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Original Article

Torsion of a Unit Curve with Constant Angle to its Binormal

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Abstract - *In this article, we characterized all unit regular curve with constant angle to its binormal in terms of its torsion. Furthermore, we proved that if α is a unit regular curve defined on open interval with constant angle to its Binormal vector, then its torsion is equal to 0. Consequently, we give a condition when the tangent vector T of α lie in a plane.*

Keywords - *Torsion, Curve, Binormal, Constant Angle, Curvature.*

1. Introduction

There are many studies and results about the torsion and curvature of a curve. In 2008, Thanh Phuong Nguyen and Isabelle Debled-Rennesson proposed a new torsion estimator for spatial curves [1]. Moreover, Lewiner et al. (2005) proposed a new scheme for estimating curvature and torsion of planar and spatial curves based on weighted least-squares fitting and local arc-length approximation [2]. There are also many applications of torsion and curvature in different fields. In physics, torsion and curvature are used in teleparallel gravity [3], axial superfield [4], extended supergravity [5], Einstein-Cartan Gravity [6], and Gauge theory [7]. In engineering, Gammack and Hydon studied the flow in pipes with non-uniform curvature and torsion [8], while Jonas Bolinder published an article regarding the first and higher-order effects of curvature and torsion on the flow in a helical rectangular duct [9]. In 2006, Robert Betchov found that helicoidal vortex filaments are elementary solutions and are unstable with the help of torsion and curvature [10]. Torsion and curvature also helped Maeck and De Roeck to predict the damage location that helped the structural integrity of the construction [11]. In astronomy, many authors used torsion and curvature in their studies. In 2020, Saridakis et al. investigate the application of $F(R,T)$ gravity with dynamical curvature and torsion in cosmology [12]. In 2019, as online transactions become necessary due to the pandemic, Lang He, Hua Ta, and Zhang-Can Huang published a paper about the efficient online signature verification based on the association of curvature and torsion feature with Hausdorff distance [13]. In 2012, the researchers in the paper entitled "Estimate Algorithms and Embedded Crafts" proposed some methods to estimate the curvature and torsion [14].

From the above, various studies regarding torsion and curvature highlighted its uses and applications in different fields of study outside mathematics. However, it is not only expanding outside of mathematics but in various areas of mathematics too. In 2017, Laurian-loan Piscoran and Vishnu Narayan Mishra investigated the S-curvature, Landsberg curvature, mean Landsberg curvature, and the Cartan torsion and mean Cartan torsion for (α,β)-metric. They prove that (α,β)-metric has bounded Cartan torsion [15]. In this paper, we are interested in the torsion of the unit curve that has a constant angle to its Binormal vector.

2. Preliminaries

A curve α on interval I is a vector function $\alpha(t)=\big(x(t),y(t),z(t)\big)$ where x,y,z are differentiable for $t\in I$ [16,17,18]. A curve α on interval I is said to be regular if and only if its derivative never vanishes [16,17,18].

If α is regular, that is, $\alpha'(t)\neq 0$ for all $t\in I$ and

$$s(t)=\int_0^t \|\alpha'(\sigma)\|d\sigma,$$

then s is a strictly monotonically increasing function, and thus, it is possible to solve t as a function of s, denoted by $t(s)$. Therefore, we define $\alpha(s)=\alpha\big(t(s)\big)$ and called this as arc length parametrization of α. Note that $\frac{ds}{dt}=\|\alpha'(t)\|$, and thus,

$$\|\alpha'(s)\|=\left\|\frac{\frac{d\alpha}{dt}}{\frac{ds}{dt}}\right\|=\left\|\frac{\alpha'(t)}{\|\alpha'(t)\|}\right\|=1.$$

For $t \in I$, we define the tangent vector of α by

$$T = \alpha'(s) = \frac{\alpha'(t)}{\|\alpha'(t)\|}.$$

The normal vector N of α is defined by

$$N = \frac{\frac{dT}{ds}}{\left\|\frac{dt}{ds}\right\|}.$$

The binormal vector B of α is defined by

$$B = T \times N.$$

Note that $\{T, N, B\}$ is an orthonormal basis, and thus, every curve α can be expressed as a linear combination of T, N and B.

Theorem 2.1 Frenet-Serret Frame [16,17,18]

Let $\alpha(s)$ be the arc length parametrization of α such that α is regular. Then,

$$\frac{dT}{ds} = \kappa N$$
$$\frac{dN}{ds} = -\kappa T + \tau B$$
$$\frac{dB}{ds} = -\tau N$$

where $\kappa = \left\|\frac{dT}{ds}\right\|$ is a curvature and $\tau = N' \cdot B$ is the torsion of α.

