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Original Article

Symmetry Reduction and Exact Solutions of A (3+1)-Dimensional Kadomtsev-Petviashvi-li-Benjamin-Bona-Mahony Equation

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Abstract *- By applying a direct symmetry method, we get the symmetry group of the (3+1)-dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KPBBM) equation. Using the associated vector fields of the obtained symmetry, we get the optimal system of group-invariant solutions. To every case of the optimal system, we derive the reductions and some exact solutions of the (3+1)-dimensional KPBBM equation.*

Keywords *- Direct symmetry method, (3+1)-dimensional KPBBM equation, Optimal system, Exact solutions.*

1. Introduction

In recent years, with the development of science and technology, nonlinear science has gradually entered people's vision. At the same time, a large number of nonlinear partial differential equations have emerged. There are many important theories and research directions in the study of nonlinear partial differential equations, among which the study of the exact solution of nonlinear partial differential equations is the focus of attention.

Up to now, there are many methods for solving nonlinear partial differential equations. Commonly used are Darboux transformation[1], Variational iteration method[2], Hirota's method[3], Lie symmetric method[4], Finite difference method[5] and so on. Some methods can be said to be enduring and have been applied by scholars in the study of the exact solutions of partial differential equations. In this paper, we study the symmetry reductions and the exact solutions of (3+1)-dimensional KPBBM equation,

$$u_{xt}+k_1u_{xx}+k_2(uu_x)_x-k_3u_{xxxt}+k_4u_{yy}+k_5u_{zz}=0\,, \tag{1.1}$$

where $u=u(x,y,z,t)$, k_1, k_2, k_3, k_4 and k_5 are arbitrary constants. Eq.(1.1) describes the fluid flow in the case of an offshore structure. When $k_4=k_5=0$, Eq.(1.1) is reduced to Benjamin-Bona-Mahony equation[6]. When $k_1=1$, $k_5=0$, Eq.(1.1) is reduced to (2+1)-dimensional KPBBM equation[7]. Lump waves and breather waves, numerical simulation and soliton solutions for Eq.(1.1) have been presented in Refs.[8-10].

The outline of this paper is as follows. In section 2, we obtain the symmetry group of (3+1)-dimensional KPBBM equation by applying a direct symmetry method. In section 3, by using the equivalent vector of the symmetry, we get the optimal system of group-invariant solutions. Based on the optimal system, some reductions and exact solutions of (3+1)-dimensional KPBBM equation are obtained. In section 4, some conclusions and discussions are given.

2. Symmerty Group

It is well known that symmetry groups provide a useful method for obtaining solutions of partial differential equation [11,12]. A growing number of mathematicians and physicists have done outstanding work on symmetry and reduction[13-20].However, there are almost always an infinite number of the subgroups, we need an optimal system to classifying all possible group-invariant solutions to the system[11]. Based on the application of classical method we consider the one-parameter group of infinitesimal transformations in (x,y,z,t) of Eq.(1.1) given by

$$
\begin{aligned}
x^* &= x + \varepsilon\xi(x, y, z, t, u) + o(\varepsilon^2), \\
y^* &= y + \varepsilon\eta(x, y, z, t, u) + o(\varepsilon^2), \\
z^* &= z + \varepsilon\varsigma(x, y, z, t, u) + o(\varepsilon^2), \\
t^* &= t + \varepsilon\tau(x, y, z, t, u) + o(\varepsilon^2), \\
u^* &= u + \varepsilon\psi(x, y, z, t, u) + o(\varepsilon^2),
\end{aligned} \tag{2.1}
$$

where ε is a group parameter. It is required that the set of Eq.(1.1) be invariant under the transformation (2.1), and this yields a system of overdetermined, linear equations for the infinitesimals ξ, η, ς, τ and ψ. Solving these equations, one can have

$$
\begin{aligned}
\xi &= c_6, \\
\eta &= k_2c_1y + k_4c_2z + c_3, \\
\varsigma &= k_2c_1z - k_5c_2y + c_4, \\
\tau &= 2k_2c_1t + c_5, \\
\psi &= 2k_2c_1u + 2k_1c_1,
\end{aligned} \tag{2.2}
$$

where $c_i (i = 1, 2, ..., 6)$ are arbitrary constants. And the associated vector fields for the one-parameter Lie group of infinitesimal transformations are $v_1, v_2, \ldots, v_6$ given by

$$
\begin{aligned}
&v_1 = \partial_x, v_2 = \partial_y, v_3 = \partial_z, v_4 = \partial_t, v_5 = k_4z\partial_y - k_5y\partial_z, \\
&v_6 = k_2y\partial_y + k_2z\partial_z + 2k_2t\partial_t + (2k_2u + 2k_1)\partial_u.
\end{aligned} \tag{2.3}
$$

