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Original Article

Some Results on Unique Fixed Point Theorems in Complete Metric Space

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***Abstract** - In this paper, we have proved the existence and uniqueness of common fixed point theorems for complete metric space. Our results generalizes fixed point results in existing literature.*

***Keywords** - Cauchy sequence, Complete metric space, Fixed point.*

Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Banach fixed point theorem[1] appeared in 1922, which is useful to solve the existence of solutions for the various nonlinear problems which used to arise in the various fields of sciences like biological, physical and social sciences[13,14].

1.1. Preliminaries

Definition 2.1 [12]: Let X be a non-empty set and consider the function $d: X \times X \to [0, \infty)$ which satisfies the following conditions:

(d_1): $d(x, x) = 0$

(d_2): $d(x, y) = d(y, x) \Longrightarrow x = y$

(d_3): $d(x, y) = d(y, x)$

(d_4): $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$

Then d is called metric on X and (X, d) is called a metric space.

Definition 2.2 [12]: A sequence $\{x_n\}$ in a metric space (X, d) is said to be convergent to z, if $\lim_{n\to\infty} d(x_n, z) = 0 = \lim_{n\to\infty} d(z, x_n)$. Here z is called limit point of a sequence$\{x_n\}$.

Definition 2.3 [12]: A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if for a given $\epsilon > 0$, there exist a $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, $d(x_n, x_m) < \epsilon$ or $d(x_m, x_n)$

Definition 2.4 [12]: A metric space (X, d) is said to be complete, if every Cauchy sequence in X is convergent to a point in X.

Definition 2.5 [12]: Let (X, d) be a metric space and let $f: X \to X$ be a mapping. Then a point $x \in X$ is a fixed point of f if $f(x) = x$.

***Theorem 2.6* [1](Banach):** Let (X, d) be a complete metric space and $f: X \to X$ be a contraction, i.e. f satisfies $d(fx, fy) \leq \alpha d(x, y)$ for all $x, y \in X$ and a fixed constant $\alpha < 1$. Then there exists a unique fixed point of f in X.
After the Banach, there was an open problem for researcher that if the map is of non-contractive type, then whether the map has a fixed point ? And the positive answer in case of complete metric space was given by by Kannan in the form of following theorem in 1968.

***Theorem 2.7* [2](Kannan):** Let $f: X \to X$, where (X, d) is a complete metric space and f satisfies the condition

$$d(fx, fy) \leq \beta[d(x, fx) + d(y, fy)],$$

where $0 < \beta < \frac{1}{2}$ and $x, y \in X$. Then f has a unique fixed point in X.
In 1999, Sarkhel [3,5], proved Kannan fixed point theorem using Banach's fixed point theorem. In 1972, the related fixed point theorem to the Kannan was given by Chatterjea as follows:

***Theorem 2.8* [4]:** Let (X, d) be a complete metric space. Let T be a Chatterjea mapping on X, i.e. there exists $r \in [0, \frac{1}{2})$ satisfying

$$d(Tx, Ty) \leq r(d(x, Ty) + d(y, Tx)), \text{ for all } x, y \in X.$$

Then T has a unique fixed point.

The generalization of Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem have been established by various authors in [7,8,9,10,11,15-25] etc.

In this paper we give two theorems which will be study of Kannan fixed point theorem and Chatterjea fixed point theorem combinely.

Main Results

Theorem 3.1: *Let (X, d) be a complete metric space and $f: X \to X$ be a mapping satisfying the condition*

$$d(fx, fy) \leq a_1[d(x, fx) + d(y, fy)] + a_2[d(x, fy) + d(y, fx)] \quad \dots (1)$$

for all $x, y \in X$, $0 < a_1,\ a_2 < \frac{1}{2}$ and $a_1 + a_2 < \frac{1}{2}$. Then f has unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X and consider the iterative sequence as

$$x_n = f^n x_0 = f x_{n-1} \text{ for all } n \in \mathbb{N}$$

Then $d(x_1, x_2) = d(fx_0, fx_1)$

$$\leq a_1[d(x_0, fx_0) + d(x_1, fx_1)] + a_2[d(x_0, fx_1) + d(x_1, fx_0)] \text{ (by equation (1))}$$

$$= a_1[d(x_0, x_1) + d(x_1, x_2)] + a_2[d(x_0, x_2) + d(x_1, x_1)]$$

$$= a_1 d(x_0, x_1) + a_1 d(x_1, x_2) + a_2 d(x_0, x_2) \qquad (as\ \ d(x_1, x_1) = 0)$$

$$\leq a_1 d(x_0, x_1) + a_1 d(x_1, x_2) + a_2[d(x_0, x_1) + d(x_1, x_2)] \text{ (by triangle inequality)}$$

