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Original Article

Z-regular Spaces in Topological Spaces

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Abstract - *The aim of this paper is to introduce and study a new class of spaces, namely Z-regular spaces by using Z-open sets. The relationships among regular, strongly rg-regular, almost regular, softly regular, weakly regular, α-regular and ξ-regular spaces are investigated. Also we obtain some characterizations of Z-regular spaces, properties of the forms of gZ-closed, Zg-closed functions and preservation theorems for Z-regular spaces.*

Keywords *- Z-open sets, Z-regular, s-regular, almost regular and softly regular spaces, gZ-closed and Z-gZ-closed functions.*

1. Introduction

O. Njastad [7] introduced and studied the notion of α-open sets. M. K. Singal and S. P. Arya [9] introduced two new classes of regular spaces, namely almost regular and weakly regular. S. S. Benchalli [1] introduced and studied the notion of α-regular spaces. M. C. Sharma, P. Sharma and M. Singh [8] introduced a new class of regular spaces called ξ-regular spaces. H. Kumar [6] obtained some more characterizations and preservation theorems for ξ-regular spaces. H. Kumar and M. C. Sharma [5] introduced two new classes of separation axioms, namely softly regular and partly regular spaces which are weaker than regular spaces.

In this paper, we utilize Z-open sets to define and study a new class of spaces, called z-regular spaces in topology. The relationships among regular, strongly rg-regular, almost regular, softly regular, weakly regular, α-regular and ξ-regular spaces are investigated. Also we obtain some characterizations and preservation theorems for Z-regular spaces.

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) always mean topological spaces X,Y and Z on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A of a space X is called δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed.

2.1 Definition. A subset A of a space X is said to be:
(1) α-open [7] if $A \subset \text{int}(\text{cl}(\text{int}(A)))$.
(2) Z-open [2] if $A \subseteq cl(int_\delta(A)) \cup int(c(A))$.

2.2 Remark. We have the following implications for the properties of subsets:

open → α-open → Z-open

Where none of the implications is reversible as can be seen from [2s].

The complement of a α-open (resp. Z-open) set is called **α-closed** (resp. Z**-closed**).

The intersection of all Z-closed sets containing A, is called the Z-closure of A and is denoted by Z-cl(A). Dually, the Z-interior of A, denoted by Z-int(A) is defined to be the union of all Z-open sets contained in A.

The family of all Z-open (resp. Z-closed) sets of a space X is denoted by Z-O(X) (resp. Z-C(X)).

2.3 Definition. A subset A of a space X is said to be
(1) Generalized Z-closed [11]
(2) Z-generalized closed [11] (briefly Zg-closed) if Z-cl(A) $\subset$ U whenever A $\subset$ U and U $\in$ Z-O(X).
The complement of gZ-closed (resp. Zg-closed) set is said to be gZ-open (resp. Zg-open).

2.4 Remark. We have the following implications for the properties of subsets:

closed $\rightarrow$ Z-closed $\rightarrow$ gZ-closed $\rightarrow$ Zg-closed

Where none of the implications is reversible as can be seen from [**11**]:

2.5 Lemma [2]. Let A be a subset of a space X and x $\in$ X. The following properties hold for z-cl(A) :
(i) x $\in$ Z-c1(A) if and only if A $\cap$ U $\neq \phi$ for every U $\in$ Z-O(X) containing x.
(ii) A is Z-closed if and only if A = Z-cl(A) .
(iii) Z-c1(A) $\subset$ Z-c1(B) if A $\subset$ B.
(iv) Z-c1(Z-c1(A)) = Z-c1(A).
(v) Z-c1(A) is Z-closed.

2.6 Lemma [11]. A subset A of a space X is gZ-open in X if and only if F $\subset$ Z-int(A) whenever F $\subset$ A and F is closed in X.

3. Z-regular spaces

3.1 Definition. A space X is said to be **Z-regular** (resp. **α-regular** [**1**], **ξ-regular [6, 8]**) if for each closed set F of X, and each point x $\in$ X − F, there exist disjoint Z-open (resp. α-open, ξ-open) set U, V such that F $\subset$ U and x $\in$ V.

3.2 Definition. A space X is said to be **softly regular** [**5**] (resp. **almost regular** [**9**], **strongly rg-regular** [**4**]) if for every π-closed (resp. regular closed, rg-closed) set F of X, and a point x$\in$X −F, there exist disjoint open sets U and V such that F $\subset$ U and x $\in$ V.