We say that α is unit if and only if $\|\alpha(t)\| = 1$ for all $t \in I$.

Definition 2.1: Let $\alpha: I \to \mathbb{R}^3$ be a unit and regular curve. Then, α has a constant angle to its binormal vector if and only if there is $\theta \in [0, \pi]$ such that $\alpha(t) \cdot B(t) = \cos\theta \,\|\alpha(t)\|\|B(t)\| = \cos\theta$, for all $t \in I$.

3. Main Results

The following results is the main results of this paper. We proved that the torsion vanished for any unit regular curve define on open interval with constant angle to its Binormal.

Theorem 3.1: Let I be an open interval. The torsion of any unit regular curve $\alpha: I \to \mathbb{R}^3$ with constant angle θ to its binormal vector is zero.

Proof: Suppose $\alpha(s)$ is the arc length parametrization of α. First, express $\alpha(s)$ as a linear combination of N, T and B, that is,

$$\alpha(s) = \lambda_1(s)N(s) + \lambda_2(s)T(s) + \lambda_3(s)B(s) \quad (1)$$

where $\lambda_1, \lambda_2, \lambda_3$ are scalar function. Thus,

$$\lambda_3 = \alpha \cdot T = \cos\theta \,\|\alpha(s)\|. \quad (2)$$

Now, differentiate α in (1) with respect to s, then we get

$$T(s) = \lambda_1(s)N'(s) + \lambda_1'(s)N(s) + \lambda_2(s)T'(s) + \lambda_2'(s)T(s) + \cos\theta\, B'(s). \quad (3)$$

Solve $T \cdot T$ in (3) and applying Frenet-Serret Frame, then we have

$$1 = T \cdot T = -\kappa\lambda_1 + \lambda_2'. \quad (4)$$

Assume $\tau(s) \neq 0$ for some s. Computing $T \cdot B$ in (3) and applying Frenet-Serret Frame, we have

$$0 = T \cdot B = \lambda_1 \tau \quad (5)$$

Let $M = \{s: \tau(s) \neq 0\} \neq \phi$. Since $\tau \neq 0$ in M, then equation (5) implies that

$$\lambda_1(s) = 0 \ \ (s \in M) \quad (6)$$

Thus, equation (6) and (4) implies that $\lambda_2'(s) = 1$ for all $s \in M$, and therefore, for $s \in M$ we have

$$\lambda_2(s) = s + C. \quad (7)$$

for some constant C. Therefore,

$$\alpha(s) = (s + C)T + \cos\theta\, B, \qquad (s \in M). \tag{8}$$

Thus, computing $\|\alpha(s)\|^2$ in (8) we obtain

$$1 = \|\alpha(s)\|^2 = (s + C)^2 + \cos^2\theta \tag{9}$$

Hence, $(s + C)^2 = 1 - \cos^2\theta$ for all $s \in M$. Therefore, $s = -C \pm |\sin\theta|$, that is, $M \subseteq \{-C - |\sin\theta|, -C + |\sin\theta|\}$. Thus, $\tau(s)$ is not zero for at most two points, which is a contradiction since τ is differentiable on open interval I with derivative $\tau' = N' \cdot B' + B \cdot N''$. Therefore, $\tau = 0$ for all s, as desired.

Corollary 3.1.1: Let I be an open interval. Every regular unit curve $\alpha: I \to \mathbb{R}^3$ with constant angle to its binormal and nonzero curvature lie in a plane.
Proof: Follow from the fact that if α is a regular curve with zero torsion, then α lie in a plane.

Corollary 3.1.2: Let $\alpha: I \to \mathbb{R}^3$ where I is an open interval. Let $F = \frac{\frac{dN}{ds}}{\left\|\frac{dN}{ds}\right\|}$ and $D = N \times F$. If there is $\theta \in [0, \pi]$ such that $T(s) \cdot D(s) = \cos\theta$, for all s, then $T(t)$ lie in a plane for all $t \in I$.
Proof: Follow directly from corollary 3.1.1.

4. Conclusion

This study proved that the torsion of any unit regular curve α defined on open interval I with constant angle to its Binormal is 0, and thus, α lie in a plane. Furthermore, we proved that if $F = \frac{\frac{dN}{ds}}{\left\|\frac{dN}{ds}\right\|}$, $D = N \times F$ and there is $\theta \in [0, \pi]$ such that $T(s) \cdot D(s) = \cos\theta$, for all s, then $T(t)$ lie in a plane for all $t \in I$.

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