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Original Article

Modification of the Japanese Theorem on Heptagon

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Abstract *- This article discusses the modification of the Japanese on Heptagon. The proof is done using Carnot's Theorem, the modification of Ptolemy's Theorem, and the Japanese Theorem. The result obtained is the proof using Carnot;s Theorem is more effective. So, the sum of the radius length of the incircle of the triangles with diagonal lines formed from all points is the same.*

Keywords – *Japanese Theorem, Carnot's Theorem, modification of Ptolemy's Theorem*.

1. Introduction

The Japanese Theorem invented by [1] in the Japanese Edo era. The Japanese discusses that any cyclic quadrilateral is formed by diagonal lines from different points to form two triangles. If a circle formed in a triangle, then the sum of radius length of the incircle of the triangles form each diagonal lines of the different points is the same.

The proof of the Japanese Theorem has been done a lot. The proof of the Japanese Theorem carried [2] uses Carnot's Theorem. Furthermore, [3] proves the Japanese Theorem using Thebault's Theorem. In [4], discusses the proof and generalization of the Japanese Theorem by including some proofs bye previous authors. The proofs by Nagasawa and Sawayama were carried out using Carnot's Theorem. The proof by Matsuo and Omori is done by using the of distance from the circumcenter to the incenter of the triangles. Furthermore, [5] proves the Japanese Theorem using modification of Ptolemy's Theorem. In addition, [6] discusses the modification of the Japanese Theorem on pentagon by forming diagonal lines from some points that can form the same triangle and quadrilateral. The proof is done by using the modification of Ptolemy's Theorem. However, this proof has a drawback, namely that it is difficult to prove when the diagonal lines are formed form only one point.

Based on the description above, this article discusses the modification of the Japanese Theorem on heptagon. If the modification of the Japanese Theorem on pentagon is done by forming diagonal lines from some points that can form the same quadrilateral and triangle, then the modification of the Japanese Theorem on heptagon is done by forming diagonal lines form only one point which will be proven by using Carnot's Theorem. In addition, this article also discusses another modification of the Japanese Theorem on heptagon for comparison. The modification is done by forming the diagonals lines from some points that can form the same quadrilateral and triangle which will be proven by using the modification of Ptolemy's Theorem and Japanese Theorem.

2. Literature Review

In this literature review, we will discuss some supporting theories related to the problem to be discussed.

2.1. Circumcircle and Incircle of Triangle

The circumcircle of a triangle is a circle that passes through the three vertice of the triangle with the circumcenter as the center of the circle. The radius length of the circumcircle of a triangle is denoted by $\boldsymbol{R}$. The incircle of a triangle is a circle that is tangent to the three sides of the triangle with the incenter as the center of the circle. The radius length of the incircle of a triangle is denoted by $\boldsymbol{r}$. The following theorem discusses the relationship between the radius length of the incircle of a triangle and the sides length of a triangle and the radius length of the circumcricle of a triangle. The proof can be seen in [7,8].

Theorem 1. Given triangle ABC with sides length BC, AC, and AB are a, b, and c, then

$$r = \frac{abc}{2R(a+b+c)}.$$

One of the theorems relating to the circumcircle and incircle of a triangle is Carnot's Theorem. Carnot's Theorem discusses the relationship between the radius length of the circumcircle and the incircle of a triangle with the length of the perpendicular lines form the circumcenter to the three sides of a triangle. Canot's Theorem can be experessed in the following two theorems.

The difference between the following two theorems is the marked distance. If a point is in the same plane as a line, the distance is positive. If a points is in a different plane form a line, the distance is negative. The proof can be seen in [7-10].

Theorem 2. Given triangle ABC with O is circumcenter which is inside the triangle. If OD, OE, and OF are perpendicular lines from point O to sides BC, AC, and AB, then

$$OD + OE + OF = R + r.$$

Theorem 3. Given triangle ABC with O is circumcenter which is outsides the triangle. If OD, OE, and OF are perpendicular lines from point O to sides BC, AC, and AB, then

$$OE + OF + (-OD) = R + r.$$

2.2. Cyclic Quadrilateral

In [7,8,11-13], cyclic quadrilateral is a quadrilateral that lies inside a circle with a all four points on the curvature of the circle. Some of the thoerems related to cyclic quadrilateral are Ptolemy's Theorem and Japanese Theorem. In [7,8,12-16], Ptolemy's Theorem discussed the relationship between the sum of the lengths of two adjacent sides and the product of the diagonals in a cyclic quadrilateral. In Ptolemy's Theorem can be modified, namely relationship betweem sides length of the triangle formed by two diagonal lines in a cyclic quadrilateral. The modification of Ptolemy's Theorem can be stated in the following theorem. The proof can be seen in [5,13].