3.3 Definition. A space X is said to be **weakly regular** [**9**] if for every point x and every regularly open set U containing x, there is an open set V such that x $\in$ V $\subset$ cl(V) $\subset$ U.

3.4 Theorem. Every regular space is Z-regular.

Proof.

By the definitions stated above, we have the following diagram:

$$\begin{array}{ccccccc} & & & & \xi\text{-regular} & & \\ & & & & \Uparrow & & \\ \text{Strongly rg-regular} & \Rightarrow & \text{regular} & \Rightarrow & \alpha\text{-regular} & \Rightarrow & \text{Z-regular} \\ & & \Downarrow & & & & \\ & & \text{Softly regular} & \Rightarrow & \text{almost regular} & \Rightarrow & \text{weakly regular} \end{array}$$

Where none of the implications is reversible as can be seen from the following examples:

3.5 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b}, {a, b}, X}. Then the space X is weakly regular. But it is neither almost regular nor softly regular.

3.6 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {a, b}, {a, b, c}, {a, b, d}, X}. Then the space X is almost regular but not strongly rg-regular.

3.7 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b}, {a, b}, {a, c}, X}. Then the space X is regular.

3.8 Example. Let $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{b, c\}, X\}$. Then the space X is regular but not strongly rg-regular. Since F = {b} is a rg-closed set such that $c \notin \{b\}$. We cannot separate c and {b} by disjoint open sets.

3.9 Theorem. The following properties are equivalent for a space X:
(a) X is Z-regular.
(b) For each $x \in X$ and each open set U of X containing x, there exists $V \in$ Z-O(X) such that $x \in V \subset$ Z-cl(V) $\subset U$.
(c) For each closed set F of X, $\cap$ {Z-cl(V) : $F \subset V \in$ Z-O(X)} = F.
(d) For each subset A of X and each open set U of X such that $A \cap U \neq \phi$, there exists $V \in$ Z- O(X) such that $A \cap V \neq \phi$ and Z-cl (V) $\subset U$.
(e) For each non empty subset A of X and each closed subset F of X such that $A \cap F = \phi$, there exist V, W $\in$ Z-O(X) such that $A \cap V \neq \phi$, $F \subset W$ and $V \cap W \neq \phi$.

Proof.
(a) $\Rightarrow$ (b). Let U be an open set containing x, then X – U is closed in X and $x \notin X - U$. By (a), there exist W, V $\in$ Z-O(X) such that $x \in V$, $X - U \subset W$ and $V \cap W = \phi$ .By Lemma 2.5, we have Z-cl(V) $\cap W = \phi$ and hence $x \in V \subset$ -Z-cl(V) $\subset U$.

(b) $\Rightarrow$ (c). Let F be a closed set of X. If $F \subset V$, then by Lemma 2.5 (iii), Z-cl(F) $\subset$ Z-cl(V) which gives $F \subset$ Z-cl(V) as $F \subset$ Z-cl(F). Therefore, $\cap$ {Z-cl(V) : $F \subset V \in$ Z-O(X)} $\supset$ F.

Conversely, let $x \notin F$. Then X – F is an open set containing x. By (b), there exists $U \in$ Z-O(X) such that $x \in U \subset$ Z-cl(U) $\subset X - F$. Put V = X – Z-cl(U). By Lemma 2.5, $F \subset V \in$ Z-O(X) and $x \notin$ Z-cl(V). This implies that $\cap$ {Z-cl(V) : $F \subset V \in$ Z-O(X)} $\subset$ F.
Hence $\cap$ {Z-cl(V) : $F \subset V \in$ Z-O(X)} = F.

(c) $\Rightarrow$ (d). Let A be a subset of X and let U be open in X such that $A \cap U \neq \phi$. Let $x \in A \cap U$, then X – U is a closed set not containing x. By (c), there exists $W \in$ Z-O(X) such that $X - U \subset W$ and $x \notin$ Z-cl(W). Put V = X – Z-cl(W). Then $V \subset X - W$. Also $x \in V \cap A$. By using Lemma 2.5, we obtain $V \in$ Z-O(X), and Z-cl(V) $\subset$ Z-cl(X – W) = X – W $\subset$ U.

