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*Original article*

# Upper semi-Advanced Mappings in a Topological Spaces

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**Abstract -** *In this paper, we introduce a new class of open sets is called $\overline{is}$-open set . Also, we present the notion of $\overline{is}$-continuous, $\overline{is}$-open, $\overline{is}$-irresolute, $\overline{is}$-totally continuous, and $\overline{is}$-contra-continuous mappings, and we investigate some properties of these mappings. Furthermore, we introduce some $\overline{is}$- separation axioms and the mappings are related with $\overline{is}$-separation axioms.*

**Keywords -** $\overline{is}$*-open set,* $\overline{is}$*-continuous,* $\overline{is}$*-open mapping,* $\overline{is}$*-irresolute,* $\overline{is}$*-totally continuous,* $\overline{is}$*-contra-continuous.*

## 1. Introduction and Preliminaries

Many researchers looking for and create many notions in topology and its applications to make new theorems that it was not prove before, we need to present another once deserving new theorems and it will related with last kinds of open and closed sets that we mentioned them in the references. The main aim of this paper is to introduce and study a new class of open sets which is called $\overline{is}$-open set and we present the notion of $\overline{is}$-continuous mapping, $\overline{is}$-totally continuity mapping and some weak separation axioms for $\overline{is}$-open sets. Furthermore, we investigate some properties of these mappings. In section 2, we define $\overline{is}$-open set, and we investigate the relationship with, open set, semi-open set, α-open set and i-open set. In section 3, we present the notion of $\overline{is}$-continuous mapping, $\overline{is}$-open mapping, is-irresolute mapping and is-homeomorphism mapping, and we investigate the relationship between $\overline{is}$-continuous mapping with some types of continuous mappings, the relationship between is-open mapping, with some types of open mappings and the relationship between $\overline{is}$-irresolute mapping with some types of irresolute mappings. Further, we compare is-homeomorphism with i-homeomorphism. In section 4, we introduce new class of mappings called $\overline{is}$-totally continuous mapping and we introduce $\overline{is}$-contra-continuous mapping and $\overline{is}$-contra-continuous mapping. Further, we study some of their basic properties. Finally, in section 5, we introduce a new weak of separation axioms for $\overline{is}$-open set and we conclude $\overline{is}$-continuous mappings effective with $\overline{is}$-separation axioms. Throughout this paper, we denote the topology spaces $(X,\tau)$ and $(Y,\sigma)$ by $X$ and $Y$. We said the following characterizations, and definitions. The closure (resp. Interior) of a subset $A$ of a topology Space $X$ is denoted by $CL(X)$ (resp.$Int(A)$).

**Definition 1.1** A subset $A$ of a topological space $X$ is said to be:
(i) semi-open set, if $A \subseteq Int(Cl(A))$ [7]
(ii) $\alpha$–open set, if $A \subseteq Int(Cl(Int(A))$ [8]
(iii) i-open set, if $A \subseteq Cl(A \cap O)$, where $\exists O \in \tau$ and $O \neq X, \emptyset$ [15]
(iv) i$\alpha$-open set $A \subseteq Cl(A \cap O)$, where $\exists O \in \alpha$ - open sets and $O \neq X, \emptyset$ [18]
(v) clopen set, if $A$ is open and closed

The family of all semi-open (resp. α-open , i-open, iα-open, clopen) sets of a topological space is denoted by $SO(X)$(resp. $\alpha O(X)$, $iO(X)$, $i\alpha O(X)$, $CO(X)$). The complement of open (resp. semi-open, α-open, i-open, iα-open) sets of a topological space $X$ is called closed (resp. semi-closed, α-closed, i-closed, iα-open) sets.

**Definition 1.2** Let $X$ and $Y$ be a topological spaces, a mapping $f: X \to Y$ is said to be:
(i) semi-continuous [7] if the inverse image of every open subset of $Y$ is semi-open set in $X$.
(ii) α-continuous [1] if the inverse image of every open subset of $Y$ is an α-open set in $X$.
(iii) i-continuous [15] if the inverse image of every open subset of $Y$ is an i-open set in $X$.
(iv) iα-continuous [15] if the inverse image of every open subset of $Y$ is an iα-open set in $X$.
(v) totally (perfectly) continuous [16] if the inverse image of every open subset of $Y$ is clopen set in $X$.
(vi) irresolute [11] the inverse image of every semi- open subset of $Y$ is semi- open subset in $X$.

(vii) α-irresolute [14] if the inverse image of every α-open subset of $Y$ is an α-open subset in $X$ .
(viii) semi α-irresolute [17] if the inverse image of every α-open subset of $Y$ is semi-open subset in $X$.
(ix) i-irresolute [15] if the inverse image of every i-open subset of $Y$ is an i-open subset in $X$.
(x) contra-continuous [4] if the inverse image of every open subset of $Y$ is closed set in $X$.
(xi) contra semi continuous [5] if the inverse image of every open subset of $Y$ is semi-closed set in $X$.
(xii) contra α-continuous [12] if the inverse image of every of open subset of $Y$ is an α-closed set in $X$.
(xiii) semi-open [6] if the image of every open set in $X$ is semi-open set in $Y$.
(xiv) α-open [1] if the image of every open set in $X$ is an α-open set in $Y$.
(xv) i-open [15] if the image of every open set in $X$ is an i-open set in $Y$.