Theorem 4. Given cyclic quadrilateral $ABCD$ with sides length AB, BC, CD, and DE are a, b, c, and d, then

$$\frac{abe}{a+b+e} + \frac{cde}{c+d+e} = \frac{bcf}{b+c+f} + \frac{adf}{a+d+f}.$$

Illustration of the Japanese Theorem can be seen in Figure 1 and stated in the following theorem. The proof is in [2-6].

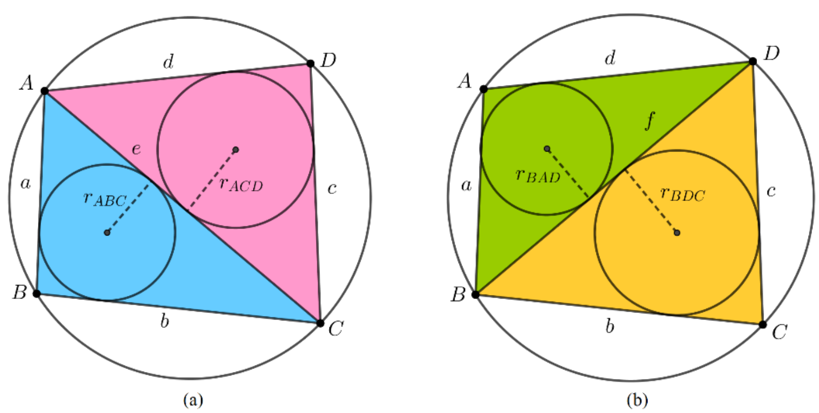

Fig. 1 Japanese Theorem on Quadriateral ABCD. (a) Diagonals AC and (b) BD

Theorem 5. Given cyclic quadrilateral $ABCD$ with sides length AB, BC, CD, and DE are a, b, c, and d and diagonals length AC and BD are e and f. If form the incircle of the triangles ABC, ACD, BAD, and BDC with the radius r_{ABC}, r_{ACD}, r_{BAD}, and r_{BDC} respectively, then

$$\boldsymbol{r_{ABC} + r_{ACD} = r_{BAD} + r_{BDC}}.$$

3. Reaserch Methodology

In this reaserch methodology, discussed the steps in solving the problem to be discussed.

3.1. Modification of the Japanese Theorem on Heptagon

The steps in proving the modification of the Japanese Theorem on heptagon is to form an arbitrary cyclic heptagon $ABCDEFG$. Furthermore, form diagonal lines from points A to other points, namely AC, AD, AE, and AF as well as point B to other points, namely BD, BE, BF, and BG. The diagonal lines from points A and B each form five triangles, namely ABC, ACD, ADE, AEF, AFG, BCD, BDE, BEF, BFG, and BGA. The triangles are formed the incircle of a triangle, say each radius are r_{ABC}, r_{ACD}, r_{ADE}, r_{AEF}, r_{AFG}, r_{BCD}, r_{BDE}, r_{BEF}, r_{BFG}, and r_{BGA}. Next, by using Theorem 2 and Theorem 3 will be proved the sum of the lengths of r_{ABC}, r_{ACD}, r_{ADE}, r_{AEF} and r_{AFG} is equal to the sum of the lengths of r_{BCD}, $\boldsymbol{r_{BDE}}$, $\boldsymbol{r_{BEF}}$, $\boldsymbol{r_{BFG}}$, and $\boldsymbol{r_{BGA}}$.