Eq.(2.3) shows that the following transformations (defined by $exp(\varepsilon v_i)$ $(i = 1, 2, ..., 6)$) of variables (x, y, z, t, u) leave the solutions of Eq.(1.1) invariant:

$$
\begin{aligned}
&exp(\varepsilon v_1): (x, y, z, t, u) \to \left(x + \varepsilon, y, z, t, u\right), \\
&exp(\varepsilon v_2): (x, y, z, t, u) \to \left(x, y + \varepsilon, z, t, u\right), \\
&exp(\varepsilon v_3): (x, y, z, t, u) \to \left(x, y, z + \varepsilon, t, u\right), \\
&exp(\varepsilon v_4): (x, y, z, t, u) \to \left(x, y, z, t + \varepsilon, u\right), \\
&exp(\varepsilon v_5): (x, y, z, t, u) \to (x, (\sqrt{k_4k_5}sin(\sqrt{k_4k_5}\varepsilon)z)/k_5 + cos(\sqrt{k_4k_5}\varepsilon)y, \\
&(-k_5sin(\sqrt{k_4k_5}\varepsilon)y)/\sqrt{k_4k_5} + cos(\sqrt{k_4k_5}\varepsilon)z, t, u), \\
&exp(\varepsilon v_6): (x, y, z, t, u) \to \left(x, ye^{k_2\varepsilon}, ze^{k_2\varepsilon}, te^{2k_2\varepsilon}, [e^{2k_2\varepsilon}(k_2u + k_1) - k_1]/k_2\right).
\end{aligned} \tag{2.4}
$$

Then the following theorem holds:

Theorem 1 If $\psi = p(x, y, z, t)$ is a solution of Eq.(1.1), then so are

$$
\begin{aligned}
\psi^{(1)} &= p(x - \varepsilon, y, z, t), \\
\psi^{(2)} &= p(x, y - \varepsilon, z, t), \\
\psi^{(3)} &= p(x, y, z - \varepsilon, t), \\
\psi^{(4)} &= p(x, y, z, t - \varepsilon), \\
\psi^{(4)} &= p(x, y, z, t - \varepsilon).
\end{aligned}
$$

3. Symmetry Reductions and Solution

In this section, we will discuss the symmetry reductions and solutions of the Eq.(1.1). In general, to each subgroup of the symmetry group, there will correspond a family of group- invariant solutions of the equation. It is too complicated to list all possible group-invariant solutions. By using the method presented in Refs.[13], we can find the optimal system of group-invariant solutions.

Applying the commutator operators $[v_m, v_n] = v_m v_n - v_n v_m$, one get the following table (the entry in row i and the column j representing $[v_i, v_j]$)

Table 1. Lie Bracket

Lie	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	0	0
v_2	0	0	0	0	$-k_5 v_4$	$k_2 v_2$
v_3	0	0	0	0	$k_4 v_2$	$k_2 v_3$
v_4	0	0	0	0	0	$2k_2 v_4$
v_5	0	$k_5 v_3$	$-k_4 v_2$	0	0	0
v_6	0	$-k_2 v_2$	$-k_2 v_3$	$-2k_2 v_4$	0	0

Therefore, there is

Proposition 1. The operators v_i $(i = 1, 2, ..., 6)$ form a Lie algebra, which is a six-dimensional symmetry algebra.

According to the Table 1, one can have the adjoint representation listed in Table 2 with the (i, j)-th entry indicating $Ad(exp(\varepsilon v_i))v_j$.

Table 2. Adjoint Representation

Ad	v_1	v_2	v_3
v_1	v_1	v_2	v_3
v_2	v_1	v_2	v_3
v_3	v_1	v_2	v_3
v_4	v_1	v_2	v_3
v_5	v_1	$cos(\sqrt{k_4 k_5}\varepsilon)v_2 - [k_2 sin(\sqrt{k_4 k_5}\varepsilon)v_3] / \sqrt{k_4 k_5}$	$[k_4 sin(\sqrt{k_4 k_5}\varepsilon)v_2] / \sqrt{k_4 k_5} + cos(\sqrt{k_4 k_5}\varepsilon)v_3$
v_6	v_1	$e^{k_2 \varepsilon} v_2$	$e^{k_2 \varepsilon} v_3$

