International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V68I6P518

Volume 68 Issue 6, 149-155, June 2022
© 2022 Seventh Sense Research Group®

Original Article

A Vertex Code of Designed Distance d_H and its Graph G

Sreedevi S. L[1], N. Suresh Babu[2]

[1]*Department of Mathematics, IARE, Hyderabad, India*
[2] *Department of Mathematics, University College, Thiruvananthapuram, India.*

Received: 03 May 2022 Revised: 16 June 2022 Accepted: 25 June 2022 Published: 30 June 2022

Abstract - *Matrices associated with the graphs were considered as a good tool to construct codes from graphs with desirable properties. We introduced a new binary code – the vertex code C from a given graph G, depending on the degree of the vertices of G, in such a way that the vertex polynomial of G is same as the weight enumerator of C in* [3]. *A detailed study on the properties of the vertex code was carried out in*[4]. *In this paper we revise the vertex code given there in with the purpose of enhancing the error correcting capability of the code and try to generalize the nature of the graph for a given code of specified size and distance.*

Keywords - *Graph, Vertex degree, Degree sequence, Arithmetic sequence, Error correcting codes, Vertex code.*

1. Introduction

Let $G = (V, E)$ be any simple graph of order p and size q. $\deg(v_i)$ denotes the degree of vertex v_i, $\delta(G)$ and $\Delta(G)$ represents the minimum and maximum degree of G. For the definitions and algorithms from graph theory we follow [1], [5]. A block code C of length n over the field $F = GF(2)$ is a collection of binary words of length n. A word is a sequence of binary digits, where the only digits are 0 and 1. The size of C denoted as $|C|$ is the number of words of C. If $C = \{c_1, c_2, c_3, \dots, c_n\}$, then the Hamming distance between any two code words c_i and c_j is defined as,

$d_H(c_i, c_j) =$ Number of coordinates in which c_i and c_j differ. The minimum distance of C is the $\min\{d_H(c_i, c_k);\ c_i \neq c_k;\ \forall\ c_i, c_k \in C\}$. The Hamming weight of a code word c_i is the number of ones in it and is denoted as $wt_H(c_i)$. The error correcting capability of code dependents directly on the minimum distance of the code. For coding concepts we refer to[2].

2. Vertex Code of a Graph

Earlier we constructed vertex code C_G of a graph G of order p, such that to every vertex u_i of G there exists a code word c_i satisfying, $wt(c_i) = deg(u_i)$, having block length n, where $\binom{n}{i} \geq \alpha_i$, α_i is the number of vertices with degree i. Also it was assumed that, $d_H(c_i, c_j) = \begin{cases} |deg(u_i) - deg(u_j)|, if\ deg(u_i) \neq deg(u_j) \\ 2,\ if\ \ deg(u_i) = deg(u_j) \end{cases}$.

The minimum distance of the code so constructed turned out to be as low as 1 or 2, there by affecting the error correcting capability of the code. The quest for good error detection and correction codes from graphs forced to think as follows:

3. Vertex Code of a Graph – Revised

Let G be a simple graph (regular graphs are not preferable here) of order p, with degree sequence $d_1, d_2, \cdots d_p$ (not all $d_i's$ distinct). If an arithmetic sequence with common difference $d \geq 3$, having s terms can be extracted from the degree sequence, rename those s vertices of G as $v_1, v_2, \cdots v_s$. Assign code words $c_1, c_2, \cdots c_s$ to these s vertices in such a way that $wt(c_i) = deg(v_i)$ $(i = 1,2, \cdots, s)$ and $d_H(c_i, c_j) = |deg(v_i) - deg(v_j)|$. The code, $C = \{c_1, c_2, \cdots c_s\}$ so obtained from G is called $vertex\ code\ C_G$ of G. Thus the $vertex\ code\ C_G$ has minimum distance $d \geq 3$.

Definition

The $vertex\ code\ C_G$ of a graph G is a code obtained by considering only those vertices of G, whose degrees are in an arithmetic sequence with common difference $d \geq 3$, where the vertices are coded in accordance with their degrees such that $deg(v_i) = wt(c_i)$ and $d_H(c_i, c_j) = |deg(v_i) - deg(v_j)|$.

Remark

If d_m denotes the minimum degree of a vertex in the considered sequence with a common difference d, then, $wt(c_k) = d_m + (k-1)d$. That is, $wt(c_i) < wt(c_j)\ for\ \forall\ i < j$.