**Definition 1.3** Let $X$ and $Y$ be a topology space, a bijective mapping $f: X \to Y$ is said to be iα [18](resp. α-open [1] , i-open[15]) homeomorphism if $f$ is an iα(resp. α-open , i-open ) continuous and iα (resp. α-open , i-open ) open mappings .

**Lemma 1.4** Every open (resp. semi–open, α-open , i-open) set in a topological space is an iα-open set [18].

## 2. Sets that are $\overline{is}$-open sets and some relations with other important sets

In this section, we introduce a new class of open sets which is called $\overline{is}$open set and we investigate the relationship with, open set, semi-open set, α-open set and i-open set.

**Definition 2.1** A subset $A$ of the topological space $X$ is said to be $\overline{is}$-open set if $A \subseteq Cl(A \cap O)$ where $\exists O \neq \emptyset$, $O \subseteq SO(X)$. The complement of the $\overline{is}$-open set is called is-closed. We denote the family of all is-open sets of a topological space $X$ by US$O$($X$).

**Example 2.2** Let X={1,3,5}, τ={∅,{5},{1,5},$X$},$SO$($X$)=$\alpha O$($X$)={∅,{1},{1,5}, {3,5}, $X$} and $USO$($X$) = iα$O$($X$)={∅,{1} ,{3},{5},{1,3},{1,5},{3,5},$X$}. Note that $SO$($X$) =$\alpha O$($X$)⊂ $USO$($X$) = iα$O$($X$).

**Example 2.3** Let $X$={1,2,3,4}, τ={∅,{1,3},{2,4}, $X$}=$SO$($X$) $USO$($X$)={∅,{1},{2},{3},{4},{1,4},{2,3},$X$}.

**Example 2.4** Let $X$={,2,3}, τ={∅,{1},{2},{1,2},$X$} $USO$($X$)={∅,{1},{2},{1,2},{1,3},{2,3},$X$}.

**Theorem 2.5** Every iα-open set in any topological space is an $\overline{is}$-open set.

**Proof.** Let $X$ be any topological space and $A \subseteq X$ be any iα-open set. Therefore, $A \subseteq Cl(A \cap O)$ here $\exists O \in \alpha O(X)$ and $O \neq X, \emptyset$ . Since, every α-open is semi-open, that implies $\exists O \in SO(X)$ . We obtain A ⊆ Cl(A ∩ O), where $\exists O \in SO(X)$ and $O \neq X, \emptyset$. Thus, $A$ is an $\bar{}is$-open set■ The following example shows that is-open set need not be i-open set.

**Example 2.6** Let $X$ ={1,2,3,4}, τ={∅,{4},$X$},$iO$($X$)={∅,{4},{1,4},{2,4},{3,4},{1,2,4}, {1,3,4} ,{2,3,4} ,$X$}⊂$US$($X$)= {∅, {1} ,{2},{3},{4},{1,2},{1,3,},{1,4},{2,3},{2,4},{3,4},{1,2,3} ,{1,2,4},{1,3,4},{2,3,4},$X$}.

**Corollary** 2.7 Every open set in any topological space is an $\overline{is}$-open set.

**Proof.** Let $X$ be any topological space and $A$⊆$X$ be any $i\alpha$-open set ,Since every open is set is iα-open set by **lemma 1.4**, then A is iα- open set, and hence $X$ $\overline{is}$–open set by Theorem 2.5, Hence, A is is-open set■

**Corollary** 2.8 Every semi-open set in any topological space is an $\overline{is}$-open set.

**Proof.** Let N be semi-open, then $N \subseteq Int(Cl(N))$. By definition of $\overline{is}$–open set N⊆ $Cl(N \cap Int(Cl(N))$ , Since $\exists N \neq X, \emptyset \in SO(X)$ . Therefore N is $\overline{is}$–open set■

**Corollary** 2.8 Every $\alpha$-open set in any topological space is an $\overline{is}$-open set.

**Proof.** Same the proof of above theorem Because every $\alpha$-open is semi-open set■

**Theorem** 2.9 Every i-open set in any topological space is an $\overline{is}$-open set.

**Proof.** Let $L$ be an i-open set. Since $L$ is i-open set, then we have $L \subseteq Cl(O \cap L)$, such that exist $O \neq \emptyset, X \in \tau$, Hence $O \in SO(X)$, because every open set is semi-open set. Therefore $L$ is $\overline{is}$-open set■

**Remark 2.10**

(i) The intersection of is-open sets is not necessary to be $i\alpha$-open set as shown in the example 2.4.