Ad	v_4	v_5	v_6
v_1	v_4	v_5	v_6
v_2	v_4	$v_5 + k_5 \varepsilon v_3$	$v_6 - k_2 \varepsilon v_2$
v_3	v_4	$v_5 - k_4 \varepsilon v_2$	$v_6 - k_2 \varepsilon v_3$

v_4	v_4	v_5	$v_6-2k_2\varepsilon v_4$
v_5	v_4	v_5	v_6
v_6	$e^{2k_2\varepsilon}v_4$	v_5	v_6

If we set $v=a_1v_1+a_2v_2+a_3v_3+a_4v_4+a_5v_5+a_6v_6$, applying the formula

$$Ad(exp(\varepsilon v))v_0-\varepsilon[v,v_0]+\frac{1}{2!}\varepsilon^2[v,[v,v_0]]-\frac{1}{3!}\varepsilon^3[v,[v,[v,v_0]]]+\dots,$$

and Proposition 1, one can get the following theorem by detailed computation:

Theorem 2. The operators generate an optimal system H

$(a)v_6+\lambda v_1+\mu v_5, a_6\neq 0$;

$(b_1)v_5+v_4+\lambda v_1, a_6=0, a_5\neq 0$;

$(b_2)v_5-v_4+\lambda v_1, a_6=0, a_5\neq 0$;

$(b_3)v_5+\lambda v_1, a_6=0, a_5\neq 0$;

$(c_1)v_3+v_4+\lambda v_1, a_5=a_6=0, a_3\neq 0$;

$(c_2)v_3-v_4+\lambda v_1, a_5=a_6=0, a_3\neq 0$;

$(c_3)v_3+v_1, a_5=a_6=0, a_3\neq 0$;

$(c_4)v_3-v_1, a_5=a_6=0, a_3\neq 0$;

$(c_5)v_3, a_5=a_6=0, a_3\neq 0$;

$(d_1)v_4+v_2+\lambda v_1, a_3=a_5=a_6=0, a_4\neq 0$;

$(d_2)v_4-v_2+\lambda v_1, a_3=a_5=a_6=0, a_4\neq 0$;

$(d_3)v_4+v_1, a_3=a_5=a_6=0, a_4\neq 0$;

$(d_4)v_4-v_1, a_3=a_5=a_6=0, a_4\neq 0$;

$(d_5)v_4, a_3=a_5=a_6=0, a_4\neq 0$;

$(e_1)v_2+v_1, a_3=a_4=a_5=a_6=0, a_2\neq 0$;

$(e_2)v_2-v_1, a_3=a_4=a_5=a_6=0, a_2\neq 0$;

$(e_3)v_2, a_3=a_4=a_5=a_6=0, a_2\neq 0$;

$(f)v_1, a_2=a_3=a_4=a_5=a_6=0, a_1\neq 0$.

Solving such reduction equations, one obtains the solutions of the Eq.(1.1).

(I)Solving the reduced equation in case (c_1), one can get

$$k_2FF_{\xi\xi}+k_2F_{\xi}^2+\lambda k_3F_{\xi\xi\xi\xi}+k_4F_{\eta\eta}+k_5F_{\alpha\alpha}+k_3F_{\xi\xi\xi\alpha}+k_1F_{\xi\xi}-\lambda F_{\xi\xi}-F_{\xi\alpha}=0, \quad (3.1)$$

where $F = F(\xi,\eta,\alpha)$, and $\xi = -\lambda t + x$, $\eta = y$, $\alpha = -t + z$. Let's solving the Eq.(3.1), we'll get the solution

$$F(\xi,\eta,\alpha) = \pm\frac{1}{2}\sqrt{\frac{C_4}{e^{(\frac{C_3}{C_2})^2} e^{(\frac{\eta}{C_2})^2}}}(e^{(\frac{C_3}{C_2})^2} e^{(\frac{\eta}{C_2})^2} + 1)[C_5 sin\frac{\sqrt{k_4}\alpha}{C_2\sqrt{k_5}} + C_6 cos\frac{\sqrt{k_4}\alpha}{C_2\sqrt{k_5}}].$$

So the solution of Eq.(1.1) is

$$u(x,y,z,t) = \pm\frac{1}{2}\sqrt{\frac{C_4}{e^{(\frac{C_3}{C_2})^2} e^{(\frac{y}{C_2})^2}}}(e^{(\frac{C_3}{C_2})^2} e^{(\frac{y}{C_2})^2} + 1)[C_5 sin\frac{\sqrt{k_4}(-t+z)}{C_2\sqrt{k_5}} + C_6 cos\frac{\sqrt{k_4}(-t+z)}{C_2\sqrt{k_5}}].$$

The corresponding projective structure figures are plotted in Fig.1(a).

