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Original Article

Existence of Chaos in the Nearest Neighbors Coupled Map Lattices

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Abstract *- This paper studies the existence of chaos in a class of Nearest Neighbor Coupled Mapping Lattice (NNCML). Prove that NNCML is chaotic in the sense of Li-Yorke or both Li-Yorke and Devaney by employing the coupled-expanding theory. At the end, two illustrative examples are provided.*

Keywords *- Chaos, Coupled-expansion theory, Coupled map lattice.*

1. Introduction

It has been found that chaos is not only challenging in mathematics theory and engineering technology, but also has a very broad prospect in the application of biological intelligence [1], computer hardware [2], communication systems [3], and other high and new technologies. As a typical class of discrete spatiotemporal systems, CML has been widely studied in application, synchronization and controlling chaos, etc. (see [4-7] and the references therein). The existence of Li-Yorke chaos of CML was investigated in recent years. In 2003, Zheng and Liu proved the existence of nonlinear solutions with time periods in one-dimensional nearest neighbor coupled mapping lattices (NNCMLs). In 2007, Tian and Chen studied some sufficient conditions of Li-Yorke chaos in NNCML [8]. In 2011, Khellat et al. studied chaotic synchronization by analyzing Lyapunov exponents of a class of CML [9]. In 2019, Nag and Poria proved that globally CML with delays is Li-Yorke chaos [10]. Recently, Liang et al. studied the chaotification of first-order partial difference equations (i.e., a special CML) and proved that it is chaotic in the sense of Li-Yorke or Devaney [11-14].

To the best of our knowledge, there are few results such that the NNCML system is chaotic either in the sense of Li-Yorke or both Li-Yorke and Devaney. This fact motivates us to explore mathematically the existence of chaos in the NNCML system. In this paper, the dynamical equation of NNCML is described as follows:

$$x(n+1,m) = (1-\varepsilon)f\big(x(n,m)\big) + 0.5\varepsilon\left(f\big(x(n,m-1)\big) + f\big(x(n,m+1)\big)\right) \quad (1)$$

where $n \geq 0$ is the time step, m is the lattice point with $m = 1,2,\cdots,N$, N is the number of the sites in the NNCML, $\varepsilon \in [0,1]$ is the coupled strength, and $f: R \to R$ is a function.

2. Preliminaries

In this section, definition of coupled-expanding map and two criteria of chaos induced by coupled-expanding maps are introduced.

Definition 1 [15]. *Let (X,d) be a metric space and $f: D \subset X \to X$ a map. If there exist $m \geq 2$ subsets $V_i (1 \leq i \leq m)$ of D with $V_i \cap V_j = \partial_D V_i \cap \partial_D V_j$ for each pair of $(i,j), 1 \leq i \neq j \leq m$, such that*

$$f(V_i) \supset \cup_{j=1}^{m} V_j, \quad 1 \leq i \leq m,$$

where $\partial_D V_i$ is the relative boundary of V_i with respect to D, then f is said to be a coupled- expanding map in V_i, $1 \leq i \leq m$. Further, the map f is said to be a strictly coupled- expanding map in V_i, $1 \leq i \leq m$., if $d(V_i, V_j) > 0$, for all $1 \leq i \neq j \leq m$.

Lemma 1 [16]. *Let (X,d) be a metric space and $V_i (1 \leq i \leq m)$ disjoint compact sets of X. If a continuous map $f: D \equiv \cup_{j=1}^{m} V_j \to X$, is a strictly coupled-expanding map in V_j, $1 \leq j \leq m$,then f is chaotic in the sense of Li-Yorke.*

Lemma 2 [17,18]. *Let (X,d) be a complete metric space and $f: D \subset X \to X$ a map. Assume that there exist k disjoint bounded and closed subsets V_i of D, $1 \le i \le k$, such that f is continuous in $\cup_{i=1}^{N} V_i$ and satisfies*

(i) f *is strictly coupled-expanding in* V_i , $1 \le i \le k$;

(ii) *there exists a constant* $\lambda > 1$ *such that*

$$d\big(f(x), f(y)\big) > \lambda d(x,y), \qquad \forall x, y \in V_i\,,\ 1 \le i \le k$$

Then, f has an invariant Cantor set $V \subset \cup_{i=1}^{N} V_i$ such that $f: V \to V$ is topologically conjugate to the subshift $\sum_k^+ \to \sum_k^+$. Consequently, f is chaotic on V in the sense of Devaney as well as Li-Yorke.