3.2. Another Modification of the Japanese Theorem on Heptagon

The steps in proving the another modification of the Japanese Theorem on heptagon is to form an arbitrary cyclic heptagon $ABCDEFG$. Furthermore, form diagonal lines from points A to other points, namely AC, AD, AE, and AF as well as point D and E, namely DB, DA, EA, and EG. The diagonal lines from points A form five triangles and points D and E also form five triangles, namely ABC, ACD, ADE, AEF, AFG, DCB, DBA, DAE, EAG, and EGF. The triangles are formed the incircle of a triangle, say each radius are r_{ABC}, r_{ACD}, r_{ADE}, r_{AEF}, r_{AFG}, r_{DCB}, r_{DBA}, r_{DAE}, r_{EAG}, and r_{EGF}. By using Theorem 4 and Theorem 5 will be proved the sum of the lengths of r_{ABC}, r_{ACD}, r_{ADE}, r_{AEF} and r_{AFG} is equal to the sum of the lengths of r_{DCB}, r_{DBA}, r_{DAE}, r_{EAG}, and r_{EGF}.

4. Results and Discussion

In the result and discussion, it will be proven the modification of the Japanese Theorem on heptagon using Carnot's Theorem and another modification using the modification of the Ptolemy's Theorem and Japanese Theorem.

4.1. Modification of The Japanese Theorem on Heptagon

Modification of the Japanese Theorem on heptagon is carried out by forming diagonal lines form points A and B on any cyclic heptagon $ABCDEFG$ as shown in Figure 2. This modification can be stated in the theorem as follows.

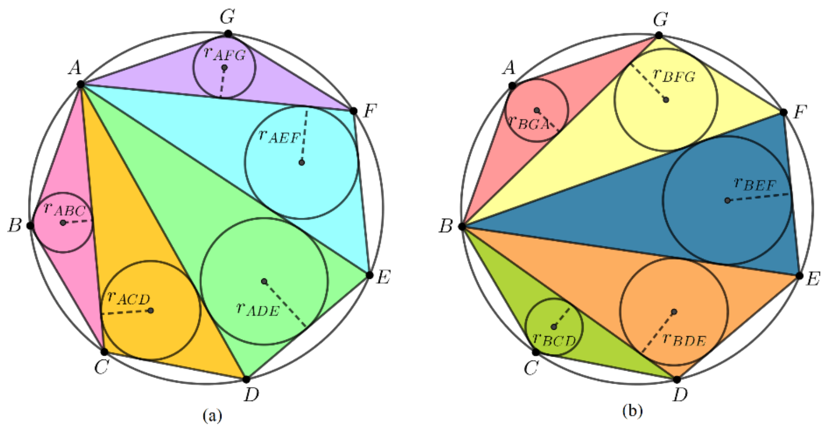

Fig. 2 Modification of the Japanese on Heptagon ABCDEFG. (a) Diagonals from point A and (b) point B.

Theorem 6. Given any cyclic heptagon $ABCDEFG$ with diagonal lines AC, AD, AE, AF, BD, BE, BF, and BG. If form the incircle of the triangles ABC, ACD, ADE, AEF, AFG, BCD, BDE, BEF, BFG, and BGA with radius r_{ABC}, r_{ACD}, r_{ADE}, r_{AEF}, r_{AFG}, r_{BCD}, r_{BDE}, r_{BEF}, r_{BFG}, and r_{BGA} respectively, then

$$r_{ABC} + r_{ACD} + r_{ADE} + r_{AEF} + r_{AFG} = r_{BCD} + r_{BDE} + r_{BEF} + r_{BFG} + r_{BGA}.$$

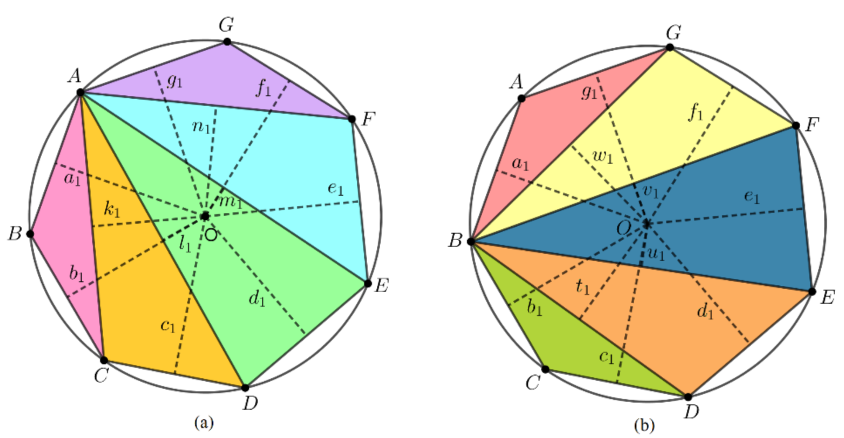

Fig. 3 Carnot's Theorem on Heptagon ABCDEFG. (a) Triangles from Diagonals point A and (b) point B.