(II)Solving the reduced equation in case (c_2), we arrive at

$$k_2 F F_{\xi\xi} + k_2 F_{\xi}^2 - k_3\lambda F_{\xi\xi\xi\xi} + k_4 F_{\eta\eta} + k_5 F_{\alpha\alpha} - k_3 F_{\xi\xi\xi\alpha} + k_1 F_{\xi\xi} + \lambda F_{\xi\xi} + F_{\xi\alpha} = 0, \tag{3.2}$$

where $F = F(\xi,\eta,\alpha)$, and $\xi = -\lambda t + x$, $\eta = y$, $\alpha = t + z$. Let's solving the Eq.(3.1), we'll get the solution

$$F(\xi,\eta,\alpha) = \frac{12k_3C_2(C_2\lambda + C_4)tanh(C_2\xi + C_3\eta + C_4\alpha + C_1)^2}{k_2} - \frac{(8C_2^4k_3 + 8C_2^3C_4k_3 + C_2^2k_1 + C_2^2\lambda + C_3^2k_4 + C_2C_4)}{k_2C_2^2}.$$

So the solution of Eq.(1.1) is

$$u(x,y,z,t) = \frac{12k_3C_2(C_2\lambda + C_4)tanh(C_2(-t\lambda + x) + C_3y + C_4(t+z) + C_1)^2}{k_2}$$
$$- \frac{(8C_2^4k_3 + 8C_2^3C_4k_3 + C_2^2k_1 + C_2^2\lambda + C_3^2k_4 + C_2C_4)}{k_2C_2^2}.$$

The corresponding projective structure figures are plotted in Fig.1(b).

(III)Solving the reduced equation in case (c_3), one arrive at

$$-F_{\xi\alpha} + k_1F_{\alpha\alpha} + k_2F_{\alpha}^2 + k_2FF_{\alpha\alpha} + k_3F_{\alpha\alpha\alpha\xi} + k_4F_{\eta\eta} + k_5F_{\alpha\alpha} = 0, \tag{3.3}$$

where $F = F(\xi,\eta,\alpha)$, and $\xi = t$, $\eta = y$, $\alpha = -x + z$. Let's solving the Eq.(3.3), we can the solution

$$F(\xi,\eta,\alpha) = \frac{-k_2(C_2\eta + C_2)^4[\frac{-8C_3 - 8\alpha}{(C_1\eta + C_2)^2}]^{\frac{3}{2}} - 32(\alpha + C_3)[\frac{k_2(C_1\eta + C_2)\sqrt{\frac{-8C_3 - 8\alpha}{(C_1\eta + C_2)^2}}}{4} - k_2 f_1(\eta) + k_1 + k_5]}{32k_2(\alpha + C_3)}$$
$$\frac{(C_1\eta + C_2)^2\sqrt{\frac{-8(\alpha + C_3)}{(C_1\eta + C_2)^2}}}{4} - f_1(\eta).$$

So one can solution of Eq.(1.1) is

$$u(x,y,z,t)=\frac{-k_2(C_2y+C_2)^4[\frac{-8C_3-8(z-x)}{(C_1y+C_2)^2}]^{\frac{3}{2}}-32(z-x+C_3)[\frac{k_2(C_1y+C_2)\sqrt{\frac{-8C_3-8(z-x)}{(C_1y+C_2)^2}}}{4}-k_2f_1(y)+k_1+k_5]}{32k_2(z-x+C_3)}$$

$$\frac{(C_1y+C_2)^2\sqrt{\frac{-8(z-x+C_3)}{(C_1y+C_2)^2}}}{4}-f_1(y).$$

The corresponding projective structure figures are plotted in Fig.1(c).