(d) $\Rightarrow$ (e). Let A be a subset of X and let F be a closed set in X such that $A \cap F = \phi$, where$A \neq \phi$. Since X – F is open in X and $A \neq \phi$, by (d), there exists $V \in$ Z-O(X) such that $A \cap V \neq \phi$ and Z-c1(V) $\subset$ X – F. Put
W = X – Z-c1(V), then $F \subset W$. Also, $V \cap W = \phi$. By Lemma 2.5, $W \in$ Z-O(X).

(e) $\Rightarrow$ (a). This is obvious.

3.10 Theorem. A topological space X is Z-regular if and only if for each closed set F of X and each $x \in X - F$, there exist Z-open sets U and V of X such that $x \in U$ and $F \subset V$ and Z-cl(U) $\cap$ Z-cl(V) = ϕ.

Proof: Let F be a closed set in X and $x \notin F$. Then there exist Z-open sets U_x and V such that $x \in U_x$, $F \subset V$ and $U_x \cap V = \phi$. This Implies that $U_x \cap$ Z-cl(V) = ϕ. Since Z-cl(V) is Z-closed and $x \notin$ Z-cl(V). Since X is Z-regular, there exist Z-open sets G and H of X such that $x \in G$, Z-cl(V) $\subset$ H and $G \cap H = \phi$. This implies Z-cl(G) $\cap H = \phi$. Take $U = U_x \cap G$. Then U and V are Z-open sets of X such that $x \in U$ and $F \subset V$ and Z-cl(U) $\cap$ Z-cl(V) = ϕ, since Z-cl(U) $\cap$ Z-cl(V) $\subset$ Z-cl(G) $\cap$ H = ϕ. Conversely, suppose for each closed set F of X and each $x \in X - F$, there exist Z-open sets U and V of X such that $x \in U$, $F \subset V$ and and Z-cl(U) $\cap$ Z-cl(V) = ϕ. Now $U \cap V \subset$ Z-cl(U) $\cap$ Z-cl(V) = ϕ. Therefore $U \cap V = \phi$. Thus X is Z-regular.

3.11 Definition. A space X is said to be Z-T_3 space if it is Z-regular as well as Z-T_1 space.

3.12 Theorem. Every Z-T_3 space is a Z-T_2 space.

Proof. Let X be Z-T_3 , so it is both Z-T_1 and Z-regular. Also X is Z-$T_1 \Rightarrow$ every singleton subset {x} of X is an Z-closed. Let {x} be an Z-closed subset of X and $y \in X - \{x\}$. Then we have $x \neq y$ since X is Z-regular, there exist disjoint Z-open sets U and V such that $\{x\} \subset U$, $y \in V$, and such that $U \cap V = \phi$ (or) U and V are disjoint Z-open sets containing x and y respectively. Since x and y are arbitrary, for every pair of distinct points, there exist disjoint Z-open sets. Hence X is Z-T_2 space.

3.13 Theorem. Every subspace of a Z-regular space is Z-regular.

Proof. Let X be a Z-regular space. Let Y be a subspace of X. Let $x \in Y$ and F be a closed set in Y such that $x \notin F$. Then there is a closed set A of X with $F = Y \cap A$ and $x \notin A$. Since X is Z-regular, there exist disjoint Z-open sets U and V such that $x \in U$ and $A \subset V$. Note that $Y \cap U$ and $Y \cap V$ are Z-open sets in Y. Also $x \in U$ and $x \in Y$, which implies $x \in Y \cap U$ and $A \subset V$ implies $Y \cap U \subset Y \cap V$, $F \subset Y \cap V$. Also, $(Y \cap U) \cap (Y \cap V) = \phi$. Hence Y is Z-regular space.
3.14 Theorem. Every Z-compact Hausdorff space is an Z-T_3 space and hence an Z-regular space.