(ii) The union of iα-open set is not necessary to be $\bar{}is$-open set as shown in the example 2.3.

## 3. Mappings that are $\overline{is}$continuous and $\overline{is}$-homeomorphism

In this section, we present the notion of $\overline{is}$-continuous mapping, $\overline{is}$-irresolute mapping and $\overline{is}$-homeomorphism mapping.

**Definition 3.1** Let $X$, $Y$ be a topological spaces, a mapping $f: X \to Y$ is said to be $\overline{is}$-continuous, if the inverse image of every open subset of $Y$ is $\overline{is}$-open set in $X$.

**Example 3.2** Let $X$=$Y$={0,2,4}, τ={∅,{2},{4},{2,4},$X$}, $USO$(X)={∅,{2},{4},{0,2} ,{0,4},{2,4},$X$} and σ={∅,{0,2},$X$}. Clearly, the identity mapping $f: X \to Y$ is an is-continuous.

**Theorem 3.3** Every iα-continuous mapping is $\overline{is}$-continuous.

**Proof.** Let $f: X \to Y$ be an iα-continuous mapping and $V$ be any open subset in $Y$. Since, $f$ is an iα-continuous, then $f^{-1}(V)$ is an iα-open set in $X$. Since, every iα-open set is an $\overline{is}$-open set by Theorem 2.5, then $f^{-1}(V)$ is an $\overline{is}$-open set in $X$. Therefore, $f$ is an $\overline{is}$-continuous■

**Corollary 3.4** Every iα-continuous mapping is $\overline{is}$-continuous.

**Corollary 3.5** Every smei-continuous mapping is $\overline{is}$-continuous.

**Corollary 3.6** Every α-continuous mapping is $\overline{is}$-continuous.

**Corollary 3.7** Every i-continuous mapping is $\overline{is}$-continuous.

**Remark 3.8** The corollaries **3.4, 3.5, 3.6** and **3.7** are the same proof of **theorem 3.3, and** the next example shows that iα-continuous mapping need not be continuous, semi-continuous, α-continuous and i-continuous mappings.

**Example 3.9** Let $X$={n,m,r} and $Y$={1,3,5}, τ={∅,{m},$X$}, $SO(X)$=$\alpha O(X)$=$iO(X)$= {∅,{m}, {n,m} ,{m,r} ,$X$} ,$USO(X)$={∅,{n},{m},{r},{n,m},{m,r},{n,r},$X$}, σ={∅,{3},$Y$}. A mapping $f: X \to Y$ is defined by $f${n}={3},$f${m}={1},$f${r}={5}. Clearly, $f$ is an $\overline{is}$-continuous, but $f$ is not continuous, $f$ is not semi-continuous, $f$ is not α-continuous and $f$ is not i-continuous because for open subset {2}, $f^{1}${3}={n}∉τ and $f^{-1}${3}={r}∉$SO$(X)=$\alpha O(X)$=$iO(X)$.

**Definition 3.10** Let $X$ and $Y$ be a topological space, a mapping $f: X \to Y$ is said to be $\overline{is}$-open, if the image of every open set in $X$ is an $\overline{is}$-open set in $Y$.

**Example 3.11** Let $X$=$Y${h,r,k}, τ={∅,{r,k},$X$}, σ={∅,{h},$Y$}, and $iSO(Y)$={ ∅,{h},{r}, {k},{h,r},{h,k},{r,k},$Y$}. Clearly, the identity mapping $f: X \to Y$ is an is-open.

**Theorem 3. 12** Every iα-open mapping is $\overline{is}$-open.

**Proof.** Let $f: X \to Y$ be an iα-open mapping and $V$ be any open set in $X$. Since, $f$ is an iα-open, then $f(V)$ is an iα-open set in $Y$. Since, every iα-open set is an $\bar{}is$-open set by Theorem 2.5, then $f(V)$is an $\bar{}is$-open set in $Y$. Therefore, $f$ is $\overline{is}$-open ■

**Corollary 3.13** Every semi-open mapping is $\overline{is}$-open.

**Corollary 3.14** Every α-open mapping is $\overline{is}$-open.

**Corollary 3.15** Every i-open mapping is $\overline{is}$-open.

**Remark 3.16** The corollaries **3.13**, **3.14,** and **3.15** are the same the proof of theorem next example shows that $\overline{is}$-open mapping need not be (resp. semi, α, i ) -open mappings.

**Example3.17** Let $X=Y=\{5,6,7\}$, $\tau=\{\emptyset,\{7\},X\}$ $\sigma=\{\emptyset,\{5\},Y\}$, $SO(Y)=\alpha O(Y)=iO(Y)$, $=\{\emptyset, \{5\},\{5,6\},\{5,7\},Y\}$, $USO(Y)=\{\emptyset,\{5\}, \{6\},\{7\},\{5,6\},\{5,7\},\{6,7\},Y\}$. A mapping $f: X \to Y$ is defined by $f(5)=6$, $f(6)=5$, $f(7)=7$. Clearly, $f$ is $\overline{is}$-open, but $f$ is not open, semi-open, α-open and $f$ is not i-open because for open subset$\{7\}$, $f^{-1}\{7\}=\{7\}\notin\sigma$ and $f^{-1}\{7\}=\{7\}\notin SO(Y)=\alpha O(Y)=iO(Y)$ .

