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On Group A-Cordial Labeling of Uniform t-Ply

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Abstract. Let A denote the multiplicative group $\{1, -1, i, -i\}$. In this paper, we prove that every uniform $t-$ply with each path individually of length at least 5 is group A-cordial and in fact we have given an explicit labeling for the same.

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1. Introduction

Cordial labelings were introduced by Cahit [8] in 1987 as a weakened version of graceful labelings.

Definition 1.1 (Cahit[8])**.** Let $f : V(G) \to \{0, 1\}$ be any function. For each edge ab assign the label $|f(a) - f(b)|$. Let $v_f(0), v_f(1)$ denote the number of vertices in G with the label 0 and 1 respectively. Let $e_f(0), e_f(1)$ denote the number of edges in G with the label 0 and 1 respectively. The function f is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. The graph G is called a *cordial graph* if it admits a cordial labeling.

Prime labelings were first introduced by Roger Entringer and discussed in a paper by Tout et al. (See [15]).

Definition 1.2. Let $G = (V, E)$ be a graph. A bijection $f : V \to \{1, 2, \ldots, |V|\}$ is called a prime labeling if for each $e = \{u, v\} \in E$, we have $\gcd(f(u), f(v)) = 1$. A graph that admits a prime labeling is called a *prime graph*.

Motivated by these two labelings, the notion of group A-cordial labeling of graphs was introduced by M. K. Karthik Chidambaram, S. Athisayanathan and R. Ponraj (See [11],[12],[13],[14]). For group $A = \{1, -1, i, -i\}$ the labeling for different graphs was considered by them. Further, for the group $A = S_3$, group A-cordial labeling in this case was studied by B. Chandra and R. Kala [9, 10]. For expository work in graph labelings you may refer to work by Andar et al. (See [1], [2], [3], [4], [5], [6], [7]).

2. Preliminaries

We begin with some necessary definitions and preliminaries in the graph labelings.

Definition 2.1. Let G be a graph and let A be a group. Let $o(a)$ denote the order of an element $a \in A$. Let $f : V(G) \to A$ be a function. For each edge uv, assign the label 1 if $\gcd\Big(o(f(u)), o(f(v))\Big) = 1$ and 0 otherwise. Let $v_f(a)$ denote the number of vertices in G labeled by f with the element a of the group A. Let $e_f(0), e_f(1)$ denote the number of edges with the label 0 and 1 respectively. The function f is called a *group A-cordial* labeling if for all $a, b \in A$, $|v_f(a) - v_f(b)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a *group A-cordial* labeling is called group *A-cordial.*

Throughout this paper, the group under consideration will be $A = \{1, -1, i, -i\}$ with respect to multiplication of complex numbers.

Definition 2.2. A *t-ply* $P_t(u, w)$ is a graph with t paths, each of length atleast two, such that no two paths have a vertex in common except for the end vertices u and w (common to all), which will also be referred to as source vertex and sink vertex respectively. We note in passing that $P_t(u, w)$ denotes an entire family of *t-ply* graphs with t paths and end vertices u, w.

Let the intermediate vertices on the x^{th} path of $P_t(u, w)$ be denoted by $v(x, 1), v(x, 2), \ldots, v(x, n_x)$, where the second coordinate denotes the distance of that vertex from the source vertex u.
Thus $V(P_t(u, w)) = \bigcup_{x=1}^{t}\{v(x, y)|1 \leq y \leq n_x\} \bigcup \{u, w\}$ and
$E(P_t(u, w)) = \{uv(x, 1)|1 \leq x \leq t\} \bigcup \{v(x, y)v(x, y+1), 1 \leq x \leq t, 1 \leq y \leq n_x - 1\} \bigcup \{v(x, n_x)w|1 \leq x \leq t\}$.
Hence $|V(P_t(u, w))| = 2 + \sum_{x=1}^{t} n_x$ and $|E(P_t(u, v)| = \sum_{x=1}^{t}(n_x + 1)$.