The following periodic boundary condition is imposed to (1):

$$x(n,0) = x(n,N), \ \ x(n,N+1) = x(n,1), \quad n \ge 0. \tag{2}$$

For any given initial condition, $x(0,m) = \phi(m), 1 \le m \le N$, where ϕ satisfies (2). Eq. (1) has a unique solution satisfying the initial condition and the boundary condition (2).

Let $x_n = \big(x(n,1), x(n,2), \cdots x(n,N)\big)^T$, then (1) with (2) can be written as

$$x_{n+1} = F(x_n), \ \ n \ge 0, \tag{3}$$

Where

$$F(x_n) = \begin{pmatrix} (1-\varepsilon) f\big(x(n,1)\big) + 0.5\varepsilon\Big(f\big(x(n,N)\big) + f\big(x(n,2)\big)\Big) \\ (1-\varepsilon) f\big(x(n,2)\big) + 0.5\varepsilon\Big(f\big(x(n,0)\big) + f\big(x(n,3)\big)\Big) \\ \vdots \\ (1-\varepsilon) f\big(x(n,N)\big) + 0.5\varepsilon\Big(f\big(x(n,N-1)\big) + f\big(x(n,1)\big)\Big) \end{pmatrix}. \tag{4}$$

System (3) is called a induced system by (1) with (2).

Definition 2 [19]. *Equation (1) with (2) is said to be chaotic in the sense of Devaney (or Li-Yorke) on $V \subset R^N$ if its induced system (3) is chaotic in the sense of Devaney (or Li-Yorke) on V.*

3. Existence of Chaos in (1)

In this section, we establish two criteria of Li-Yorke and Devaney chaos for (1) in the two cases of $0 \le \varepsilon < 1$ and $\varepsilon = 1$.

Theorem 1 ($\mathbf{0 \le \varepsilon < 1}$). Consider (1) with (2). Assume that

(i) $f(0) = 0$ and there exist positive constants r and L, such that

$$|f(x) - f(y)| \le L|x - y|, \qquad \forall x, y \in [-r, r]; \tag{5}$$

(ii) there exist constants a, b, c, d with $-r < a < 0 < b < c < d < r$, such that

$$f(a) \cdot f(b) < 0, f(c) \cdot f(d) < 0, \ \ |f(j)| > \mu, j = a, b, c, d, \tag{6}$$

where $\mu = \max\{|a|, b\}$.

Then for any ε satisfying

$$0 \le \varepsilon \le \frac{\mu - \gamma}{\mu(1+L)},$$

where $\gamma = \min\{|a|, b\}$, there exists a Cantor set $\Lambda \subset [a,b]^N \cup [c,d]^N$ such that (1) is chaotic on Λ in the sense of Li-Yorke.

Furthermore, suppose that

(iii) there exist constants $\lambda > 1$ such that

$$|f(x) - f(y)| \ge \lambda|x - y|, \qquad \forall x, y \in [a,b] \cup [c,d]. \tag{7}$$

then for any ε satisfying

$$0 \le \varepsilon \le \min\left\{\frac{\mu-\gamma}{\mu(1+L)}, \frac{\lambda-1}{\lambda+L}\right\},$$

there exists a Cantor set $\Lambda_1 \subset [a,b]^N \cup [c,d]^N$, such that (1) is chaotic on Λ_1 in the sense of both Li-Yorke and Devaney.

Proof. Lemmas 1 and 2 are used to prove the theorem.

Set

$$\mathrm{V}_1 = [a,b]^N, \quad \mathrm{V}_2 = [c,d]^N,$$

then $\mathrm{V}_1, \mathrm{V}_2 \subset [-R,R]^N$, V_1 and V_2 are compact sets, and

$$\mathrm{d}(\mathrm{V}_1,\mathrm{V}_2) = \inf\{||x-y||: x \in \mathrm{V}_1, y \in \mathrm{V}_2\} = c - b > 0.$$

Now, we prove that F satisfies all the assumptions of Lemmas 1 and 2 on V_1 and V_2. The whole proof is divided into two parts.

Step 1. F is strictly coupled-expanding in V_1 and V_2.

By assumption (ii), there are four cases. $Case\ I$: $f(a) < -\mu, f(b) > \mu, f(c) < -\mu, f(d) > \mu$; $Case\ II$: $f(a) > \mu, f(b) < -\mu, f(c) > \mu, f(d) < -\mu$; $Case\ III$: $f(a) < -\mu, f(b) > \mu, f(c) > \mu, f(d) < -\mu$; $Case\ IV$: $f(a) > \mu, f(b) < -\mu, f(c) < -\mu, f(d) > \mu$. Next, we will prove that (1) is Li-Yorke chaos in this four cases.

