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Original Article

Modification of the Cross Theorem on Non-Convex Quadrilateral

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Abstract – *This article discusses the modification of the Cross Theorem on a non-convex quadrilateral by expanding the outward square twice, thus forming four new two-dimentional figure. The proof is done using the sine and cosine rules. The result obtained is that the difference in the area of the two-dimentional figure that faces each other obtained from the Cross Theorem on a non-convex quadrilateral is equal to four times the area of the original quadrilateral.*

Keywords *– Non-Convex, Sine and Cosine, Cross Theorem.*

1. Introduction

Cross Theorem was discovered by David Cross. The Cross Theorem was first put forward by [1] which states that a triangle on each side is constructed of squares pointing outward, then connected to the points of the adjacent square so that a new triangle will be obtained with the area of the new triangle will be equal to the original triangle. The proof of the Cross theorem on triangles has been explained by [2,3].

In general, the Cross Theorem applies to triangles. However, much has been done on the Cross Theorem. In [4,5], discussed the development of the Cross Theorem on triangles using rectangles. Furthermore, [6] discusses the development of the Cross Theorem on convex quadrilaterals. Next, [7] discusses the modification of the Cross Theorem on triangles by expanding the square outward twice, thus forming three new two-dimentional figure that have an area equal to five times the area of the original triangle. The proof of the Cross Theorem was carried out by [1,2,6] using the sine and cosine rules. Furthermore, [3,7] discuss the proof of the Cross Theorem using the congruence rule.

In [8,9] many ideas about the concept of proving the area of a triangle using the sine and cosine rules. For quadrilaterals, you can also prove using the sine and cosine rules, by dividing the quadrilateral into two triangular parts. On for quadrilateral have been discussed various theorem. Including, the Theorem Napoleon's [10,11,12], Euler's quadrilateral Theorem [13] and Cross Theorem [6].

The Cross Theorem on a convex quadrilateral states that, if any quadrilateral on each side is constructed of outward-pointing squares and the vertices of the squares are connected, it will form four new triangles whose area is the sum of the areas of the opposite triangles equal to the area of the original quadrilateral. Seeing the relationship between the area formed from the new flat shape and the original convex quadrilateral, in this article, we discuss the modification of the Cross Theorem on non-convex quadrilaterals. If the development of the Cross Theorem on a convex quadrilateral is carried out with one square expansion pointing outward, then the modification of the Cross Theorem on a non-convex quadrilateral is done by expanding the square pointing out twice which will be proven by using the sine and cosine rules.

2. Literature Review

The Cross Theorem was discovered by David Cross and was first put forward by Faux which states that a triangle ABC on each of its sides is constructed of outward-facing squares, then connected with the points of the adjacent square so that a new triangle will be obtained, then the area of the new triangle will be equal to triangle ABC. In general Cross' theorem applies to triangles, but some authors have developed Cross Theorem on triangles using rectangles and Cross Theorem on quadrilaterals [6]. Cross Theorem and some of the proofs have been discussed that triangles are formed has the same area as the initial triangle [2, 3]. For clarity, the following will be attached to the Cross Theorem on the triangle in Theorem 1.

Theorem 1. Let ΔABC denote any triangle, and construct squares on each side of the triangle outward which are $\square ABIH$, $\square BCGF$ and $\square ACDE$. If line EH, IF and DG are constructed, then will form ΔAEH, ΔBIF and ΔCDG which have an area same as ΔABC, as shown in Figure 1.

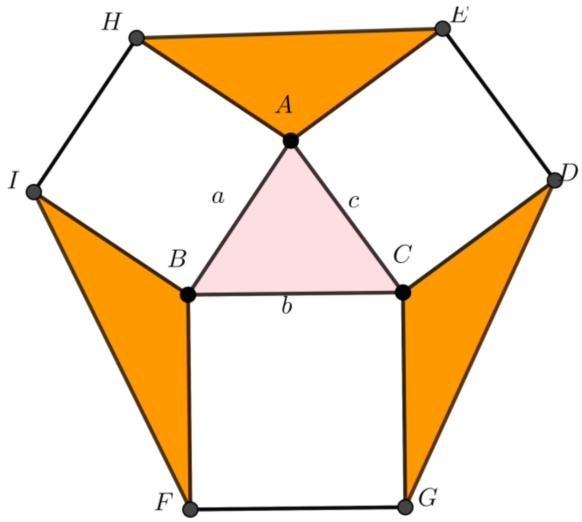

Fig. 1 Cross Theorem on Triangles.

Proof. The proof can be seen in [1,2,3].

Further in [7], the Cross Theorem on triangles has been modified, namely by constructing once again the square leading out from the sides of the triangle formed from the Cross Theorem in Theorem 1, then the area of the two-dimentional figure formed from the modification of the Cross Theorem on a triangle is equal to five the area of the original triangle, as discussed in Theorem 2.