Definition 2.3. A path on the t-ply $P_t(u, w)$ is said to be of type l if the length of the path is congruent to l modulo 4, $l = 1, 2, 3, 4$.
We call $P_t(u, w)$ a uniform $t-$ply of type l if all t paths are of type l i.e. length of each path is congruent to l modulo 4. We denote such a $t-$ply by $P_t^l(u, w)$.

We further observe that in a uniform $t-ply$ of type l, the x^{th} path is of length n_x+1 where $n_x + 1 \equiv l \pmod 4$.
Let $n_x + 1 = 4k_x + l$; then
$|V(P_t^l(u, w))| = 2 + 4\sum_{x=1}^{t} k_x + (l-1)t$ and $|E(P_t^l(u, w))| = 4\sum_{x=1}^{t} k_x + lt$.

Remark 2.4. We keep ready two labelings given below which will be used to partially label the intermediate vertices on the paths in a t-ply.

For any path $P = \{v_1, v_2, \ldots, v_n\}$ of length $n - 1$; we take $n = 4k + m$, where $m = 0, 1, 2, 3$.
Let $g : V(P) \to A = \{1, -1, i, -i\}$ be a map defined as follows:
For $1 \leq j \leq 4k$,

$$\begin{aligned} g(v_j) &= i, \text{ if } j \equiv 0 \pmod 4 \\ &= 1, \text{ if } j \equiv 1 \pmod 4 \\ &= -1, \text{ if } j \equiv 2 \pmod 4 \\ &= -i, \text{ if } j \equiv 3 \pmod 4. \end{aligned}$$

Likewise, we define a function h as

$$\begin{aligned} h(v_j) &= i, \text{ if } j \equiv 0 \pmod 4 \\ &= -1, \text{ if } j \equiv 1 \pmod 4 \\ &= -i, \text{ if } j \equiv 2 \pmod 4 \\ &= 1, \text{ if } j \equiv 3 \pmod 4. \end{aligned}$$

Observe that g and h label only the first $4k$ vertices on the path.

3. Group A cordiality of uniform $t-$ply

In what follows, we shall prove that every uniform $t-$ply $P_t^l(u, w)$ with each path of length at least 5 is group A cordial by giving an explicit labeling in each case.

Theorem 3.1. *Every uniform $t-$ply with each path of length at least 5 is group A cordial.*

Proof. A uniform $t-$ply is of type l where l is $1, 2, 3$ or 4. For each type or value of l, we exhibit a labeling for $P_t^l(u, w)$ which is group A cordial. We achieve this by carrying out the labeling in three stages which will be described in the sequel below.
For $l = 1, 3$; we will be taking the number of paths congruent modulo 2 i.e. we take $t = 2s + r$; where $r = 0, 1$. We use the function g on the intermediate vertices $v(x, 1), v(x, 2), \dots, v(x, 4k_x)$ of the first s paths i.e. for $1 \leq x \leq s$ and the function h on the intermediate vertices of the paths from $s + 1$ to $2s$.
For $l = 2, 4$; we will be taking the number of paths congruent modulo 4; i.e. we take $t = 4s + r$; where $r = 0, 1, 2, 3$. We use the function g on the first $2s$ paths i.e. for $1 \leq x \leq 2s$ and the function h on the intermediate vertices of the next 2s paths i.e. $2s < x \leq 4s$. This is stage I of the labeling. In this stage the labels for the vertices and the edges are equitably distributed.
There now remain $l - 1$ vertices for labeling on each of the paths that have received labelings in stage I. We label these $2s(l - 1)$ or $4s(l - 1)$ vertices as the case maybe in stage II which will be discussed casewise. In stage II of the labeling we will ensure that the vertex labels and the edge labels are equitably distributed.
Next, we look at the last r paths which have not received any labels so far. In stage III, we label all vertices on such paths. We label the vertex u with -1 and w with 1, in all cases. In each case the resulting labeling for $P_t^l(u, w)$ will be denoted by α_{lr}.