Since we are considering only those vertices whose degrees are in an arithmetic sequence with common difference$d \geq 3$, the minimum distance of the code also turns out to be greater than or equal to 3, and in turn the code can correct at least 1 error.

Theorem

If $C = \{c_1, c_2, \cdots c_M\}$ is a code with minimum distance d_H and if $wt(c_1) = d_m$, then the minimum required block length n of C is $n = d_m + (M-1)d_H$ (1)

Proof: We have $C = \{c_1, c_2, \cdots c_M\}$ and $wt(c_1) = d_m$.

We prove the theorem using Mathematical Induction.

When $M = 1$, that is the code contains only one word, as there are no other words to be compared with it, assume it to be with minimum weight d_m. Then the minimum required block length of the code is equal to the weight of the given code, that is, $n = d_m$.

When $M = 2$, the code words c_1 and c_2 differ by a distance of d_H and the block length $n = d_m$ is not sufficient to represent these two words, and $n = d_m + d_H$, turns out to be the minimum to satisfy the coding condition.

Assume the given value of n is true for $M = k$. That is for $C = \{c_1, c_2, \cdots c_k\}$,

$n = d_m + (k-1)d_H$.

When $M = k+1$. That is, $C = \{c_1, c_2, \cdots c_k, c_{k+1}\}$. Since the distance between code words c_k and c_{k+1},i.e. $d(c_k, c_{k+1}) \geq d_H$, according to our construction,

$$n \geq d_m + (k-1)d_H + d_H$$
$$= d_m + k\ d_H$$

Since we are interested in minimum block length we take, $n = d_m + k\ d_H$

Thus, $n = d_m + (M-1)d_H$, for any code C having size M and distance d_H.

3.1. Constructing a code from a given graphG.

We proceed with an example. Consider the graph G_1 in figure 1.

Here G_1 has the degree sequence 16,15,14,,13,13,12,11,10,9,9,8,7,6,5,4,2,2 and the corresponding vertices are named as $v_{17}, v_{16}, v_{15}, v_{14}, v_{13}, v_{12}, v_{11}, v_{10}, v_9, v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_1$, respectively. G_1 can be vertex coded as defined in [2]. But the minimum distance $d_H$turns out to be 1. So consider the subsets of vertices $V_{S_1}, V_{S_2}, V_{S_3}, V_{S_4}, V_{S_5}$ with $C_{S_1}, C_{S_2}, C_{S_3}, C_{S_4}, C_{S_5}$ as the corresponding codes respectively. Each case is separately dealt with as follows:

(i) $V_{S_1} = \{v_1, v_4, v_7, v_{11}, v_{14}\}$, corresponding to the vertices with degrees $(2, 5, 8, 11, 14)$

Here, $|C_{S_1}| = 5$, $d_H = 3$ and $wt(c_1) = 2$. The minimum required block length of this code, $n = 5 \times 3 + (2-3) = 14$.