Theorem 2. Let ΔAEH, ΔBIF and ΔCDG are triangles formed from Cross' theorem triangle ABC. Furthermore, $EHRQ$, $IFPO$ and $DGMN$ are constructed on each side triangle Cross' Theorem, if vertices squares are connected then it will form a two-dimetional figure and have an area of 5 $\Delta LABC$, shown in Figure 2.

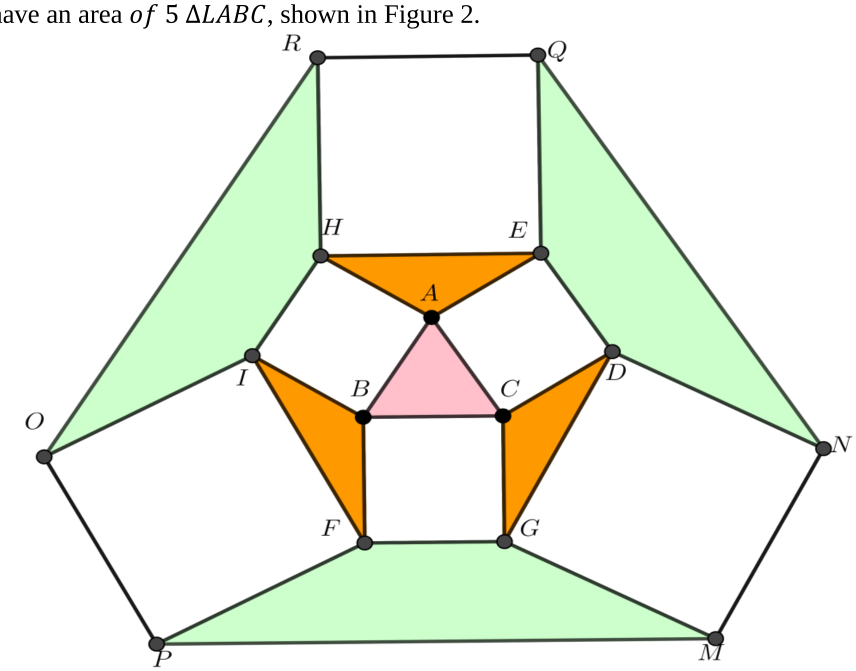

Fig. 2 Modification Theorem Cross on Triangle.

Proof. The proof can be seen in [7**].**
In [6] the Cross Theorem on quadrilaterals is an expansion of the Cross Theorem which shows that the sum of the areas of two pairs of new two-dimentional figure of opposite is equal to the area of the original quadrilateral, as discussed in Theorem 3.

Theorem 3. Let $ABCD$ denote any quadrilateral, square construction on each side of the quadrilateral outward $\square ADKL$, $\square ABFE$, $\square BCHG$ and $\square CDJI$. If line EH, FG, HI and JK are constructed, then will form ΔAEL, ΔBFG, ΔCHI and ΔDKJ which have area $L\Delta AEL + L\Delta CHI = L\square ABCD$ and $L\Delta BFG + L\Delta DKJ = L\square ABCD$, as shown in Figure 3.

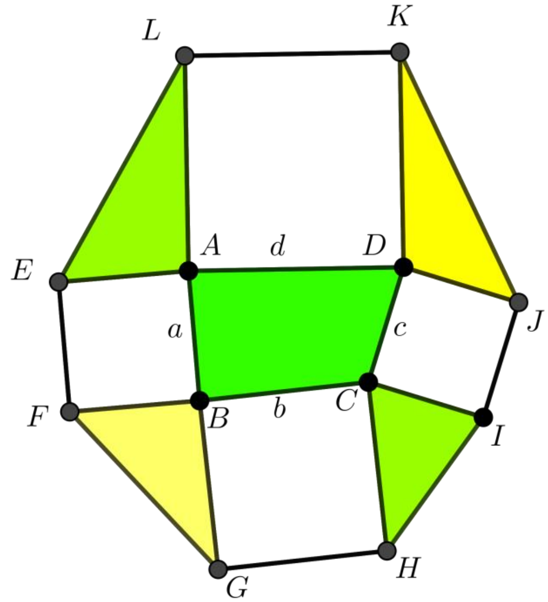

Fig. 3 The Cross Theorem on Quadrilateral.

Proof. The proof can be seen in [6].