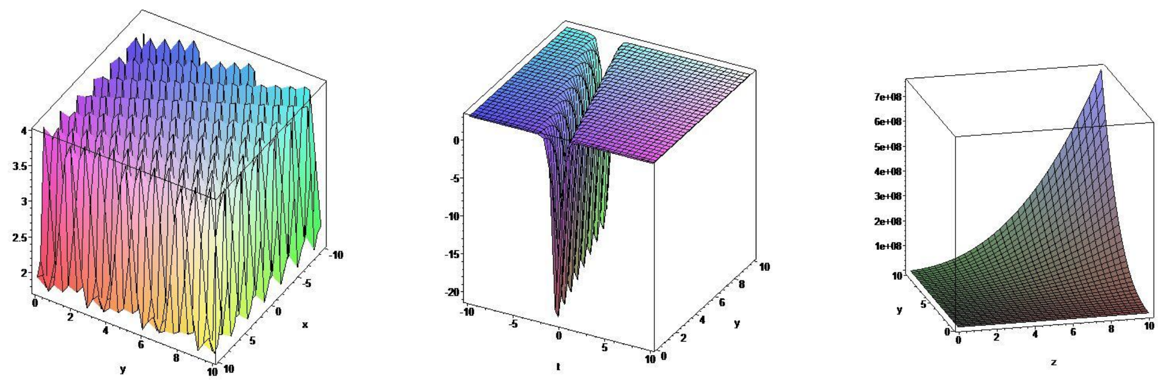

Fig. 1 (a) 3D-plot (y=0); (b) 3D-plot (x=z=0); (c) 3D-plot (x=0).

(IV)Solving the reduced equation in case (c_5), one arrive at

$$F_{\xi\eta}+k_1F_{\eta\eta}^2+k_2F_\eta^2+k_2FF_{\eta\eta}-k_3F_{\eta\eta\eta\xi}+k_4F_{\alpha\alpha}=0, \tag{3.4}$$

where $F=F(\xi,\eta,\alpha)$, and $\xi=t$, $\eta=x$, $\alpha=y$. Let's solving the Eq.(3.4), we can the solution

$$F(\xi,\eta,\alpha)=-\sqrt{2C_1\eta+2C_2}\alpha+C_3\sqrt{C_1\eta+C_2}-\frac{k_1}{k_2}.$$

So one can have the solution of Eq.(1.1)

$$u(x,y,z,t)=-\sqrt{2C_1x+2C_2}y+C_3\sqrt{C_1x+C_2}-\frac{k_1}{k_2}.$$

The corresponding projective structure figures are plotted in Fig.2(d).

(V)Solving the reduced equation in case (d_3), one can have

$$-F_{\xi\xi}+k_1F_{\xi\xi}+k_2F_\xi^2+k_2FF_{\xi\xi}+k_3F_{\xi\xi\xi\xi}+k_4F_{\eta\eta}+k_5F_{\alpha\alpha}=0, \tag{3.5}$$

where $F=F(\xi,\eta,\alpha)$, and $\xi=x-t$, $\eta=y$, $\alpha=z$. Let's solving the Eq.(3.5), we can the solution

$$F(\xi,\eta,\alpha)=(C_1\eta+C_3)\alpha+C_2\eta+C_4.$$

So we can have the solution of Eq.(1.1)

$$u(x,y,z,t)=(C_1y+C_3)z+C_2y+C_4\,.$$

The corresponding projective structure figures are plotted in Fig.2(e).

Base on theorem1, we can know that

$$u(x,y,z,t)=[C_1(\frac{\sqrt{k_4k_5}sin(\sqrt{k_4k_5}\varepsilon)z}{k_5}+cos(\sqrt{k_4k_5}\varepsilon)y)+C_3][\frac{-k_5sin(\sqrt{k_4k_5}\varepsilon)y}{\sqrt{k_4k_5}}+cos(\sqrt{k_4k_5}\varepsilon)z]$$

$$u(x,y,z,t)=C_2[\frac{-k_5sin(\sqrt{k_4k_5}\varepsilon)y}{\sqrt{k_4k_5}}+cos(\sqrt{k_4k_5}\varepsilon)z]+C_4\,.$$

is also the solution of Eq.(1.1).

(VI)Solving the reduced equation in case (d_5), one can have

$$k_1F_{\xi\xi}+k_2F_{\xi}^2+k_2FF_{\xi\xi}+k_4F_{\eta\eta}+k_5F_{\alpha\alpha}=0\,, \tag{3.6}$$

where $F=F(\xi,\eta,\alpha)$, and $\xi=x$, $\eta=y$, $\alpha=z$. Let's solving the Eq.(3.6), we get the solution

$$F(\xi,\eta,\alpha)=\frac{\sqrt{2k_2\xi(\frac{C_2\sqrt{k_4k_5}}{-k_4}\eta+C_2\alpha+C_1)+2k_2\,f_1(\frac{\sqrt{-k_4k_5}\alpha+k_5\eta}{\sqrt{-k_4k_5}})+k_1^2-k_1}}{k_2}\,.$$

So we arrive at the solution of Eq.(1.1)

$$u(x,y,z,t)=\frac{\sqrt{2k_2x(\frac{C_2\sqrt{k_4k_5}}{-k_4}y+C_2z+C_1)+2k_2\,f_1(\frac{\sqrt{-k_4k_5}z+k_5y}{\sqrt{-k_4k_5}})+k_1^2-k_1}}{k_2}\,.$$

The corresponding projective structure figures are plotted in Fig.2(f).