Case I: $f(a) < -\mu, f(b) > \mu, f(c) < -\mu, f(d) > \mu$;

For each $x \in \mathrm{V}_1$ with $x(i) = a, i = 1,2,\cdots,N$, it follows from (2) and (5) that

$$\begin{aligned} F_i(x) &= (1-\varepsilon)f(a) + 0.5\varepsilon\left(f\big(x(i-1)\big) + f\big(x(i+1)\big)\right) \\ &\le (1-\varepsilon)f(a) + \varepsilon L max\{|a|, b\} \\ &< -(1-\varepsilon)\mu + \varepsilon L\mu \le a; \end{aligned} \tag{8}$$

and for each $x \in \mathrm{V}_1$ with $x(i) = b, i = 1,2,\cdots,N$, (2) and (5) yields that

$$F_i(x) \ge (1-\varepsilon)f(b) - \varepsilon L max\{|a|, b\} > (1-\varepsilon)\mu - \varepsilon L\mu \ge d. \tag{9}$$

For each $x \in \mathrm{V}_2$ with $x(i) = c, i = 1,2,\cdots,N$, by (2) and (5),

$$F_i(x) \le (1-\varepsilon)f(c) + \varepsilon Ld < -(1-\varepsilon)\mu + \varepsilon L\mu \le a; \tag{10}$$

and for each $x \in \mathrm{V}_2$ with $x(i) = d, i = 1,2,\cdots,N$,one has

$$F_i(x) \ge (1-\varepsilon)f(d) - \varepsilon Ld > (1-\varepsilon)\mu - \varepsilon L\mu \ge d. \tag{11}$$

It follows from the assumption (i) that F is continuous in $\mathrm{V}_1 \cup \mathrm{V}_2$. By the intermediate value theorem and (8) - (11), one has $F(\mathrm{V_i}) \supset \mathrm{V}_1 \cup \mathrm{V}_2$ for $i = 1, 2$.

Case II: $f(a) > \mu, f(b) < -\mu, f(c) > \mu, f(d) < -\mu$;

For each $x \in \mathrm{V}_1$ with $x(i) = a, i = 1,2,\cdots,N$ it follows from (2) and (5) that

$$F_i(x) > (1-\varepsilon)\mu - \varepsilon L\mu \ge d; \tag{12}$$

and for each $x \in \mathrm{V}_1$ with $x(i) = b, i = 1,2,\cdots,N$,

$$F_i(x) < -(1-\varepsilon)\mu + \varepsilon L\mu \le a. \tag{13}$$

For each $x \in \mathrm{V}_2$ with $x(i) = c, i = 1,2,\cdots,N$,

$$F_i(x) > (1-\varepsilon)\mu - \varepsilon L\mu \ge d; \tag{14}$$

and for each $x \in \mathrm{V}_2$ with $x(i) = d, i = 1,2,\cdots,N$,

$$F_i(x) < -(1-\varepsilon)\mu + \varepsilon L\mu \le a. \tag{15}$$

By the continuity of F in $\mathrm{V}_1 \cup \mathrm{V}_2$, (12) - (15), and the intermediate value theorem, one has$F(\mathrm{V_i}) \supset \mathrm{V}_1 \cup \mathrm{V}_2$ for $i = 1, 2$.

Case III: $f(a) < -\mu, f(b) > \mu, f(c) > \mu, f(d) < -\mu$. In this case, F satisfies (8), (9), (14) and (15). Hence, $F(V_i) \supset V_1 \cup V_2$ for $i = 1, 2$.

Case IV: $f(a) > \mu, f(b) < -\mu, f(c) < -\mu, f(d) > \mu$. In this case, F satisfies (10) -(13). With a similar discussion to Cases I and II, we have $F(V_i) \supset V_1 \cup V_2$ for $i = 1, 2$.

By the above discussion, F is strictly coupled-expanding in V_1 and V_2. Therefore, by Lemma 1, (1) with (2) is chaotic in the sense of Li-Yorke.