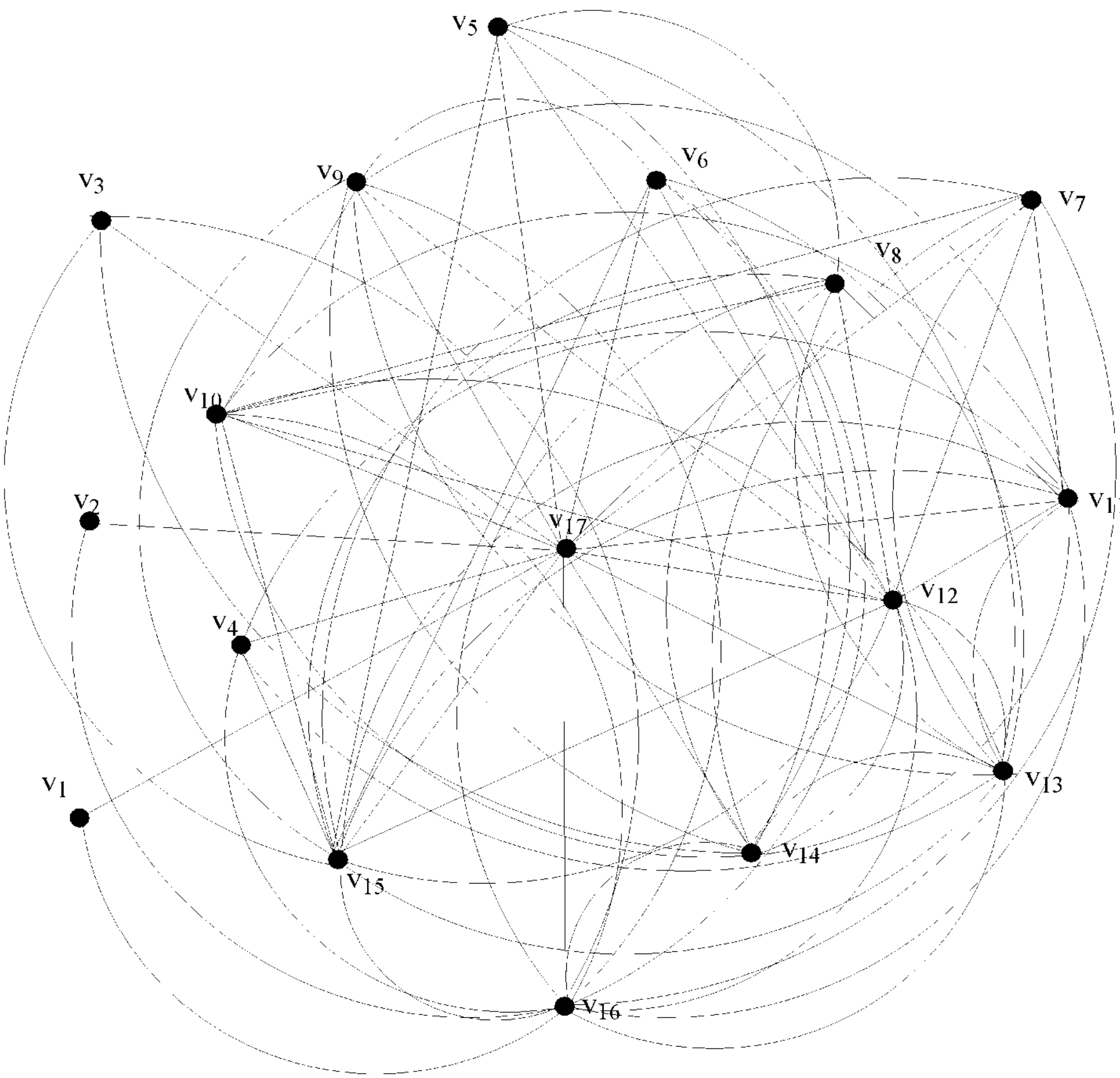
(ii) $V_{S_2} = \{v_3, v_7, v_{12}, v_{17}\}$, corresponding to the vertices with degrees $(4,8,12,16)$.
The code C_{S_2} has size 4 and minimum distance 4 and has the block length of 16.

(iii) The code C_{S_3} with respect to $V_{S_3} = \{v_5, v_8, v_{12}, v_{16}\}$, corresponding to the vertices with degrees $(6, 9, 12, 15)$ has size 4 and minimum distance 3 and block length 15.

(iv) The code C_{S_4} with respect to $V_{S_4} = \{\, v_4,\ v_9, v_{13}\}$, has size 3 and minimum distance 4 and block length 13.

(v) $V_{S_5} = \{v_3, v_6, v_{10}, v_{13}, v_{17}\}$ corresponding to the vertices with degrees $(4,7,10,13,16)$ has, $|C_{S_5}| = 5$, $d_H = 3$ and , $n = 16$

Thus from a given simple graph we can construct a number of sub graphs and corresponding $vertex\ codes$ with varying size, distance and block length, so as to meet error correction parameters.



G_1
Fig. 1

3.2. Construction of the graph G_C from a given code C

If we are given a code C of block length n with $|C| = M$ and minimum distance d_H, such that, $d(c_i, c_j) = k \times d_H, k \in$ I, and $\forall$ $c_i, c_j \in C$, then we can construct the graph G of minimal order and size in an unique way.
[Assumptions followed on constructing codes from graphs are applied here in reverse manner.]
Step by step procedure to construct a graph G_C from given code C:

1. Rewrite the code words $c_1, c_2, \cdots c_M$ of C in such a way that $wt(c_i) < wt(c_{i+1})$
2. If c_M is an code word with weight n, according to our assumption there exists a vertex v_M of G_C such that $deg(v_M) = n$. Since G is a simple graph, v_M has n distinct neighbours. Thus the required G_C contains at least $n+1$ vertices.
3. Plot $n+1$ points in a plane and mark the vertex v_M out of those $n+1$ points and draw its n adjacencies.
4. Next choose a vertex v_l corresponding to the code word c_l of just lower weight than that of c_M and draw its adjacencies so that, $wt(c_l) = deg(v_l)$.
5. Choose a vertex v_k from any of the adjacent vertices of v_l and draw all remaining adjacencies so as to match the codeword c_k, such that $wt(c_k) = deg(v_k)$
6. Repeat the process in step.5, choosing the next vertex with having maximum degree.
7. Continue the process till all the vertices with respect to the code words in C are marked along with their adjacencies.

The graph thus obtained is a minimal one with respect to the given code C.

Definition

A graph G_C with respect to a given (n, M, d_H) code C, with d_m as the minimum weight of code word in C, is said to be a ***minimal graph*** if it is of order p, with $p = n + 1$ and size q, where

$$q = \begin{cases} |M| \times n - \sum_{i=1}^{|M|-1} i(d_H + 1); if\ |M| \le d_m & (a) \\ (|M| - 1) \times n - \sum_{i=1}^{|M|-2} i(d_H + 1); if\ |M| > d_m & (b) \end{cases} \qquad (2)$$

3.2.1. Consider the following examples of drawing graphs from given $vertex\ codes$ C_G

(i) Case I $|M| < d_m$

Consider the given code $\{11110000, 11111111\}$. Here $d_m = 4$ and $|M| = 2$
The corresponding minimal graph following the step by step procedure discussed earlier is shown in figure 2, with $p = 9$ and $q = 11$ and which satisfy (a) of (2)

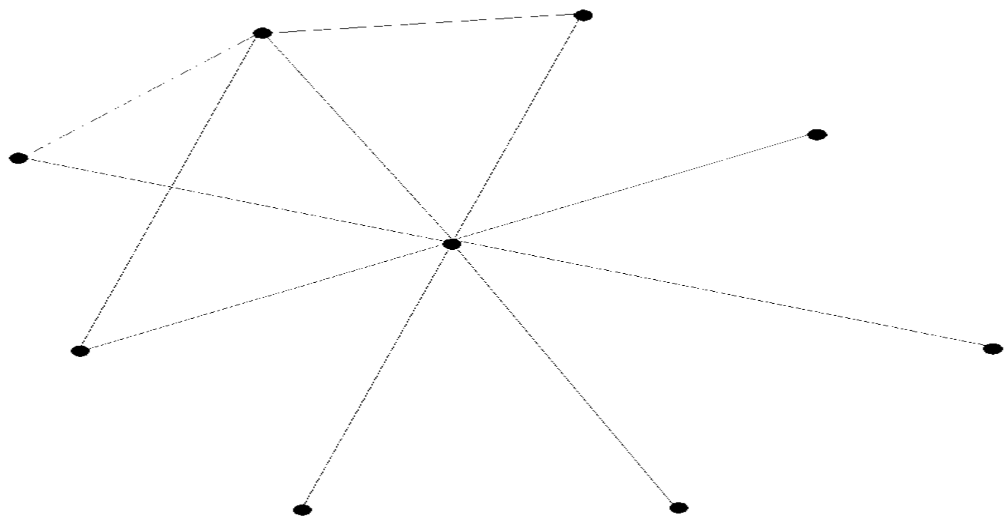


Fig. 2

(ii) Case II $|M| = d_m$

Consider the given code {111000000,111000111, 111111111}. Here $d_m = 3$ and $|M| = 3$.

In figure 3, the vertices h, $d, and\ j$ with degrees $3, 6, and\ 9$ corresponds to the given code words 111000000, 111000111 and 111111111 $p = 10$ and $q = 15$, which satisfies (a) of (2).