**Definition 3.18** Let $X$ and $Y$ be a topological space, a mapping $f: X \to Y$ is said to be $\overline{is}$- irresolute, if the inverse image of every $\overline{is}$- open subset of $Y$ is $\overline{is}$-open subset in $X$.

**Example 3.19** Let $X=Y=\{d,e,f\}$, $\tau=\{\emptyset,\{e\},X\}$,$iSO(X)= \{\emptyset,\{d\}, \{e\},\{f\} ,\{d,e\}, \{d,f\} ,\{e,f\},X\}$, $\sigma=\{\emptyset,\{f\},Y\}$ and $iSO(Y)=\{\emptyset,\{d\},\{e\},\{f\},\{d,e\},\{d,f\},\{e,f\},Y\}$. Clearly, the identity mapping $f: X \to Y$ is $\overline{is}$-irresolute.

**Theorem 3.20** Every iα-irresolute mapping is $\overline{is}$-irresolute.

**Proof.** Let $f: X \to Y$ be an iα-irresolute mapping and $V$ be any iα-open set in $Y$ and Since every iα-open set is an $\overline{is}$ – open set Theorem 2.5, then is $\overline{is}$- open set. Since, $f$ is an iα-irresolute, then $f^{-1}(V)$ is $\overline{\text{is}}$-open set in $X$. Since every iα-open set is $\overline{\text{is}}$ –open set. Hence, $\overline{\text{is}}$-open set in $X$ by Theorem 2.5. Therefore, $f$ is $\overline{\text{is}}$-irresolute■

**Corollary 3.14** Every irresolute mapping is $\overline{\text{is}}$-irresolute.

**Corollary 3.15** Every α-irresolute mapping is $\overline{\text{is}}$-irresolute.

**Corollary 3.16** Every i-irresolute mapping is $\overline{\text{is}}$-irresolute.

**Remark 3.17** The corollaries of **3.14, 3.15, and 3.16** are the same proof of Theorem **3.13, and the** next example shows that is-irresolute mapping need not be irresolute, semi α-irresolute, α-irresolute and i-irresolute mappings.

**Example 3.18** Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},X\}$, $SO(X)=\alpha O(X) =iO(X)=\{\emptyset,\{a\} ,\{a,b\}, \{a,c\},X\}$, $USO(X)=\{\emptyset,\{a\} ,\{b\} ,\{c\} ,\{a,b\},\{a,c\},\{b,c\},X\}$, $\sigma=\{\emptyset,\{c\},Y\}$, $SO(Y)=\alpha O(Y)= iO(Y)=\{\emptyset,\{c\}, \{a,c\}, \{b,c\},Y\}$ and $iSO(Y)=\{\emptyset,\{a\} ,\{b\},\{c\}$, S$\{a,b\},\{a,c\},\{b,c\},Y\}$. Clearly , the identity mapping $f: X \to Y$ is an $\bar{}is$-irresolute, but $f$ is not irresolute, $f$ is not α-irresolute, $f$ is not semi α-irresolute and $f$ is not i-irresolute because for semi-open, α-open and i-open subset $\{c\}$, $f^{-1}\{c\}=\{c\} \notin SO(X)=\alpha O(X)=iO(X)$.

**Theorem 3.19** Every iα-irresolute mapping is $\overline{is}$-continuous.

**Proof.** Let $f: X \to Y$ be an iα-irresolute mapping and $V$ be any open set in $Y$. Since, every iα-open set is $\overline{is}$- open set. Since, $f$ is an iα-irresolute, then $f^{-1}(V)$ is $\overline{\text{is}}$-open set in $X$. Therefore fis $\overline{\text{is}}$-continuous■ The converse of the above proposition need not be true as shown in the following example

**Example 3.20** Let $X=Y=\{a,b,c\}$,$\tau=\{\emptyset, \{a,b\}, X\}$,$USO(X)=\{\emptyset ,\{a\},\{b\},\{a,b\} ,\{a,c\} ,\{b,c\} ,X\}$ ,$\sigma= \{a,c\} ,Y\}$ and $iSO(Y)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},Y\}$. Clearly, the identity mapping $f: X \to Y$ is $\overline{is}$-continuous, but $f$ is not $\overline{is}$-irresolute because for $\overline{is}$-open set $\{c\}$, $f^{-1}\{c\}=\{c\} \notin USO(X)$.

**Definition 3.21** Let $X$ and $Y$ be a topological space, a bijective mapping $f: X \to Y$ is said to be $\overline{is}$-homeomorphism if $f$ is $\overline{is}$-continuous and $\bar{}is$-open.