Proof. Let X be a compact Hausdorff space, that is an Z-T_3 space. But every Z-T_2 space is Z-T_1. To prove that it is Z-T_3 space, it is sufficient to prove that it is Z-regular. Let F be a closed subset of X, and $x \notin F$. Now $x \in X - F$ so that any point $y \in F$ is a point of X which is different from x. Since X is an Z-T_2 space corresponding to x and y, there exists two Z-open sets H_y and G_y such that $G_y \cap H_y = \phi$ where $x \in H_y$ and $y \in G_y$. Now let $\Im^*$ denote the relative topology for F so that the collection $C^* = \{F \cap H_y : y \in F\}$ is an ii-$\Im^*$ open cover of F. But F is closed and since X is Z-compact $(F, \Im^*)$ is also Z-compact. Hence a finite subcover of F (or) there exist points $y_1, y_2, \ldots y_n$ in F such that $C^* = \{F \cap H_{yi} : i = 1, 2, \ldots n\}$ is a finite sub cover for F. Now $\mathbf{F} = \cup \{F \cap H_{yi} : i = 1, 2, \ldots n\}$ or $F = F \cap \{\cup \{ H_{yi} : i = 1, 2, \ldots n\}\}$
Hence $F \subset \cup \{ H_{yi} : i = 1, 2, \ldots n\}$ or $F \subset H$ where $H = \cup \{H_{yi} : i = 1, 2, \ldots n\}$ is Z-open set containing H, being the union of Z-open sets. Again G_{yi} for $i = 1, 2, 3, \ldots n$ is Z-open set containing x and hence $G = \cap \{G_{yi} : i = 1, 2, \ldots n\}$ is also an Z-open set containing x.
Also $G \cap H = \phi$, otherwise $G_{yi} \cap H_{yi} \neq \phi$ for some i. Hence corresponding to each closed set F and an element x in $X - F$ we have two Z-open sets G and H such that $x \in G$, $F \subset H$ and $G \cap H = \phi$. Hence X is Z-regular. Since it is Z-T_2 so Z-T_1 and hence X isZ-T_3.

4. Some related functions with Z-regular spaces

4.1 Definition. A function $f : X \to Y$ is said to be Z-closed [3] if for each closed set F of X, f(F) is Z-closed in Y**.**
4.2 Definition. A function $f : X \to Y$ is said to be
(i) generalized Z-closed [11] (briefly gZ-closed) if for each closed set F of X, f (F) is gZ-closed in Y.
(ii) Z-generalized Z-closed [11] (briefly Z-gZ-closed) if for each ii-closed set F of X, f (F) is gZ-closed in Y.

4.3 Remark. Every closed function is Z-closed but not conversely. Also, every Z-closed function is gZ-closed because every Z-cosed set is gZ-closed. It is obvious that both Z-closedness and Z-gZ-closedness imply gZ-closedness.

4.4 Theorem. A surjective function $f : X \to Y$ is gZ-closed (resp. Z-gZ-closed) if and only if for each subset B of Y and each open (resp. Z-open) set U of X containing $f^{-1}(B)$, there exists a gZ-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof. Suppose that f is gZ-closed (resp. Z-gZ-closed). Let B be any subset of Y and U be open (resp. Z-open) set of X containing $f^{-1}(B)$. Put $V = Y - f(X - U)$. Then the complement V^c of V is $V^c = Y - V = f(X - U)$. Since $X - U$ is closed in X and f is gZ-closed, $f(X - U) = V^c$ is gZ-closed. Therefore, V is gZ-open in Y. It is easy to see that $B \subset V$ and $f^{-1}(V) \subset U$.

Conversely, let F be a closed (resp. Z-closed) set of X. Put $B = Y - f(F)$, then we have $f^{-1}(B) \subset X - F$ and $X - F$ is open (resp. Z-open) in X. Then by assumption, there exists a gZ-open set V of Y such that $B = Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Now $f^{-1}(V) \subset X - F$ implies $V \subset Y - f(F) = B$. Also $B \subset V$ and so $B = V$. Therefore, we obtain $f(F) = Y - V$ and hence f(F) is gZ-closed in Y. This shows that f is gZ-closed (resp. Z-gZ-closed).

4.5 Remark. We can prove the necessity part of the above theorem by replacing each set to closed set in the form of the proposition given below:

4.6 Proposition. If a surjective function $f : X \to Y$ is gZ-closed (resp. Z-gZ-closed) then for a closed set F of Y and for any open (resp. Z-open) set U of X containing $f^{-1}(F)$, there exists an Z-open set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$.

Proof. By **Theorem 4.4**, there exists a gZ-open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset U$. Since F is closed, by Lemma 2.6 we have $F \subset$ Z-int(W). Put V = Z-int(W). Then $V \in$ Z-O(Y), F V and $f^{-1}(V) \subset U$.