Proof: In this proof using Carnot's Theorem by forming perpendicular lines from the circumcenter to each sides and diagonals on the cyclic heptagon $ABCDEFG$, namely a_1, b_1, c_1, d_1, e_1, f_1, g_1, k_1, l_1, m_1, n_1, t_1, u_1, v_1, and w_1 as shown in Figure 3. By considering ΔABC, ΔACD, ΔADE, ΔAEF, and ΔAFG, based on Theorem 2 and Theorem 3 we get

$$R + r_{ABC} = a_1 + b_1 - k_1, \tag{1}$$
$$R + r_{ACD} = k_1 + c_1 - l_1, \tag{2}$$
$$R + r_{ADE} = l_1 + d_1 + m_1, \tag{3}$$
$$R + r_{AEF} = e_1 + n_1 - m_1, \tag{4}$$
$$R + r_{AFG} = f_1 + g_1 - n_1. \tag{5}$$

If equation (1) is added with equation (2), (3), (4), and (5) we get

$$r_{ABC} + r_{ACD} + r_{ADE} + r_{AEF} + r_{AFG} = a_1 + b_1 + c_1 + d_1 + e_1 + f_1 + g_1 - 5R. \tag{6}$$

Using the same method for ΔBCD, ΔBDE, ΔBEF, ΔBFG, and ΔBGA we get

$$r_{BCD} + r_{BDE} + r_{BEF} + r_{BFG} + r_{BGA} = a_1 + b_1 + c_1 + d_1 + e_1 + f_1 + g_1 - 5R. \tag{7}$$

From equation (6) and (7) obtained similarities, namely

$$r_{ABC} + r_{ACD} + r_{ADE} + r_{AEF} + r_{AFG} = r_{BCD} + r_{BDE} + r_{BEF} + r_{BFG} + r_{BGA}.$$

Therefore, Theorem 6 is proven. □

Using same method as Theorem 6, we get the following corollary.

Corollary 1. In any cyclic heptagon $ABCDEFG$ diagonal lines are formed from each points so that it forms five triangles from each points. If form incircle of the triangles with r_1, r_2, r_3, r_4, r_5, r_6, and r_7 are the sum of the radius lengths of the incircle of the triangles formed from the diagonal lines of points A, B, C, D, E, F, and G, then

$$r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = r_7.$$

4.2. Another Modification of The Japanese Theorem on Heptagon

Another modification of the Japanese Theorem on heptagon is done by forming diagonal lines from points A, D, and E in any cyclic heptagon $ABCDEFG$ so as to form the same quadrilateral and triangle, namely $\square ABCD$, $\square AEFG$, and ΔADE as shown in Figure 4. This modification can be stated in theorem as follows.

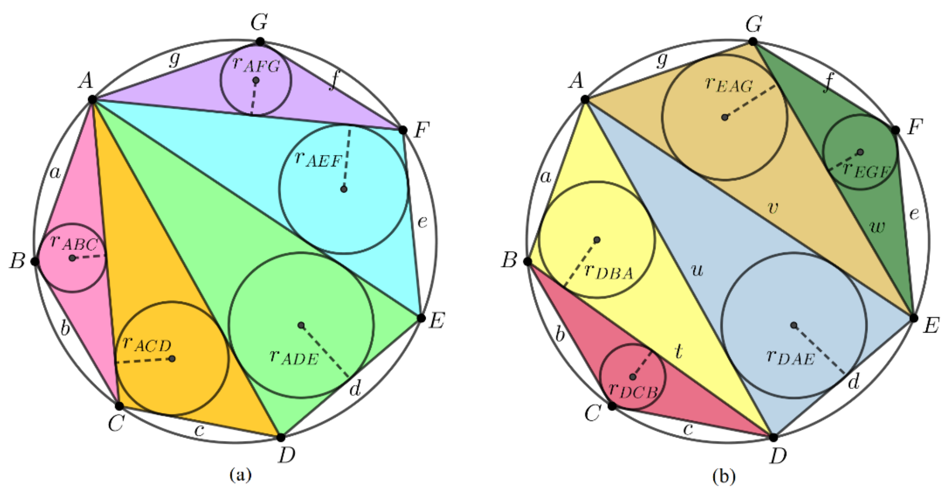

Fig. 4 Another Modification of the Japanese on Heptagon ABCDEFG. (a) Diagonals from point A, (b) points D and E.