**Theorem 3.22** If $f: X \to Y$ is an $i\alpha$-homomorphism, then $f: X \to Y$ is $\overline{is}$-homomorphism.

**Proof.** Let $f: X \to Y$ is an $i\alpha$-homomorphism , Since every iα -continuous mapping is $\overline{is}$-continuous by theorem 3.3. Also, since every iα-open mapping is $\overline{is}$-open by theorem 3.12. Furthermore, $f$ $\overline{is}$ bijective. Therefore, $f$ is $\overline{is}$-homomorphism ■

**Corollary 3.23** If $f: X \to Y$ is an i-homomorphism, then $f: X \to Y$ is $\overline{is}$-homomorphism.

**Corollary 3.24** If $f: X \to Y$ is α-homomorphism, then $f: X \to Y$ is $\overline{is}$-homomorphism.

**Remark 2.25** The next example shows that $\overline{is}$-homomorphism need not to be $s$-homomorphism, and α-homomorphism

**Example 3.26** Let $X$=$Y$={a,b,c},τ={∅,{a},$X$}, α($X$)=$iO(X)$={∅,{a}, {a,b}, {a,c},$X$}, USO($X$)= {∅, {a}, {b},{c} ,{a,b} ,{a,c}, {b,c},$X$},σ ={∅,{b},$Y$ }, α($Y$)=$iO(Y)$={∅,{b},{a,b} ,{b,c},$Y$} and USO($Y$)={∅,{a},{b},{c}, {a,b}, {a,c}, {b,c},$Y$}. Clearly, the identity mapping $f: X \to Y$ is $\overline{is}$-homomorphism, but it is not i-homomorphism and it is not α-homomorphism because $f$ is not i-continuous and α-continuous , since for open subset {b}, $f^{-1}${b}={b} ∉$iO$($X$)= α($X$).

## 4. Mappings that are $\overline{is}$-totally continuous and $\overline{is}$-contra-continuous

In this section, we introduce new classes of mappings called $\overline{is}$-totally continuous and $\overline{is}$-contra-continuous.

**Definition 4.1** Let $X$ and $Y$ be a topological space, a mapping $f: X \to Y$ is said to be $\overline{is}$-totally continuous, if the inverse image of every $\overline{is}$-open subset of $Y$ is clopen set in $X$.

**Example 4.2** Let $X$=$Y$={a,b,c},τ={∅,{a},{b,c},$X$}, σ={∅,{a}, $Y$} and USO($Y$)= {∅,{a},{b} ,{c}, {a,b} ,{a,c},{b,c},$Y$}. The mapping $f: X \to Y$ is defined by $f${a}={a} ,$f${b}=$f${c}=b. Clearly, $f$ is $\overline{is}$-totally continuous mapping.

**Theorem 4.3** Every iα-totally continuous mapping is $\overline{is}$-totally continuous.

**Proof.** Let f: X → Y be iα-totally continuous and $V$ be any open set in $Y$. Since, every open set is an iα-open set, then $V$ is an iα-open set in $Y$. Since, $f$ is an iα-totally continuous mapping, then $f^{-1}$($V$) is clopen set in $X$. Therefore, $f$ is totally continuous■ The converse of the above theorem need not be true as shown in the following example.

**Example 4.4** Let $X$=$Y$={a,b,c},τ={∅,{a},{b,c},$X$} σ={∅,{a},$Y$} and USO($Y$)= {∅,{a}, {b},{c} ,{a,b}, {a,c} ,{b,c},$Y$}. Clearly, the identity mapping f: X → Y is totally continuous, but $f$ is not $\overline{is}$-totally continuous because for $\overline{is}$-open set {a,c}, $f^{-1}${a,c}={a,c}∉$CO$($X$).

**Corollary 4.5** Every totally continuous mapping is $\overline{is}$-totally continuous.

**Theorem 4.6** Every $\overline{is}$-totally continuous mapping is $\overline{is}$-irresolute.

**Proof.** Let f: X → Y be is-totally continuous and $V$ be an is-open set in $Y$. Since, $f$ is an is-totally continuous mapping, then $f^{-1}$($V$) is clopen set in $X$, which implies $f^{-1}$($V$) open, it follow $f^{-1}$($V$) is-open set in $X$. Therefore, $f$ is an is-irresolute■ The converse of the above theorem need not be true as shown in the following example

**Example 4.7** Let $X$=$Y$={1,2,3},τ={∅,{2},$X$}, USO($X$)={∅,{1},{2},{3},{1,2} ,{1,3} ,{2},{3}, {1,2}, {1,3},{2,3},$X$} σ={∅ ,{1,2},$Y$} and USO($Y$)={∅, {1},{2} ,{1,2},{1,3} ,{2,3},$Y$}. Clearly, the identity mapping $f: X \to Y$ is $\overline{is}$-irresolute, but $f$ is not $\overline{is}$-totally continuous because for $\overline{is}$-open subset{1,3}, $f^{-1}${1,3}={1,3}∉$CO$($X$).