4.7 Proposition. If $f : X \to Y$ is continuous Z-gZ-closed and A is gZ-closed in X, then f(A) is gZ-closed in Y .

Proof. Let V be a open set of Y containing f(A). Then $A \subset f^{-1}(V)$. Since f is continuous, $f^{-1}(V)$ is open in X. Since A is gz-closed in X, by a definition, we get $z\text{-c1}(A) \subset f^{-1}(V)$ and hence $f(z\text{-c1}(A)) \subset V$. Since f is Z-gZ-closed and Z-c1(A) is Z-closed in X, f(Z-c1(A)) is gZ-closed in Y sand hence we have $Z\text{-c1}(f(Z\text{-c1}(A))) \subset V$. By definition of the Z-closure of a set, $A \subset Z\text{-c1}(A)$ which implies $f(A) \subset f(Z\text{-c1}(A))$ and using Lemma 2.5, $Z\text{-c1}(f(A)) \subset Z\text{-c1}(f(Z\text{-c1}(A))) \subset U$. That is $Z\text{-c1}(f(A)) \subset U$. This shows that f (A) is gZ-closed in Y .

4.8 Definition. A function $f : X \to Y$ is said to be Z-irresolute [3] if for each $V \in Z\text{-O}(Y)$, $f^{-1}(V) \in Z\text{-O}(X)$.

4.9 Proposition. If $f : X \to Y$ is an open Z-irresolute bijection and B is gZ-closed in Y, then $f^{-1}(B)$ is gZ-closed in X .
Proof. Let U be a open set of X containing $f^{-1}(B)$. Then $B \subset f(U)$ and f(U) is open in Y. Since B is gZ-closed in Y, $Z\text{-c1}(B) \subset f(U)$ and hence we have $f^{-1}(Z\text{-c1}(B)) \subset U$. Since f is Z-irresolute,
$f^{-1}(Z\text{-c1}(B))$ is Z-closed in X (Theorem 2.5 (i) and (v)),we have $Z\text{-c1}(f^{-1}(B)) \subset f^{-1}(Z\text{-c1}(B) \subset U$. This shows that $f^{-1}(B)$ is gZ-closed in X.

4.10 Theorem. Let $f : X \to Y$ and $h : Y \to Z$ be the two functions, then
(i) If $hof : X \to Z$ is gZ-closed and if $f : X \to Y$ is a continuous surjection, then $h : X \to Z$ is gZ-closed .
(ii) If $f : X \to Y$ is gZ-closed with $h : Y \to Z$ is continuous and Z-gZ-closed, then $hof : X \to Z$ is gZ-closed.
(iii) If $f : X \to Y$ is closed and $h : Y \to Z$ is gZ-closed, then $hof : X \to Z$ is gZ-closed.

Proof.
(i) Let F be a closed set of Y. Then $f^{-1}(F)$ is closed in X since f is continuous. By hypothesis $(hof)(f^{-1}(F))$ is gZ-closed in Z. Hence h is gZ-closed.
(ii)The proof follows from the Proposition 4.7.
(iii)The proof is obvious from definitions.

4.11 Theorem. The following properties are equivalent for a space X :
(a) X is Z-regular.
(b) For each closed set F and each point $x \in X - F$, there exists $U \in Z\text{-O}(X)$ and a gZ-open set V such that $x \in U$ and $F \subset V$ and $U \cap V = \phi$.
(c) For each subset A of X and each closed set F such that $A \cap F = \phi$, there exist $U \in Z\text{-O}(X)$ and a gZ-open set V such that $A \cap U \neq \phi$, $F \subset V$ and $U \cap V = \phi$.
(d) For each closed set F of X, $F = \cap \{Z\text{-c1}(V) : F \subset V \text{ and } V \text{ is gZ-open}\}$.

Proof.
(a) $\Rightarrow$ (b). The proof is obvious since every Z-open set is gZ-open.