Step 2. There exists $(1-\varepsilon)\lambda - \varepsilon L > 1$, such that

$$\|F(x) - F(y)\| \geq \big((1-\varepsilon)\lambda - \varepsilon L\big)\|x - y\|, \ \ \forall x, y \in V_i\,, i = 1, 2$$

It follows from $\|x - y\| = \max\{|x(i) - y(i)|, i = 1, 2, \cdots, N\}$ that there exists a constant, $k \in \{i = 1, 2, \cdots, N\}$, such that

$$\|x - y\| = |x(k) - y(k)|. \tag{16}$$

Therefore, by (5), (7) and (16), for all $x, y \in V_1$ and $x, y \in V_2$, one has

$$\|F(x) - F(y)\| = \max\{|F_i(x) - F_i(y)|, i = 1, 2, \cdots, N\}$$

$$= \max\left\{\left|(1-\varepsilon)\Big(f\big(x(i)\big) - f\big(y(i)\big)\Big) + 0.5\varepsilon\Big(f\big(x(i-1)\big) - f\big(y(i-1)\big) + f\big(x(i+1)\big) - f\big(y(i+1)\big)\Big)\right|, i = 1, 2, \cdots, N\right\}$$

$$\geq (1-\varepsilon)\big|f\big(x(k)\big) - f\big(y(k)\big)\big| - 0.5\varepsilon\big|f\big(x(k-1)\big) - f\big(y(k-1)\big)\big| - 0.5\varepsilon\big|f\big(x(k+1)\big) - f\big(y(k+1)\big)\big|$$

$$\geq (1-\varepsilon)\,\lambda|x(k) - y(k)| - 0.5\varepsilon L|x(k-1) - y(k-1)| - 0.5\varepsilon L|x(k+1) - y(k+1)|$$

$$\geq (1-\varepsilon)\,\lambda\|x - y\| - \varepsilon L\|x - y\|$$

$$= ((1-\varepsilon)\lambda - \varepsilon L)\|x - y\|.$$

Note that $(1-\varepsilon)\lambda - \varepsilon L > 1$.

Together with the proof of Step 1, F satisfies all the assumptions of Lemma 2. Hence, (1) with (2) is chaotic in the sense of both Li-Yorke and Devaney. The entire proof is complete.

Theorem 2 (ε = 1). *Suppose that f satisfies the assumption (i) of Theorem 1, and there exist constants a, b, c, d with $-r < a < 0 < b < c < d < r$, such that*

$$f(a) \cdot f(b) < 0, f(c) \cdot f(d) < 0, \ \ |f(j)| > (2 + L)\mu.$$

Then there exists a Cantor set $\Lambda_2 \subset [a, b]^N \cup [c, d]^N$, such that (1) is chaotic in the sense of Li-Yorke.

Proof. The proof is similar to that of Theorem 1. The differences are as follows.

Case I: $f(a) < -(2 + L)\mu, f(b) > (2 + L)\mu, f(c) < -(2 + L)\mu, f(d) > (2 + L)\mu$;

For each $x \in V_1$ with $x(i-1) = a, i = 1,2, \cdots, N$, it follows from (2) and (5) that

$$F_i(x) = 0.5f(a) + 0.5f\big(x(i+1)\big) \leq 0.5f(a) + 0.5Lmax\{|a|, b\}$$

$$< -0.5(2 + L)\mu + 0.5L\mu = -\mu \leq a; \tag{17}$$

and for each $x \in V_1$ with $x(i-1) = b, i = 1,2, \cdots, N$ one has

$$F_i(x) \geq 0.5f(b) - 0.5Lmax\{|a|, b\} > 0.5(2 + L)\mu - 0.5L\mu = \mu \geq d. \tag{18}$$