**Theorem 4.8** The composition of two $\overline{is}$-totally continuous mapping is also $\overline{is}$-totally continuous.

**Proof.** Let f: X → Y and $g: Y \to Z$ be any two iα-totally continuous. Let $V$ be any $\overline{is}$-open in $Z$. Since, $g$ is an is-totally continuous, then $g^{-1}$($V$) is clopen set in $Y$, which implies $f^{-1}$($V$) open set, it follow $f^{-1}$($V$) $\overline{is}$-open set. Since, $f$ is an is-totally continuous, then $f^{-1}$( $g^{-1}$($V$))=($g$ o $f$ )$^{-1}$($V$) is clopen in $X$. Therefore, $g \circ f: Y \to Z$ is $\overline{is}$-totally continuous■

**Theorem 4.9** If $f: X \rightarrow Y$ be $\overline{is}$-totally continuous and $g: Y \rightarrow Z$ be $\overline{is}$-irresolute, then $g \circ f: Y \rightarrow Z$ is $\overline{is}$-totally continuous.

**Proof.** Let $f: X \rightarrow Y$ be $\overline{is}$-totally continuous and $g: Y \rightarrow Z$ be $\overline{is}$-irresolute . Let *V* be $\overline{is}$-open set in *Z*. Since, *g* is an is-irresolute, then $g^{-1}(V)$ is an $\overline{is}$-open set in *Y*. Since, *f* is $\overline{is}$-totally continuous, then $f^{-1}((g^{-1}(V))=(g \text{ o } f)^{-1}(V)$ is clopen set in *X*. Therefore, $g \circ f: Y \rightarrow Z$ is $\overline{is}$-totally continuous■

**Theorem 4.10** If $f: X \rightarrow Y$ is $\overline{is}$-totally continuous and g: Y → Z is $\overline{is}$-continuous, then $g \circ f: Y \rightarrow Z$ is totally continuous.

**Proof.** Let f: X → Y be $\overline{is}$-totally continuous and g: Y → Z is $\overline{is}$-continuous . Let *V* be an open set in *Z*. Since, *g* is $\overline{is}$-continuous, then $g^{-1}(V)$ is $\overline{is}$-open set in *Y*. Since, *f* is $\overline{is}$-totally continuous, then $f^{-1}(g^{-1}(V))=(g \text{ o } f)^{-1}(V)$ is clopen set in *X*. Therefore, $g \circ f: Y \rightarrow Z$ is totally continuous■

**Definition 4.11** Let *X*, *Y* be a topological spaces, a mapping $f: X \rightarrow Y$ is said to be $\overline{is}$-contra-continuous, if the inverse image of every open subset of *Y* is $\overline{is}$-closed set in *X*.

**Example 4.12** Let *X*=*Y*={a,b,c},τ={∅,{a},*X*}, σ={∅,{c},*Y*} and US*O*(*X*)= {∅,{a},{b},{c} ,{a,b}, {a,c} , {b,c},*X*}. Clearly, the identity mapping $f: X \rightarrow Y$ is $\overline{is}$-contra-continuous.

**Theorem 4.13** Every i$\alpha$-contra-continuous mapping is $\overline{is}$-contra-continuous.

**Proof.** Let f: X → Y be i$\alpha$-contra continuous mapping and *V* any open set in *Y*. Since, *f* is an i$\alpha$-contra continuous, then $f^{-1}(V)$ is an i$\alpha$ -closed sets in *X*. Since, every i$\alpha$-closed set is i$\alpha$ -closed set, then $f^{-1}(V)$ is i$\alpha$ -closed set in *X*. Therefore, *f* is i$\alpha$ -contra-continuous■ Similarly we have the following results.

**Corollary 4.14** Every contra-continuous mapping is an is-contra-continuous.

**Corollary 4.15** Every i-contra-continuous mapping is an is-contra-continuous.

**Corollary 4.16** Every contra semi-continuous mapping is an i-contra-continuous.

**Corollary 4.17** Every contra α-continuous mapping is an is-contra-continuous.

**Remark 4.18** The following example shows that iα-contra-continuous mapping need not be contra-continuous, contra semi-continuous, contra-α-continuous and i-contra-continuous mappings.

**Example4.19** Let *X*=*Y*={a,b,c},τ={∅,{a,c},*X*}, *USO*(*X*)={∅,{a},{c} ,{a,b},{a,c} ,{b,c}, *X*} and σ={∅,{c},*Y*}. Clearly, the identity mapping $f: X \rightarrow Y$ is $\overline{is}$- contra continuous, but *f* is not contra-continuous, *f* is not contra semi-continuous, *f* is not contra α-continuous because for open subset $f^{-1}${c}={c} is not closed in *X*, $f^{-1}${c} ={c}is not semi-closed in *X* and $f^{-1}${c}={c}is not α-closed in *X*.