$$\therefore [1 - (a_1 + a_2)] d(x_1, x_2) \leq (a_1 + a_2) d(x_0, x_1)$$

$$d(x_1, x_2) \leq \left[\frac{a_1 + a_2}{1 - (a_1 + a_2)}\right] d(x_0, x_1)$$

$$d(x_1, x_2) \leq \beta d(x_0, x_1) \text{ , where } \beta = \frac{a_1 + a_2}{1 - (a_1 + a_2)}$$

$$d(x_2, x_3) \leq \beta^2 d(x_0, x_1)$$

$$d(x_3, x_4) \leq \beta^3 d(x_0, x_1)$$

In general, if n is any positive integer, then

$$d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1)$$

If p is any positive integer, then by triangle inequality we have,

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})$$

$$\leq \beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1) + \cdots + \beta^{n+p-1} d(x_0, x_1)$$

$$=(\beta^n + \beta^{n+1} + \cdots + \beta^{n+p-1})d(x_0, x_1)$$

$$=\beta^n(1 + \beta + \cdots + \beta^{p-1})d(x_0, x_1)$$

$$\leq \frac{\beta^n}{1-\beta}\, d(x_0, x_1) = \frac{\beta^n}{1-\beta}\, d(x_0, fx_0)$$

Since, $0 < a_1, a_2 < \frac{1}{2}$, and $a_1 + a_2 < \frac{1}{2}$, we have $0 < \beta < 1$ and so

$d(x_n, x_{n+p}) \to 0$ as $n \to \infty$ ... (2)

Therefore, $\{x_n\}$ is a Cauchy sequence. As X is complete metric space, we have

$$\lim_{n\to\infty} x_n = z \in X \qquad \text{... (3)}$$

Now, we will show that this z is a fixed point of f.

For this, we have by triangle inequality

$d(z, fz) \leq d(z, x_n) + d(x_n, fz)$

$\leq d(z, x_n) + d(fx_{n-1}, fz)$

$\leq d(z, x_n) + a_1[d(x_{n-1}, fx_{n-1}) + d(z, fz)] + a_2[d(x_{n-1}, fz) + d(z, fx_{n-1})]$ (by equation (1))

$\leq d(z, x_n) + a_1 d(x_{n-1}, fx_{n-1}) + a_1 d(z, fz) + a_2 d(x_{n-1}, fz) + a_2 d(z, fx_{n-1})$

$\leq d(z, x_n) + a_1 d(x_{n-1}, fx_{n-1}) + a_1 d(z, fz) + a_2[d(x_{n-1}, z) + d(z, fz)] + a_2 d(z, fx_{n-1})$

(by triangle inequality)

$\therefore [1 - (a_1 + a_2)]d(z, fz) \leq d(z, x_n) + a_1 d(x_{n-1}, x_n) + a_2 d(x_{n-1}, fx_{n-1})$

$= d(z, x_n) + a_1 d(x_{n-1}, x_n) + a_2 d(x_{n-1}, x_n)$

$\leq d(z, x_n) + (a_1 + a_2) d(x_{n-1}, x_n)$

As $n \to \infty$ by equations (2) and (3), we have

$[1 - (a_1 + a_2)]d(z, fz) \leq 0.$

But as $1 - (a_1 + a_2) > 0$ this implies $d(z, fz) \leq 0$ and by definition of metric space we have $d(z, fz) \geq 0$

Therefore we get, $d(z, fz) = 0$ i.e. $fz = z$ and z is a fixed point of f.

Uniqueness:

Now, we will prove that z is a unique fixed point of f. For this, assume that z and z_1 are two distinct fixed points of f. Then

$d(z, z_1) = d(fz, fz_1) \leq a_1[d(z, fz) + d(z_1, fz_1)] + a_2[d(z, fz_1) + d(z_1, fz)]$ (by equation (1))

$= a_1 d(z, z) + a_1 d(z_1, z_1) + a_2 d(z, z_1) + a_2 d(z_1, z)$

$= 2a_2 d(z, z_1)$ (as $d(z, z) = d(z_1, z_1) = 0$ and $d(z, z_1) = d(z_1, z)$)

implies $(1 - 2a_2)d(z, z_1) \leq 0$ i.e. $d(z, z_1) \leq 0$ as $(1 - 2a_2) > 0$.

Therefore, $d(z, z_1) = 0$ i.e. $z = z_1$, which is a contraction.

Hence z is a unique fixed point of f.

