33 Example in example 3.16, h is a r^g-homeomorphism and also βwg**-homeomorphism but not a gg homeomorphism as $h^{-1}\{1,2\} = \{1,3\} \notin gg - C(M,\tau)$ for $\{1,2\} \in C(N,\sigma)$

3.34 Remark the results discussed above are shown in the diagram as follows

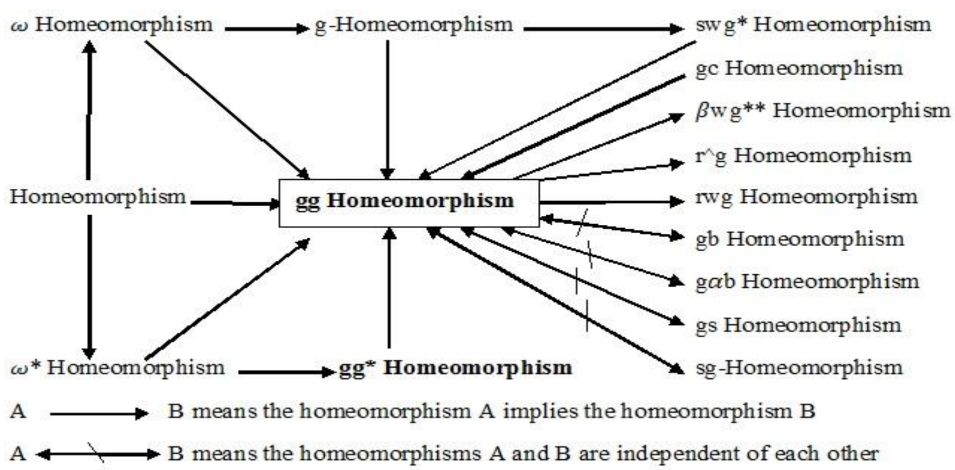

Fig. 1 Relation between gg-homeomorphism with various homeomorphisms in topological spaces

3.35 Theorem Any two gg*-homeomorphisms are preserved under composition.
Proof. Consider two gg*-homeomorphisms, $h: M \to N$and $g: N \to P$. Let V$\in C(P)$. Then, $g(V) \in gg - C(N)$ and $h^{-1}\big(g^{-1}(V)\big) \in gg - C(M)$. But $h^{-1}\big(g^{-1}(V)\big) = (goh)^{-1}(V)$, Therefore goh is gg-irresolute. Take W$\in C(M)$. Then, $h(W) \in gg - C(N)$ and $g(h(W)) \in gg - C(P)$. But $(goh)(W) = g(h(W))$.. Therefore goh is gg-irresolute. Hence the theorem.

3.36 Remark in the collection of all topological spaces, gg*-homeomorphism is an equivalence relation.
Proof. the reflexive and symmetric relations are follows from the known facts and transitive property follows from theorem 3.35

Conclusion

In this work, a new-class of homeomorphism namely gg-homeomorphisms are defined in topological spaces by using gg-closed maps and gg-continuous maps. Also, the basic properties of these homeomorphisms are discussed in detail. the stronger, weaker, and independent form of gg-homeomorphisms with various homeomorphisms in topological spaces are established and shown in the figure. Also, introduced and studied the basic properties of gg*-homeomorphisms in topological spaces. This research work can further continue in soft topological spaces and fuzzy topological spaces. There is a possibility of studying this work in digital plane.

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