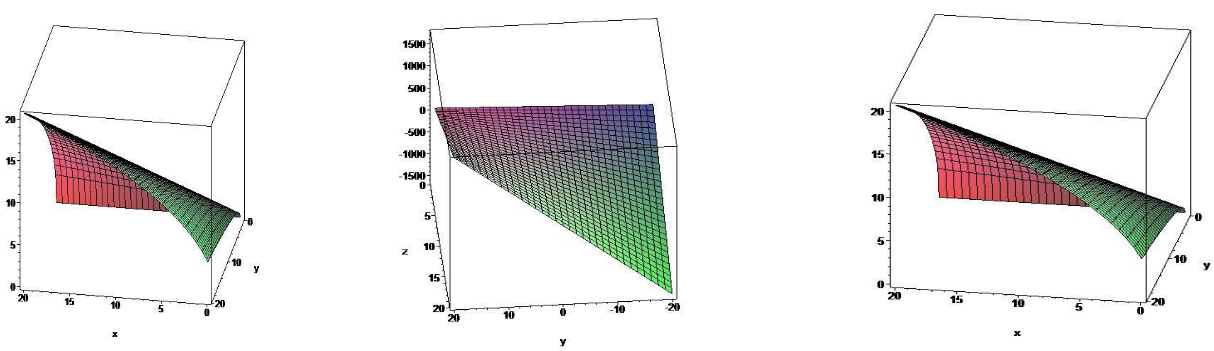

Fig. 2 (d) 3D-plot ; (e) 3D-plot (x=z=0); (f) 3D-plot (z=0).

(VII)Solving the reduced equation in case (e_1), one can have

$$-F_{\xi\eta}+k_1F_{\eta\eta}^2+k_2F_{\eta}^2+k_2FF_{\eta\eta}+k_3F_{\eta\eta\eta\xi}+k_4F_{\eta\eta}+k_5F_{\alpha\alpha}=0, \tag{3.7}$$

where $F=F(\xi,\eta,\alpha)$, and $\xi=x$, $\eta=y$, $\alpha=z$. Let's solving the Eq.(3.1), we get the solution

$$F(\xi,\eta,\alpha)=\frac{\eta}{-k_2\xi+C_1}+\alpha f_1(\xi)+f_2(\xi).$$

So we can obtain the solution of Eq.(1.1)

$$u(x,y,z,t)=\frac{-x+y}{-k_2t+C_1}+\alpha f_1(t)+f_2(t).$$

The corresponding projective structure figures are plotted in Fig.3(h).

(VII)Solving the reduced equation in case (e_2), we arrive at

$$F_{\xi\eta}+k_1F_{\eta\eta}+k_2F_{\eta}^2+k_2FF_{\eta\eta}-k_3F_{\eta\eta\eta\xi}+k_4F_{\eta\eta}+k_5F_{\alpha\alpha}=0, \tag{3.8}$$

where $F=F(\xi,\eta,\alpha)$, and $\xi=t$, $\eta=x+y$, $\alpha=z$. One can get the solution of Eq.(3.8).

$$F(\xi,\eta,\alpha)=\frac{1}{12k_5}[-2\alpha^3 f_1^{'}(\xi)-6\alpha^2 f_2^{'}(\xi)-k_2\alpha^4 f_1(\xi)^2+(-4k_2\alpha^3 f_2(\xi)+12k_5\eta\alpha) f_1(\xi)$$
$$-6k_2\alpha^2 f_2(\xi)+12k_5\eta f_2(\xi)+12k_5(\alpha f_3(\xi)+f_4(\xi))].$$

So one arrive at the solution of Eq.(1.1)

$$F(\xi,\eta,\alpha)=\frac{1}{12k_5}[-2\alpha^3 f_1^{'}(t)-6\alpha^2 f_2^{'}(t)-k_2\alpha^4 f_1(\xi)^2+(-4k_2\alpha^3 f_2(t)+12k_5(x+y)z) f_1(t)$$
$$-6k_2z^2 f_2(t)+12k_5(x+y) f_2(t)+12k_5(\alpha f_3(t)+f_4(t))].$$

The corresponding projective structure figures are plotted in Fig.3(i).