For each $x \in V_2$ with $x(i-1) = c, i = 1,2, \cdots, N$, by (2) and (5),

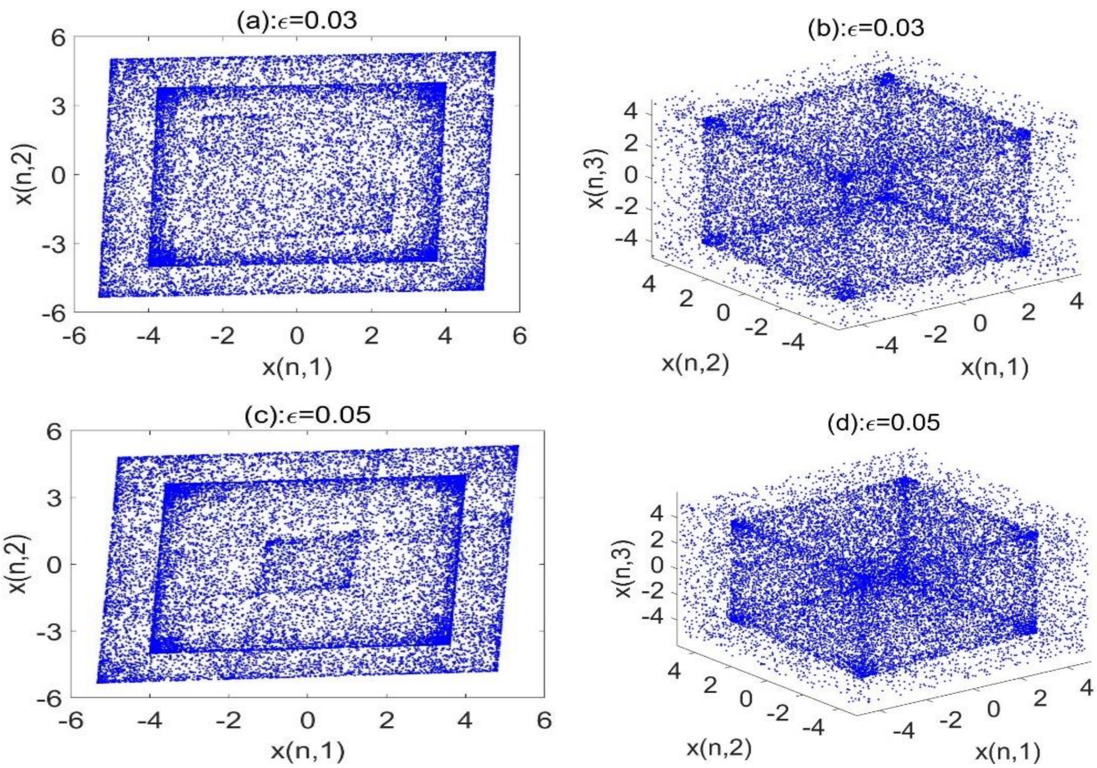

Fig. 1 2-D and 3-D computer simulations for system (1) with (2), where $n = 0, 1, \cdots, 20000$. In the 2-D graphs, $N = 2$, the initial value is taken as $x(1) = 0.1$ and $x(2) = 0.2$. In the 3-D graphs, $N = 3$, the initial value is $x(1) = 0.1$, $x(2) = 0.2$, $x(3) = 1$.

$$F_i(x) \leq 0.5f(c) + 0.5Ld < -0.5(2 + L)\mu + 0.5L\mu = -\mu \leq a; \tag{19}$$

and for each $x \in V_2$ with $x(i-1) = d, i = 1,2,\cdots,N,$

$$F_i(x) \geq 0.5f(d) - 0.5Ld > 0.5(2 + L)\mu - 0.5L\mu = \mu \geq d. \tag{20}$$

By the intermediate value theorem and (17) - (20), one has $F(V_i) \supset V_1 \cup V_2$ for $i = 1, 2$.

For the other three cases, we can similarly prove that $F(V_i) \supset V_1 \cup V_2$ for $i = 1, 2$.

In summary, F satisfies all the assumptions of Lemma 1. So, (1) is Li-Yorke chaos.

Remark 1. *In the cases of $\varepsilon = 1$, we can't be sure that (1) is chaotic in the sense of Devaney or not.*

4. Examples

In this section, two examples are discussed with computer simulations.

Example 4.1. *Consider the NNCML (1) with (2), where*

$$f(x) = \begin{cases} -\frac{1}{3}x^3 + 4x, & -4 \leq x \leq 4 \\ 4sinx, & otherwise. \end{cases}$$

Obviously, $f(0) = 0, f(-1.2) \cdot f(1.8) < 0, f(2.1) \cdot f(3.9) < 0$. Therefore, f satisfies assumptions (i) and (ii) of Theorem 1 with $r = 4, a = -1.2, b = 1.8, c = 2.1, d = 3.9$, and $L = 12$. Thus, by Theorem 1, for any constants $\varepsilon < 0.053$, there exists a Cantor set $\Lambda \subset [-1.2, 1.8]^N \cup [2.1, 3.9]^N$ such that (1) is chaotic on Λ in the sense of Li-Yorke.

For computer simulation, we take $N = 2, 3$ and $\varepsilon = 0.03, 0.05$, respectively. The simulation results in the two-dimensional *space $(x(\cdot,1), x(\cdot,2))$, and three-dimensional space $(x(\cdot,1), x(\cdot,2), x(\cdot,3))$ are shown in Fig.1, which indicates that (1) has very complicated dynamical behaviors on* Λ.