**Theorem 4.20** Every totally continuous mapping is $\overline{is}$-contra continuous.

**Proof.** Let $f: X \rightarrow Y$ be totally continuous and *V* be any open set in *Y*. Since, *f* is an totally continuous mapping, then $f^{-1}(V)$ is clopen set in *X*, and hence closed, it follows $\overline{is}$-closed set. Therefore, *f* is $\overline{is}$-contra-continuous■ The converse of the above theorem need not be true as shown in the following example

**Example 4.21** Let *X*=*Y*={a,b,c},τ={∅,{c},*X*}, σ={∅,{a},*Y*} and US*O*(*X*)={∅,{a},{b}, {c},{a,b} ,{a,c},{b,c},*X*}. Clearly, the identity mapping f: X → Y is $\overline{is}$-contra-continuous, but *f* is not totally continuous because for open subset $f^{-1}${a}={a}∉C*O*(*X*).

## 5. Separation axioms with $\overline{\overline{is}}$-open Set

In this Section, we introduce some new weak of separation axioms by $\overline{is}$-open sets.

**Definition 5.1** A topology space *X* is said to be:

(i) $\overline{is}$-$T_o$ if for each pair distinct points of *X*, there exists $\overline{is}$-open set containing one point but not the other.
(ii) $\overline{is}$-$T_1$ (resp. clopen -$T_1$ [3]) if for each pair of distinct points of *X*, there exists two $\overline{is}$-open (resp. clopen) sets containing one point but not the other .
(iii) $\overline{is}$-$T_2$ (resp. ultra hausdorff (U$T_2$)[10]) if for each pair of distinct points of *X* can be separated by disjoint $\overline{is}$-open (resp. clopen ) sets.
(iv) $\overline{is}$–regular (resp. ultra regular [9]) if for each closed set *F* not containing a point in *X* can be separated by disjoint $\overline{is}$-open (resp. clopen) sets.
(v) clopen regular [10] if for each clopen set *F* not containing a point in *X* can be separated by disjoint open sets.
(vi) $\overline{is}$–normal (resp. ultra normal[10], s-normal[13], α-normal[2]) if for each of non-empty disjoint closed sets in *X* can be separated by disjoint $\overline{is}$-open (resp. clopen, semi-open, α-open) sets.
(vii) clopen normal [10] if for each of non-empty disjoint clopen sets in *X* can be separated by disjoint open sets.
(viii) $\overline{is}$-$T_{1/2}$ if every is-closed is i-closed in *X*.

**Remark 5.2** The following example shows that is-normal need not be normal, s-normal, α-normal spaces

**Example 5.3** Let $X$={1,2,3,4,5}, τ={∅,{1,2,3},{1,2,3,4},{1,2,3,5},$X$} and USO($X$)= {∅, {1} ,{2} ,{3},{4}, {5}, {1,2} ,{1,3},{1,4},{1,5},{2,3},{2,4},{2,5},{3,4},{3,5},{4,5},{1,2,3},{1,2,4},{1,2,5},{1,3,4},{1,3,5},{1,4,5},{2,3,4},{2,3,5},{2, 4,5},{3,4,5},{1,2,3,4},{1,2,3,5},{1,3,4,5},{1,2,4,5},{2,3,4,5},$X$}. Clearly, the space *X* is is-$T_o$, is-$T_1$, is-$T_2$, is–regular, is–normal and is-$T_{1/2}$, but *X* is not normal, s-normal and α-normal.

**Theorem 5.4** If $f: X \to Y$ is $\overline{is}$-totally continuous injection mapping and *Y* is $\overline{is}$-$T_1$, then *X* is clopen-$T_1$.

**Proof.** Let *x* and *y* be any two distinct points in *X*. Since, *f* is an injective, we have $f(x)$ and $f(y) \in Y$ such that $f(x) \neq f(y)$. Since, *Y* is $\overline{is}$-$T_1$, there exists $\overline{is}$-open sets *U* and *V* in *Y* such that $f(x)\in U$, $f(y)\notin U$ and $f(y)\in V$, $f(x)\notin V$. Therefore, we have $x\in f^{-1}(U)$,$y\notin f^{-1}(U)$ and $y\in f^{-1}(V)$ and $x\notin f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are clopen subsets of *X* because *f* is an is-totally continuous. This shows that *X* is clopen-$T_1$■

**Theorem 5.5** If $f: X \to Y$ is $\overline{is}$-totally continuous injection mapping and *Y* is $\overline{is}$-$T_o$, then *X* is ultra-Hausdorff (U$T_2$).

**Proof.** Let *a* and *b* be any pair of distinct points of *X* and *f* be an injective, then $f(a)\neq f(b)$ in *Y*. Since *Y* is an is-$T_o$, there exists $\overline{is}$-open set *U* containing $f(a)$ but not $f(b)$, then we have $a\in f^{-1}(U)$ and $b\notin f^{-1}(U)$. Since, *f* is $\overline{is}$-totally continuous, then $f^{-1}(U)$ is clopen in *X*. Also $a\in f^{-1}(U)$ and $b\in X$-$f^{-1}(U)$. This implies every pair of distinct points of *X* can be separated by disjoint clopen sets in *X*. Therefore, *X* is ultra-Hausdorff■

**Theorem 5.6** Let $f: X \to Y$ be a closed is-continuous injection mapping. If *Y* is $\overline{is}$–normal, then *X* is $\overline{is}$–normal.