(IX)Solving the reduced equation in case (f), we arrive at

$$F_{\xi\eta}+k_1F_{\eta\eta}+k_1F_{\eta}^2+k_2FF_{\eta\eta}-k_3F_{\eta\eta\eta\xi}+k_4F_{\eta\eta\eta\eta}+k_5F_{\alpha\alpha}=0, \tag{3.9}$$

where $F=F(\xi,\eta,\alpha)$, and $\xi=t$, $\eta=y$, $\alpha=z$. One can get the solution of Eq.(3.9).

$$F(\xi,\eta,\alpha)=\eta\alpha f_1(\xi)+\eta f_2(\xi)+\alpha f_3(\xi)+f_4(\xi).$$

So one arrive at the solution of Eq.(1.1)

$$u(x,y,z,t)=yzf_1(t)+yf_2(t)+zf_3(t)+f_4(t).$$

The corresponding projective structure figures are plotted in Fig.3(j).

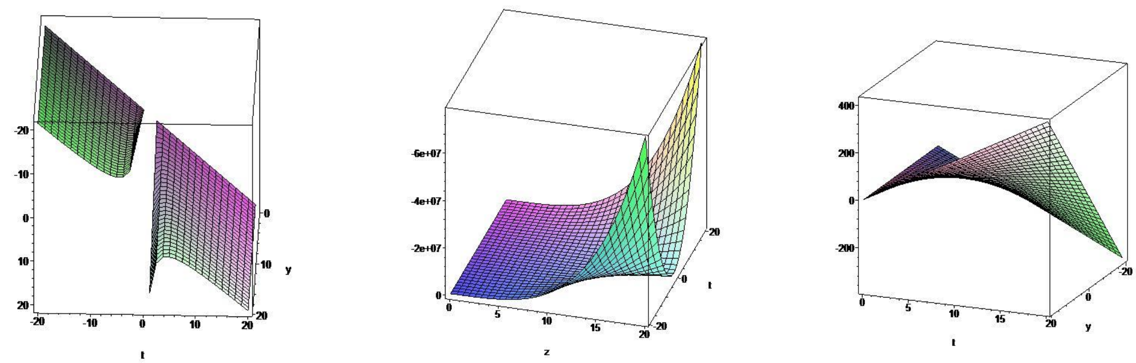

Fig. 2 (h) 3D-plot (x=z=0); (i) 3D-plot (x=y=0); (j) 3D-plot (z=0).

4. Conclusion

By means of a direct symmetry method[21-25], we investigate a (3+1)-dimensional KPBBM equation. The symmetry group is obtained and its corresponding group invariant solutions are constructed. Then we give the optimal system by using the equivalent vector of the obtained symmetry. To every case of the optimal system, we find reductions and obtain some new explicit solutions of the (3+1)-dimensional KPBBM equation.

References

[1] S. Xu, J. He and Wang L, "The Darboux Transformation of the Derivative Nonlinear Schrodinger Equation," *Journal of Physics A Mathematical and Theoretical*, vol. 44, no. 30, pp. 6629-6636, 2011.

[2] A. Barari, A.R. Ghotbi, and F. Farrokhzad, "Variational Iteration Method and Homotopy-Perturbation Method for Solving Different Types of Wave Equations," *Journal of Applied Sciences,* vol. 8, no. 1, pp. 120-126, 2008.

[3] H. Alatas, A.A. Kandi, and A.A. Iskandar, "New Class of Bright Spatial Solitons Obtained by Hirota's Method from Generalized Coupled Mode Equations of Nonlinear Optical Bragg Grating," *Journal of Nonlinear Optical Physics and Materials*, vol. 17, no. 2, pp. 225-233, 2008.

[4] X.W. Chen, Y.M. Li, and Y.H. Zhao, "Lie Symmetries, Perturbation to Symmetries and Adiabatic Invariants of Lagrange System," *Physics Letters A*, vol. 37, no. 4-6, pp. 274-278, 2005.

[5] T, Mookum, and M, Khebchareon, "Finite Difierence Methods for Finding a Control Parameter in Two-Dimensional Parabolic Equation with Neumann Boundary Conditions," *Thai Journal of Mathematics,* vol. 6, no. 1, pp. 117-137, 2008.

[6] H, Zhang. G.M. Wei, and Y.T. Gao, "On the General Form of the Benjamin-Bona-Mahony Equation in Fluid Mechanics," *The Journal of Chemical Physics,* vol. 52, pp. 373-377, 2002.

[7] S. Ming, C. Yang, and B, Zhang, "Exact Solitary Wave Solutions of the Kadomtsov-Petviashvili-Benjamin-Bona-Mahony Equation," *Applied Mathe- Matics and Computation,* vol. 217, no. 4, pp. 1334-1339, 2010.

[8] Y. Yin, B. Tian, and X.Y. Wu, "Lump Waves and Breather Waves for a (3+1)-Dimensional Generalized Kadomtsev-Petviashvili-Benjamin-Bona-Mahony Equation for an Offshore Structure," *Modern Physics Letters B*, vol. 32, no. 10, pp. 1850031, 2018.