Case 1: $l = 1$

In this case $n_x \equiv 0 \pmod 4$. Let $n_x = 4k_x$ and $t = 2s + r$; $r = 0, 1$. As mentioned above in the strategy, we use the function g on the first s paths and the function h on the next s paths in stage I of the labeling. Both functions label vertices on groups of 4 on each path, hence on the $2s$ paths dealt with so far, there are no intermediate vertices left for labeling. We go directly to stage III in this case and deal with the r paths that are untouched so far.

Case 1(a): $r = 0$

If $r = 0$, there will be no path remaining to be the labeled in stage III. The resulting labeling is α_{10}, for which $v_{\alpha_{10}}(-1) = v_{\alpha_{10}}(1) = v_{\alpha_{10}}(i) + 1 = v_{\alpha_{10}}(-i) + 1$ and $e_{\alpha_{10}}(0) = e_{\alpha_{10}}(1)$.

Case 1(b): $r = 1$

On the one remaining path, we use the function g. We get the labeling α_{11}, for which $v_{\alpha_{11}}(-1) = v_{\alpha_{11}}(1) = v_{\alpha_{11}}(i) + 1 = v_{\alpha_{11}}(-i) + 1$ and $e_{\alpha_{11}}(1) = e_{\alpha_{11}}(0) + 1$.

Case 2: $l = 2$

Now $n_x \equiv 1 \pmod 4$. Hence we take $n_x = 4k_x + 1$ and $t = 4s + r$; $r = 0, 1, 2, 3$. In stage I, we use the function g on the first $2s$ of the paths and h on the next $2s$ paths. This leaves 1 vertex on each of the $4s$ paths. We label these vertices using a function f defined below:

$$
\begin{aligned}
f(v(x, n_x)) &= i, \text{ for } 1 \leq i \leq s \\
&= -1, \text{ for } s < i \leq 2s \\
&= -i, \text{ for } 2s < i \leq 3s \\
&= 1, \text{ for } 3s < i \leq 4s.
\end{aligned}
$$

We now proceed to stage III and consider the vertices on the remaining r paths.