(b) $\Rightarrow$ (c). Let A be a subset of X and let F be a closed set in X such that $A \cap F = \phi$. For a point $x \in A$, $x \in X - F$ and hence there exists $U \in Z\text{-O}(X)$ and a gZ-open set V such that $x \in U$ and $F \subset V$ and $U \cap V = \phi$. Also $x \in A$, $x \in U$ implies $x \in A \cap U$. So $A \cap U \neq \phi$.
(c) $\Rightarrow$ (a). Let F be a closed set and let $x \in X - F$. Then, $\{x\} \cap F = \phi$ and there exist $U \in Z\text{-O}(X)$ and a gZ-open set W such that $x \in U$, $F \subset W$ and $U \cap W = \phi$. Put V = Z-int(W), then by Lemma 2.6, we have $F \subset V$, $V \in Z\text{-O}(X)$ and $U \cap V = \phi$. Therefore X is Z-regular.

(a) $\Rightarrow$ (d). For a closed set F of X, by Theorem 3.9, we obtain
$$F \subset \cap \{Z\text{-c1}(V) : F \subset V \text{ and } V \text{ is gZ-open}\}$$
$$\subset \cap \{Z\text{-c1}(V) : F \subset V \text{ and } V \in Z\text{-O}(X)\} = F$$
Therefore, $F = \cap \{Z\text{-c1}(V) : F \subset V \text{ and } V \text{ is gZ-open}\}$.

(d) $\Rightarrow$ (a). Let F be a closed set of X and $x \in X - F$. by (d), there exists a gZ-open set W of X such that $F \subset W$ and $x \in X - Z\text{-c1}(W)$. Since F is closed, $F \subset Z\text{-int}(W)$ by Lemma 2.6. Put V = Z-int(W), then $F \subset V$ and $V \in Z\text{-O}(X)$. Since $x \in X - Z\text{-c1}(W)$, $x \in X - Z\text{-c1}(V)$. Put $U = X - Z\text{-c1}(V)$ then, $x \in U$, $U \in Z\text{-O}(X)$ and $U \cap V = \phi$. This shows that X is Z-regular.

4.12 Definition. A function $f : X \to Y$ is said to be Z-open [3] if for each open set U of X, $f(U) \in Z\text{-O}(Y)$.

4.13 Theorem. If $f : X \rightarrow Y$ is a continuous Z-open gZ-closed surjection and X is regular, then Y is Z-regular.

Proof. Let $y \in Y$ and let V be an open set of Y containing y. Let x be a point of X such that $y = f(x)$. By the regularity of X, there exists an open set U of X such that $x \in U \subset c1(U) \subset f^{-1}(V)$. We have $y \in f(U) \subset f(c1(U)) \subset V$. since f is Z-open and gZ-closed, $f(U) \in$ Z-O(Y) and f(c1(U)) is gZ-closed in Y. So, we obtain, $y \in f(U) \subset$ Z-c1(f(U)) $\subset$ Z-cl(f(c1(U))) $\subset V$. It follows from Theorem 4.11 that Y is Z-regular.

4.14 Definition. A function $f : X \rightarrow Y$ is said to be pre Z-open [3] if for each Z-open set U of X, $f(U) \in$ Z-O(Y).

4.15 Theorem. If $f : X \rightarrow Y$ is a continuous pre Z-open Z-gZ-closed surjection and X is Z-regular, then Y is Z-regular.

Proof. Let F be any closed set of Y and $y \in Y - F$. Then $f^{-1}(Y) \cap f^{-1}(F) = \phi$ and $f^{-1}(F)$ is closed in X. Since X is Z-regular, for a point $x \in f^{-1}(y)$, there exist U, V $\in$ Z-O(X) such that $x \in U$, $f^{-1}(F) \subset V$ and $U \cap V = \phi$. Since F is closed in Y, by Proposition 4.6, there exists W $\in$ Z-O(Y) such that $F \subset W$ and $f^{-1}(W) \subset V$. Since f pre Z-open, we have $y = f(x) \in f(U)$ and $f(U) \in$ Z-O(Y). Since $U \cap V = \phi$, $f^{-1}(W) \cap U = \phi$ and hence $W \cap f(U) = \phi$. This shows that Y is Z-regular.

5. Conclusion

In this paper, we introduce and study a new class of spaces, namely Z-regular spaces by using Z-open sets. The relationships among regular, strongly rg-regular, almost regular, softly regular, weakly regular, α-regular, ξ-regular and Z-regular spaces are investigated. Also we obtained some characterizations of Z-regular spaces, properties of the forms of gZ-closed, Zg-closed functions and preservation theorems for Z-regular spaces. Of course, the entire content will be a successful tool for the researchers for finding the way to obtain the results in the context of such types of regular spaces.

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