[9] A. Mekki, and M.M, Ali, Numerical Simulation of Kadomtsev-Petviashvili-Benjamin-Bona-Mahony Equations Using Finite Difference Method," *App- lied Mathematics and Computation*, 2013.

[10] K.H. Tariq, A.R, "Soliton Solution of (3+1)-Dimensional Korteweg-De Vries Benjiamin-Mahony, Kadomtsov-Petviashvili-Benjamin-Bona-Mahony and Modified Korteweg-de Vries-Zakharov-Kuznetsov Equations and Their Applications in Water Waves," *Journal of King Saud University Science,* vol. 31, no. 1, pp. 8-13, 2019.

[11] K.H. Tariq, and A.R. Seadawy, "Soliton Solutions of (3+1)-Dimensional Korteweg-de VriesBenjiamin-Bona-Mahony, Kadomtsev-Petviashvili-Benj- amin-Bona-Mahony and Modified Korteweg-de Vries-Zakharov-Kuznetsov Equations and Their Applications in Water Waves," *Journal of King Saud University Science*, vol. 31, no. 1, pp. 8-13, 2019.

[12] C. Hu, R. Wang, and D.H. Ding, "Symmetry Groups, Physical Property Tensors, Elasticity and Dislocations in Quasicrystals," *Reports on Progress in Physics,* vol. 63, no. 1, pp. 1, 2000.

[13] H.C. Ma, and S.Y. Lou, "Non-Lie Symmetry Groups of (2+1)-Dimensional Nonlinear Systems," *Communications in Theoretical Physics,* vol. 46, no. 12, pp. 1005-1010, 2006.

[14] Z.Z. Dong, Y. Chen, and L. Wang, "Similarity Reductions of (2+1)-Dimensional Multicomponent Broer Kaup System," *Communications in Theoretical Physics,* vol. 50, pp. 803-808, 2008.

[15] X.P. Xin, X.Q. Liu, and L.L. Zhang, "Symmetry Reductions and Exact Solutions of a (2+1)-Dimensional Nonlinear Evolution Equation," *Journal of Jinggangshan University,* vol. 29, no. 4, pp. 411-416, 2012.

[16] H.C. Hu, J.B. Wang, and H.D. Zhu, "Symmetry Reduction of (2+1)-Dimensional Lax Kadomtsev-Petviashvili Equation," *Communications in Theoretical Physics*, 2015.

[17] K. Sachin, and R. Setu, "Study of Exact Analytical Solutions and Various Wave Profiles of a New Extended (2+1)-Dimensional Boussinesq Equation Using Symmetry Analysis," *Journal of Ocean Engineering and Science,* 2021.

[18] K. Ssachin, A. Hassan, and K.D. Shubham, "A Study of Bogoyavlenskiis (2+1)-Dimensional Breaking Soliton Equation: Lie Symmetry, Dynamical Behaviors and Closed-form Solutions," *Results in Physics*, vol. 29, pp. 2211-7797, 2021.

[19] J. Zhuang, Y Liu, and P. Zhuang, "Variety Interaction Solutions Comprising Lump Solitons for the (2+1)-Dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada Equation", *AIMS Mathematics*, vol. 6, no. 5, pp. 5370-5386, 2021.

[20] C.J. Bai, and H. Zhao, "A New Rational Approach to Find Exact Analytical Solutions to a (2+1)-Dimensional Symmetry," *Communications in Theoretical Physics*, vol. 48, pp. 801-810, 2007.

[21] S.Y. Wang, F.X. Mei, "Form Invariance and Lie Symmetry of Equations of Non-Holonomic Systems," *Chinese Physics*, vol. 11, no. 1, pp. 5, 2002.

[22] R. K. Gazizov, N.H. Ibragimov, "Lie Symmetry Analysis of Differential Equations in Finance," *Nonlinear Dynamics*, vol. 17, no. 4, pp. 387-407, 1998.

[23] X.M. Feng, "Lie Symmetry and the Conserved Quantity of a Generalized Hamiltonian System," *Acta Physica Sinica*, vol. 52, no. 5, pp. 1048-1050, 2003.

[24] S.K. Luo, "Mei Symmetry, Noether Symmetry and Lie Symmetry of Hamiltonian System," *Acta Physica Sinica,* 2003.

[25] M. Craddock, K.A. Lennox, "The Calculation of Expectations for Classes of Diffusion Processes by Lie Symmetry Methods," *The Annals of Applied*, 2021.