**Proof.** Let $F_1$and $F_2$ be disjoint closed subsets of *X*. Since, *f* is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of *Y*. Since, *Y* is $\overline{\text{is}}$–normal , $f(F_1)$ and $f(F_2)$ are separated by disjoint $\overline{is}$-open sets $V_1$ and $V_2$ respectively. Therefore, we obtain, $F_1\subset f^{-1}(V_1)$ and $F_2\subset f^{-1}(V_2)$. Since, *f* is an is-continuous, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are is-open sets in *X*. Also, $f^{-1}(V_1)\cap f^{-1}(V_1)= f^{-1}(V_1\cap V_2)=\emptyset$. Thus, for each pair of non-empty disjoint closed sets in *X* can be separated by disjoint is-open sets. Therefore, *X* is an is–normal ■

**Theorem 5.7** If $f: X \to Y$ is $\overline{is}$-totally continuous closed injection mapping and *Y* is $\overline{is}$–normal, then *X* is ultra-normal.

**Proof.** Let $F_1$and $F_2$ be disjoint closed subsets of *X*. Since, *f* is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of *Y*. Since, *Y* is $\overline{\text{is}}$–normal , $f(F_1)$ and $f(F_2)$ are separated by disjoint is-open sets $V_1$and $V_2$ respectively. Therefore, we obtain, $F_1\subset f^{-1}(V_1)$ and $F_2\subset f^{-1}(V_2)$. Since, *f* is $\overline{is}$-totally continuous, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are clopen sets in *X*. Also, $f^{-1}(V_1)\cap f^{-1}(V_2)= f^{-1}(V_1 \cap V_2)=\emptyset$. Thus, for each pair of non-empty disjoint closed sets in *X* can be separated by disjoint clopen sets in *X*. Therefore, *X* is ultra-normal■

**Theorem 5.8** Let $f: X \to Y$ be a totally continuous closed injection mapping, if *Y* is $\overline{is}$–regular, then *X* is ultra-regular.

**Proof.** Let *F* be a closed set not containing *x*. Since, *f* is closed, we have *f* (*F*) is a closed set in *Y* not containing *f* (*x*). Since, *Y* is $\overline{is}$–regular, there exists disjoint $\overline{is}$-open sets *A* and *B* such that *f* (*x*)∈ *A* and *f* (*F*)⊂*B*, which imply *x*∈ $f^{-1}$(A) and *F*⊂ $f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are clopen sets in *X* because *f* is totally continuous. Moreover, since *f* is an injective, we have $f^{-1}(A) \cap f^{-1}(B)= f^{-1}(A\cap B)= f^{-1}(\emptyset)=\emptyset$. Thus, for a pair of a point and a closed set not containing a point in *X* can be separated by disjoint clopen sets. Therefore, *X* is ultra-regular■

**Theorem 5.9** If $f: X \to Y$ is totally continuous injective $\overline{is}$-open mapping from a clopen regular space *X* into a space *Y*, then *Y* is $\overline{is}$–regular.

**Proof.** Let *F* be a closed set in *Y* and y∉ *F*. Take *y* =*f* (*x*). Since, *f* is totally continuous, $f^{-1}(F)$ is clopen in *X*. Let $G=f^{-1}(F)$, then we have *x*∉ *G*. Since, *X* is clopen regular, there exists disjoint open sets *U* and *V* such that *G*⊂*U* and *x*∈ *V*. This implies *F*= *f* (*G*)⊂ *f* (*U*) and *y* =*f* (*x*)∈*V*. Further, since *f* is an injective and $\overline{is}$-open, we have $f(U) \cap f(V)= f(U\cap V)= f(\emptyset)=\emptyset$ , *f* (*U*) and *f* (*V*) are $\overline{is}$-open sets in *Y*. Thus, for each closed set *F* in *Y* and each y∉*F*, there exists disjoint is-open sets *f* (*U*) and *f* (*V*) in *Y* such that *F*⊂ *f* (*U*) and *y* ∈*f* (*V*). Therefore, *Y* is $\overline{is}$–regular■

**Theorem 5.10** If $f: X \to Y$ is a totally continuous injective and $\overline{is}$-open mapping from clopen normal space *X* into a space *Y*, then *Y* is $\overline{is}$–normal.

**Corollary 5.11** If $f: X \to Y$ is a totally continuous injective and is-open mapping from clopen normal space *X* into a space *Y*, then *Y* $\overline{is}$is–normal.

**Proof.** Clear, since every iα-open sets is $\overline{is}$-open sets ■

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