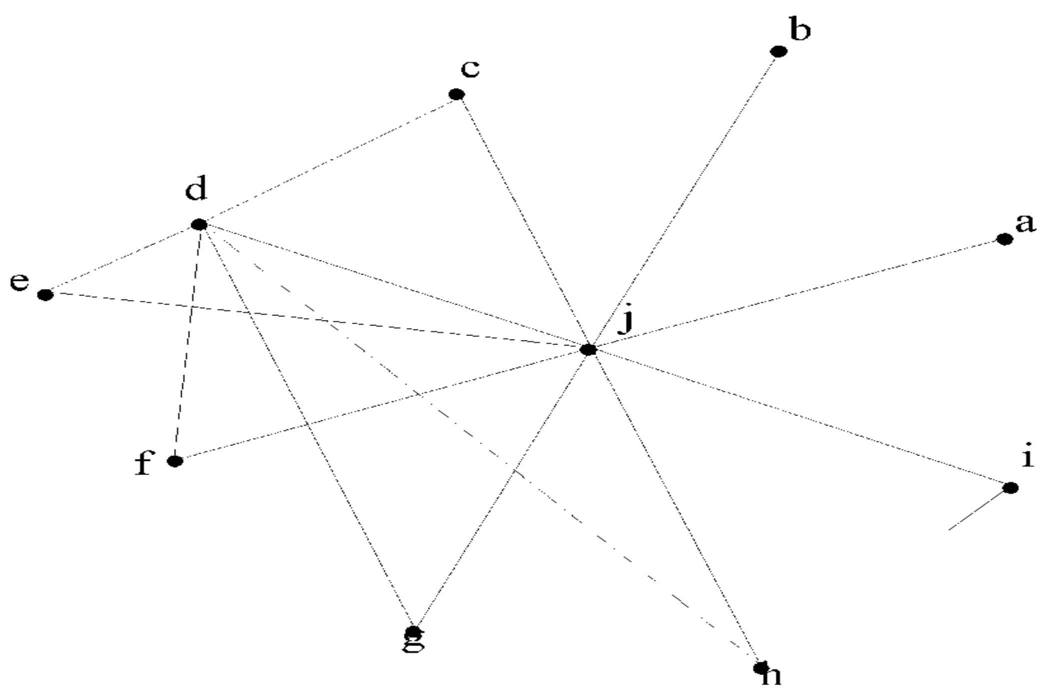


Fig. 3

(iii) Case III $|M| > d_m$

Consider the given code $C = \begin{Bmatrix} 101010000000000, 101010101010000, \\ 101010101010111, 111110101011111, \\ 111111111111111 \end{Bmatrix}.$

Here $d_m = 3$ and $|M| = 5$.

The vertices $v_3,\ v_6,\ v_9\ , v_{12}$ and v_{15} corresponds to the code words in C, in given order. As the maximum weight of a word in C is 15, there must be a vertex of degree 15. Since the graphs under our consideration are simple, the required graph should have at least 16 vertices. The graph drawn as per the step by step procedure discussed above is given in figure 4. This is a minimal graph with order $p = 16$ and size $q = 36$. Here (b) of (2) is satisfied.