Case 2(a): $r = 0$
We have obtained the labeling α_{20}, for which $v_{\alpha_{20}}(-1) = v_{\alpha_{20}}(1) = v_{\alpha_{20}}(i) + 1 = v_{\alpha_{20}}(-i) + 1$ and $e_{\alpha_{20}}(0) = e_{\alpha_{20}}(1)$.
Case 2(b): $r = 1$
On the remaining path, we use g on the first $4k_t$ vertices and label the last vertex by i. The resulting labeling is α_{21}, for which $v_{\alpha_{21}}(-1) = v_{\alpha_{21}}(1) = v_{\alpha_{21}}(i) = v_{\alpha_{21}}(-i)+1$ and $e_{\alpha_{21}}(1) = e_{\alpha_{21}}(0)$.
Case 2(c): $r = 2$
On the two remaining paths, we use g and we label the last vertex on one of these paths by i and the other by $-i$. The resulting labeling is α_{22}, for which $v_{\alpha_{22}}(-1) = v_{\alpha_{22}}(1) = v_{\alpha_{22}}(i) = v_{\alpha_{22}}(-i)$ and $e_{\alpha_{22}}(1) = e_{\alpha_{22}}(0)$.
Case 2(d): $r = 3$
Out of the three remaining paths, on two of the paths we use the labeling as described in Case 2(c) and on the third we use the function h followed by the label '$1'$' for the last vertex. We thus obtain the labeling α_{23}, for which $v_{\alpha_{23}}(1) = v_{\alpha_{23}}(-1) + 1 = v_{\alpha_{23}}(i) + 1 = v_{\alpha_{23}}(-i) + 1$ and $e_{\alpha_{23}}(1) = e_{\alpha_{23}}(0)$.
Case 3: $l = 3$
Here we take $n_x = 4k_x + 2$ and $t = 2s + r,\ r = 0, 1$.
On the first s paths where g has been used, we define
$f(v(x, n_x - 1)) = i$ and $f(v(x, n_x)) = 1$. On the second set of s paths i.e. for $s < x \leq 2s$ where we have used h, we define,
$f(v(x, n_x - 1)) = -1$ and $f(v(x, n_x)) = -i$.
Case 3(a): $r = 0$
There are no vertices remaining for stage III of the labeling. Hence we obtain the labeling α_{30}, for which $v_{\alpha_{30}}(1) = v_{\alpha_{30}}(-1) = v_{\alpha_{30}}(i) + 1 = v_{\alpha_{30}}(-i) + 1$ and $e_{\alpha_{30}}(1) = e_{\alpha_{30}}(0)$.
Case 3(b): $r = 1$
On the one remaining path, use g followed by the labels i and $-i$ to get the labeling α_{31} of $P_t(u, v)$. Thus we have, $v_{\alpha_{31}}(-1) = v_{\alpha_{31}}(1) = v_{\alpha_{31}}(i) = v_{\alpha_{31}}(-i)$ and $e_{\alpha_{31}}(0) = e_{\alpha_{31}}(1) + 1$.
Case 4: $l = 4$
We take $n_x = 4k_x + 3$ and $t = 2s + r,\ r = 0, 1, 2$.
For $1 \leq x \leq s$, paths where g has been used, we define
$f(v(x, n_x - 2)) = i$, $f(v(x, n_x - 1)) = -1$ and $f(v(x, n_x)) = -i$.
On the second set of s paths i.e. for $s < x \leq 2s$, where we still have used g, we define
$f(v(x, n_x - 2)) = 1$, $f(v(x, n_x - 1)) = -i$ and $f(v(x, n_x)) = -1$.
For $2s < x \leq 3s$, where h has been used we define
$f(v(x, n_x - 2)) = 1$, $f(v(x, n_x - 1)) = -i$ and $f(v(x, n_x)) = i$.
For $3s < x \leq 4s$, paths where we still have used h, we define
$f(v(x, n_x - 2)) = 1$, $f(v(x, n_x - 1)) = -1$ and $f(v(x, n_x)) = i$.
Case 4(a): $r = 0$
We have obtained the labeling α_{40}, for which $v_{\alpha_{40}}(1) = v_{\alpha_{40}}(-1) = v_{\alpha_{40}}(i) + 1 = v_{\alpha_{40}}(-i) + 1$ and $e_{\alpha_{40}}(0) = e_{\alpha_{40}}(1)$.
Case 4(b): $r = 1$
We use g on the first $4k_t$ vertices and on the last three we use $i, -i, 1$. The labeling obtained is α_{41}, for which $v_{\alpha_{41}}(1) = v_{\alpha_{41}}(-1) + 1 = v_{\alpha_{41}}(i) + 1 = v_{\alpha_{41}}(-i) + 1$ and $e_{\alpha_{41}}(0) = e_{\alpha_{41}}(1)$.

Case 4(c): $r = 2$

On $(t-1)^{th}$ path we use g followed by $1, i, -i$; and on the t^{th} path we use h followed by $i, -1, -i$. The labeling obtained is α_{42}, for which $v_{\alpha_{42}}(-1) = v_{\alpha_{42}}(1) = v_{\alpha_{42}}(i) = v_{\alpha_{42}}(-i)$ and $e_{\alpha_{42}}(0) = e_{\alpha_{42}}(1)$.

Case 4(d): $r = 3$

On $(t-2)^{th}$ path we use g followed by $1, i, -i$; on $(t-1)^{th}$ path we use h followed by $i, -1, -i$; and on the t^{th} path we use g followed by $i, -1, 1$. The labeling thus obtained is α_{43}, for which $v_{\alpha_{43}}(1) = v_{\alpha_{43}}(-1) = v_{\alpha_{43}}(i) = v_{\alpha_{43}}(-i) + 1$ and $e_{\alpha_{43}}(1) = e_{\alpha_{43}}(0)$. □

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