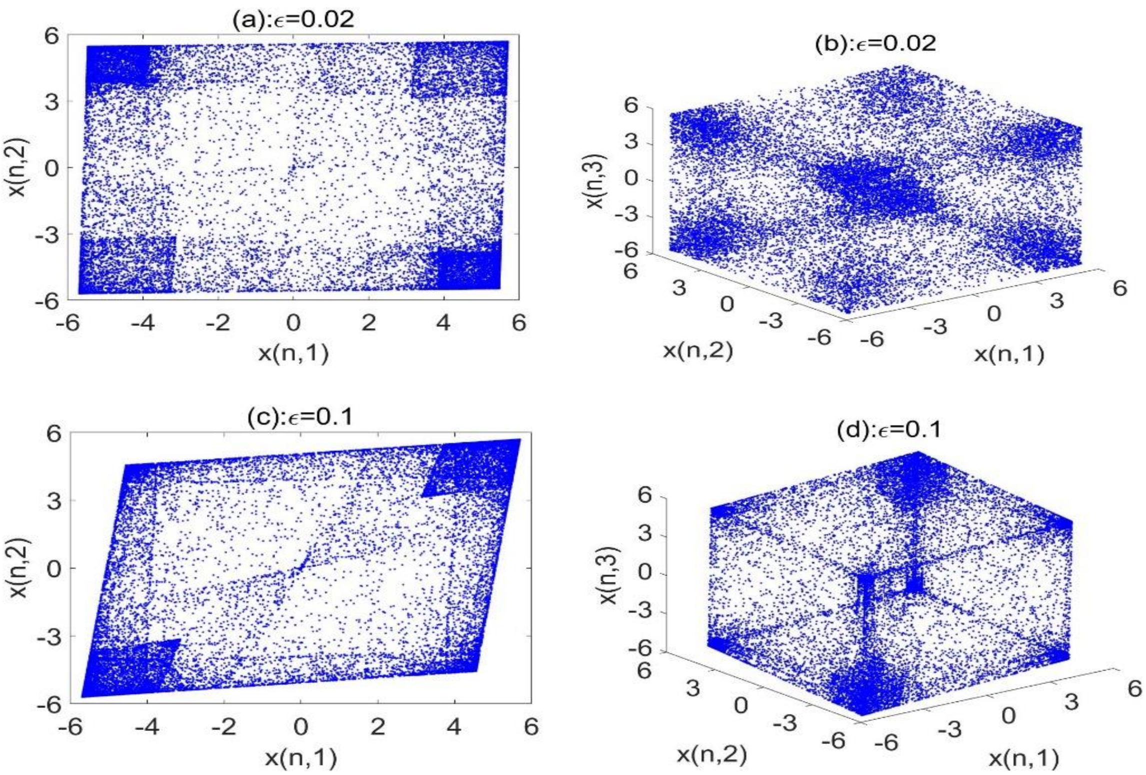

Fig. 2 Simulations for (1) with (2), where e $\mathbf{n = 0, 1, \cdots, 20000}$. In the 2-D graphs, $\mathbf{N = 2}$, the initial value is taken as $\mathbf{x(1) = 0.1}$ and $\mathbf{x(2) = 0.2}$. In the 3-D graphs, $\mathbf{N = 3}$, the initial value is $\mathbf{x(1) = 0.1}$, $\mathbf{x(2) = 0.2}$, $\mathbf{x(3) = 1}$.

Example 4.2. *Consider the NNCML (1) with (2), where*

$$f(x) = 5.7 \sin x.$$

Obviously, $f(0) = 0, f\left(-\frac{\pi}{4}\right) \cdot f\left(\frac{\pi}{4}\right) < 0, f\left(\frac{3\pi}{4}\right) \cdot f\left(\frac{5\pi}{4}\right) < 0$. *Therefore, f \satisfy all the assumptions of Theorem 1 with* $r = 3\pi, a = -\frac{\pi}{4},\ b = \frac{\pi}{4},\ c = \frac{3\pi}{4}, d = \frac{5\pi}{4},\ \lambda\ = 4.03,$ *and* $L = 5.7$. *Thus, by Theorem 1, for any constant* $\varepsilon \leq 0.119$, *there exists a Cantor set* $\Lambda \subset \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]^N \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]^N$ *such that (1) is chaotic on* Λ *in the sense of both Li-Yorke and Devaney.*

For computer simulation, we take $N = 2, 3$ *and* $\varepsilon = 0.02, 0.1$, *respectively. The simulation results are shown in Fig.2.*

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