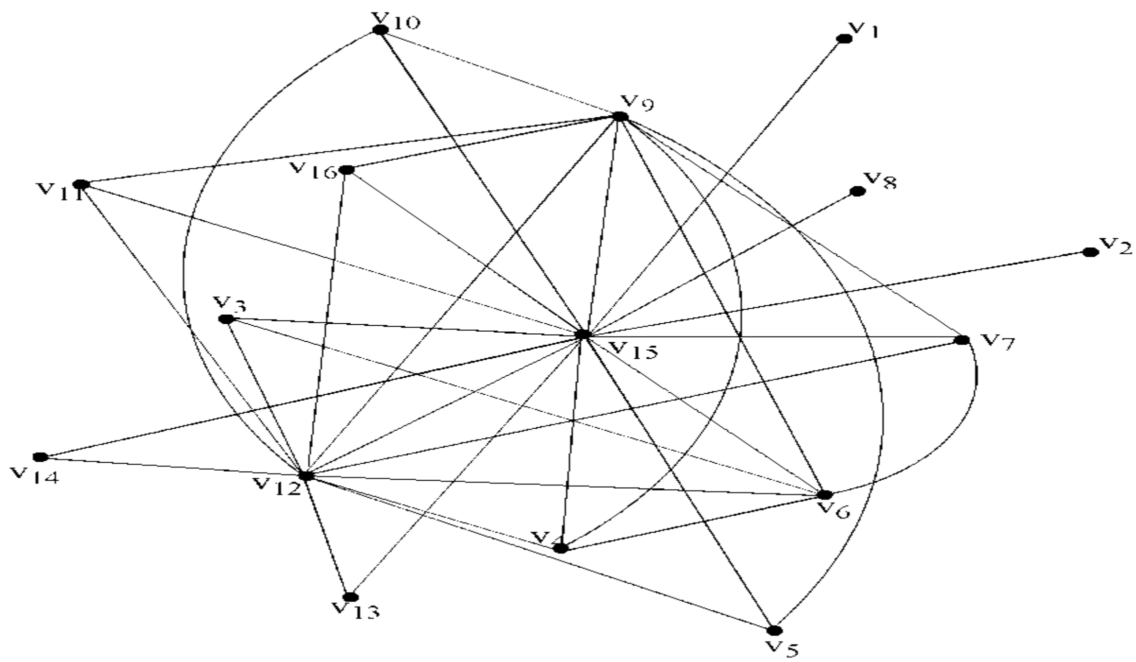


Fig. 4

3.3. Properties of Minimal graph

1. The vertices corresponding to M code words form a Hamiltonian circuit.
2. The vertices corresponding to code words of C act as cut vertices.
3. The vertex corresponding to the code word with maximum weight is a minimum vertex cut.

4. Conclusion

In this paper we constructed a vertex code from a given graph with enhanced error correction capability. A minimal graph G_C with respect to a given vertex code is also constructed.

References

[1] R. Balakrishnan, K. Ranganathan, A Textbook of Graph Theory, *Springer* (India) Private Limited, New Delhi 110001, (2008).
[2] R. S. Bapat, Graphs and Matrices, *Springer*, Co-publication with Hindustan Book Agency, New Delhi,2015.
[3] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Elsevier Science Publishing Co., 52 Vanderbilt Avenue, New York, NY 10017, (1976).
[4] P. J. Cameron and Van Lint, Designs, Graphs, Codes and their Links, London Mathematical Society, Student Texts 22, Cambridge University Press, (1991).
[5] Frank Harary and Gordon W. Wilcox, Boolean Operations on Graphs, Math. SCAND, 20 (1967) 41-51.
[6] Frank Harary, Graph Theory, © Narosa / Addison- Wesley, Indian Student Edition, (1990).

[7] D. G. Hoffman, D. A. Leonard, C. C. Lindner, K.T. Phelps, C. A. Rodger, J. R. Wall, Coding Theory- The Essentials, © Marcel Dekker, INC (1991).

[8] Jasan I. Brown, Richard J. Nowakowski, The Neighbourhood Polynomial of a Graph, Australian Journal of Combinatorics, 42(2008) 55-68.

[9] Kaveh, Optimal Analysis of Structures by concepts of Symmetry and Regularity, DOI 10.1007/978-3-7091-1565-7_2©Springer-Verlang Wein (2013).

[10] F. J. MacWilliams, NJA Sloane, The Theory of Error Correcting Codes, Elsevier Science Publishers, B.V., P. O. Box 1991, 1000BZ, Amsterdam, The Netherlands, (1988).

[11] Narsing Deo, Graph Theory - with Applications to Engineering and Computer Science, PHI, Pvt. Ltd, M-97, Cannaught Circus, New-Delhi 110001, (1997).

[12] Norman Biggs, Algebraic Graph Theory, Second edition, Cambridge University Press, (1993).

[13] San Ling, Chaoping Xing, Coding Theory; A First Course, Cambridge University Press, (2004).

[14] S. Sedghi, N. Shobe, M. A. Salahshoor, The Polynomials of a Graph, *Iranian Journal of Mathematical Sciences and Informatics*, 3(2) (2008) 55-67.

[15] N. Suresh Babu, G. Suresh Singh, Sreedevi S.L. Graphs and Codes, *International Journal of Mathematics Trends and Technology*, 56(2) (2018) 120- 128.

[16] N. Suresh Babu, Sreedevi S.L. Vertex Codes of Graphs, *Far East Journal of Applied Mathematics*, 100(3) (2018) 197- 215.

[17] G. Suresh Singh, Graph Theory, PHI, New Delhi, (2010).

[18] G. Suresh Singh, Sreedevi S. L., Vertex Polynomials Derived Through Various Graph Theoretical Operations, *Indian Journal of Mathematics and Mathematical Sciences*, 13(2) (2017) 357-367.

[19] Vladimir D. Tonchev, Error Correcting Codes from Graphs, Elsevier, Discrete Mathematics 257 (2002) 549-557.