Theorem 7. Given any cyclic heptagon $ABCDEFG$ with diagonal lines AC, AD, AE, AF, DB, DA, EA, and EG. If form the incircle of the triangles ABC, ACD, AEF, AFG, DCB, DBA, DAE, EAG, and EGF with radius r_{ABC}, r_{ACD}, r_{ADE}, r_{AEF}, r_{AFG}, r_{DCB}, r_{DBA}, r_{DAE}, r_{EAG}, and r_{EGF} respectively, then

$$r_{ABC} + r_{ACD} + r_{ADE} + r_{AEF} + r_{AFG} = r_{DCB} + r_{DBA} + r_{DAE} + r_{EAG} + r_{EGF}.$$

Proof: In Figure 4, suppose a, b, c, d, e, f, and g are lengths of sides AB, BC, CD, DE, EF, FG, and GA respectively. Next, suppose k, l, m, n, t, u, v, and w are lengths of diagonals AC, AD, AE, AF, DB, DA, EA, and EG.

Alternative 1. In this alternative proof, it will be carried out using the modification of Ptolemy's Theorem (Theorem 4). By considering ΔABC, ΔACD, ΔDCB, and ΔDBA, based on Theorem 1 we get

$$r_{ABC} = \frac{abk}{2R(a+b+k)},$$
$$r_{ACD} = \frac{kcl}{2R(k+c+l)},$$
$$r_{DCB} = \frac{bct}{2R(b+c+t)},$$
$$r_{DBA} = \frac{tau}{2R(t+a+u)}.$$

By considering □$ABCD$, based on Theorem 4 on ΔABC, ΔACD, ΔDCB, and ΔDBA we get

$$\frac{abk}{a+b+k} + \frac{kcl}{k+c+l} = \frac{bct}{b+c+t} + \frac{tau}{t+a+u}. \quad (8)$$

If both sides of equation (8) are multiplied by $1/2R$ we get

$$\frac{abk}{2R(a+b+k)} + \frac{kcl}{2R(k+c+l)} = \frac{bct}{2R(b+c+t)} + \frac{tau}{2R(t+a+u)},$$
$$r_{ABC} + r_{ACD} = r_{DCB} + r_{DBA}. \quad (9)$$

Using the same method for □$AEFG$ we get

$$r_{AEF} + r_{AFG} = r_{EAG} + r_{EGF}.$$

Because $\Delta ADE = \Delta DAE$, then

$$r_{ADE} = r_{DAE}.$$

If equation (9) is added with equation (10) and (11) we get

$$r_{ABC} + r_{ACD} + r_{ADE} + r_{AEF} + r_{AFG} = r_{DCB} + r_{DBA} + r_{DAE} + r_{EAG} + r_{EGF}.$$

Therefore, Theorem 7 is proven. □

Alternative 2. In this alternative proof, it will be proved by using Japanese Theorem (Theorem 5). By considering □$ABCD$ and □$AEFG$, based on Theorem 5 we get

$$r_{ABC} + r_{ACD} = r_{DCB} + r_{DBA}, \quad (12)$$
$$r_{AEF} + r_{AFG} = r_{EAG} + r_{EGF}. \quad (13)$$

Because $\Delta ADE = \Delta DAE$, then

$$r_{ADE} = r_{DAE}. \quad (14)$$

If equation (12) is added with equation (13) and (14) we get

$$r_{ABC} + r_{ACD} + r_{ADE} + r_{AEF} + r_{AFG} = r_{DCB} + r_{DBA} + r_{DAE} + r_{EAG} + r_{EGF}.$$

Therefore, Theorem 7 is proven. □

5. Conclusion

Based on result and discussion, it can be concluded the Japanese Theorem can be modified in the heptagon. The proof is done using Carnot's Theorem, the modification of Ptolemy's Theorem and the Japanese Theorem. The result obtained from the proof using Carnot's Theorem is the sum of the radius length of the incircle of the triangles with diagonal lines formed from all points are the same. While the proof by using the modification of the Ptolemy's Theorem and Japanese Theorem is done by forming the same quadrilateral and triangles so that the sum of the radius length of the incircle of the triangles are the same.

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