Theorem 3.2: *Let (X,d) be a complete metric space. Suppose that $f_1, f_2 : X \to X$ are continuous self-mappings satisfying the following conditions:*

$$d(f_1x, f_2y) \le a_1[d(x, f_1x) + d(y, f_2y)] + a_2[d(x, f_2y) + d(y, f_1x)] \ldots (1)$$

for all $x, y \in X$, $0 < a_1,\ a_2 < \frac{1}{2}$ and $a_1 + a_2 < \frac{1}{2}$. Then f_1 and f_2 have a common unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X and consider the iterative sequence as

$$x_0, x_1 = f_1x_0, x_2 = f_1x_1, \ldots, x_{2n+1} = f_1x_{2n}$$

$$x_2 = f_2x_1, x_3 = f_2x_2, \ldots, x_{2n} = f_2x_{2n-1} \quad \text{for all } n \in \mathbb{N}$$

Now consider,

$d(x_{2n+1}, x_{2n+2}) = d(f_1x_{2n}, f_2x_{2n+1})$

$\le a_1[d(x_{2n}, f_1x_{2n}) + d(x_{2n+1}, f_2x_{2n+1})] + a_2[d(x_{2n}, f_2x_{2n+1}) + d(x_{2n+1}, f_1x_{2n})]$ (by equation (1))

$= a_1[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + a_2[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})]$

$\le a_1d(x_{2n}, x_{2n+1}) + a_1d(x_{2n+1}, x_{2n+2}) + a_2d(x_{2n}, x_{2n+1}) + a_2d(x_{2n+1}, x_{2n+2})$

(as $d(x_{2n+1}, x_{2n+1}) = 0$ and by triangle inequality)

$[1 - (a_1 + a_2)]d(x_{2n+1}, x_{2n+2}) \le (a_1 + a_2)d(x_{2n}, x_{2n+1})$

$\therefore d(x_{2n+1}, x_{2n+2}) \le \frac{a_1+a_2}{1-(a_1+a_2)} d(x_{2n}, x_{2n+1}) = \beta d(x_{2n}, x_{2n+1})$

where $\beta = \frac{a_1+a_2}{1-(a_1+a_2)} < 1$ as $a_1 + a_2 < \frac{1}{2}$

Similarly, $d(x_{2n}, x_{2n+1}) \le \beta d(x_{2n-1}, x_{2n})$, so we have

$$d(x_{2n+1}, x_{2n+2}) \le \beta^2 d(x_{2n-1}, x_{2n})$$

Continuing in this way we get,

$$d(x_n, x_{n+1}) \le \beta^n d(x_0, x_1)$$

Using triangle inequality for $n, p \in \mathbb{N}$ with $p > n$, we have

$$d(x_n, x_p) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})$$

$$\le \beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1) + \cdots + \beta^{n+p-1} d(x_0, x_1)$$

$$\le \beta^n[1 + \beta + \beta^2 + \cdots + \beta^{p-1}]d(x_0, x_1)$$

$$\le \frac{\beta^n}{1-\beta} d(x_0, x_1)$$

For $r > 0$, we can choose a positive integer n_0 such that, $\frac{\beta^{n_0}}{1-\beta} d(x_0, x_1) < r$

For any $n, p \ge n_0$, we have $\quad d(x_n, x_p) \le \frac{\beta^n}{1-\beta} d(x_0, x_1) \le \frac{\beta^{n_0}}{1-\beta} d(x_0, x_1) < r$

As $r \to 0$, $\quad d(x_n, x_p) \to 0$

Therefore, $\{x_n\}$ is a Cauchy sequence in a complete metric space (X, d).

Hence, there exist a point $z \in X$, such that $\lim_{n\to\infty} x_n = z \in X$.

Here, the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of the sequence $\{x_n\}$ also converges to z.

As f_1 is continuous mapping so

$$\lim_{n\to\infty} x_{2n+1} = z \Longrightarrow \lim_{n\to\infty} f_1 x_{2n+1} = f_1 z \Longrightarrow \lim_{n\to\infty} x_{2n+2} = f_1 z$$

This implies, $f_1 z = z$ and hence z is a fixed point of f_1.

Similarly, using the continuity of f_2, one can show that $f_2 z = z$

Therefore z is a common fixed point of $f_1$and f_2.

Uniqueness:

Let z and z_1 be two distinct common fixed points of f_1 and f_2.

Now, consider

$$d(f_1 z, f_2 z_1) \leq a_1[d(z, f_1 z) + d(z, f_2 z_1)] + a_2[d(z, f_2 z_1) + d(z_1, f_1 z)] \quad \text{(by equation (1))}$$

$$d(z, z_1) \leq a_1[d(z, z) + d(z_1, z_1)] + a_2[d(z, z_1) + d(z_1, z)]$$

$$\text{(as } f_1(z) = z, f_2(z) = z, f_1(z_1) = z_1, f_2(z_1) = z_1 \text{)}$$

$$d(z, z_1) \leq 2a_2 d(z, z_1) \quad \text{(as } d(z, z_1) = d(z_1, z))$$

$\therefore (1 - 2a_2) d(z, z_1) \leq 0.$

But $(1 - 2a_2) > 0$ and so $d(z, z_1) \leq 0$ and hence $d(z, z_1) = 0$, $\therefore z = z_1$ which is a contradiction.

Therefore, z is a unique common fixed point of f_1 and f_2.

2. Conclusion

In this research article, we proved two fixed point theorems with the help of Kannan fixed point theorem and Chatterjea fixed point theorem.

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