3. Research Methodology

The steps in solving the modification of the Cross Theorem on a non-convex quadrilateral are to form an arbitrary non-convex quadrilateral $ABCD$ with side lengths $AB = a$, $BC = b$, $CD = c$ and $AD = d$. Next, from each side of the non-convex quadrilateral $ABCD$, a square is formed pointing outward so that $AB = AE = BF = EF = a$, $BC = BG = CH = GH = b$, $CD = CI = DJ = JI = c$ and $AD = AL = DK = KL = d$. Next, a line is drawn from the outer corner points of the flat shapes that are close together to form a triangle of EAL, FBG, CHI and DKJ. After obtaining four new triangles, namely triangle EAL, FBG, CHI dan DKJ, from side EL, FG, HI and JK an equilateral polygon is formed so that we get $EL = EN = LM = MN$, $FG = FO = GP = OP$, $HI = HQ = IR = QR$ and $JK = JS = KT = TS$. Next, a line is drawn from the outer corner points of a new flat shape so that it forms a new data shape $EFON$, $GHQP$, $IJSR$ and $KLMT$. Next, a diagonal line is drawn, for each quadrilateral $EFON$, $GHQP$, $IJSR$ and $KLMT$ successively from point E to point O, point G to Q, point I to point S and from point K to point M. So that each new flat shape will become two triangles, namely the quadrilateral $EFON$ into triangle ENO and EFO, then quadrilateral $GHQP$ becomes triangle GHQ and GPQ, then the $IJSR$ quadrilateral becomes the IJS and ISR triangle and for the $KLMT$ quadrilateral, it becomes the KLM and KMT triangle. Furthermore, using the sine and cosine rules, it will be proven that the difference in the area of the $EFON$ quadrilateral with the $IJSR$ quadrilateral is equal to four times the area of the $ABCD$ quadrilateral and the difference in the area of the $KLMT$ quadrilateral with the $GHQP$ quadrilateral is equal to four times the area of the $ABCD$ quadrilateral.

4. Results and Discussion

The proof of Theorem 3 does not apply to non-convex quadrilaterals. because when one of the angles of the quadrilateral becomes projection, the sine of that angle will be negative. For the Cross theorem of a non-convex quadrilateral, the areas of the opposite planes are not added together, but the two-dimentional figure of opposite the projective angle will be subtracted by the projective angle, as seen in Figure 4, the following will be given the Cross Theorem on a non-convex quadrilateral.

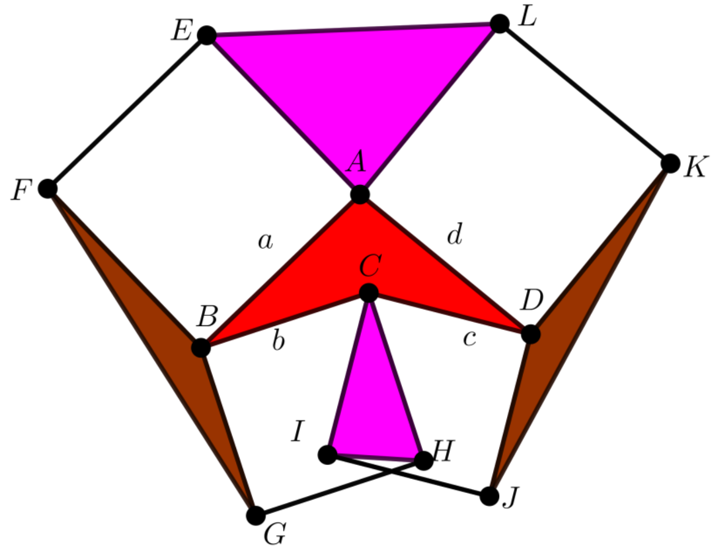

Fig. 4 The Cross Theorem in a Non-Convex Quadrilateral.

Theorem 4. Suppose that the quadrilateral $ABCD$ is any concave quadrilateral where BCD is a projective angle and L represents the area of a two-dimentional figure. Furthermore, $\square ADKL$, $\square ABFE$, $\square BCHG$ and $\square CDJI$ are constructed pointing out. If a line is drawn EL, FG, HI and JK then it will form ΔAEL, ΔBFG, ΔCHI and ΔDJK where $L\Delta AEL - L\Delta CHI = L\square ABCD$ and $L\Delta BFG + L\Delta DJK = L\square ABCD$.

Proof: By considering to Figure 4. Let $AB = a$, $BC = b$, $CD = c$ and $AD = d$, and let $\angle BAD = \alpha$, $\angle CBA = \beta$, $\angle BCD = \lambda$ dan $\angle ADC = \sigma$. Obtained $L\square ABCD = L\Delta BAD - L\Delta BCD$ and $L\square ABCD = L\Delta ABC + L\Delta ADC$. Using the sine and cosine rules to calculate the area of the quadrilateral $ABCD$, it is obtained that the area $\Delta BAD = \frac{1}{2}ad\sin\alpha$, $\Delta ABC = \frac{1}{2}ab\sin\beta$, $\Delta BCD = \frac{1}{2}bc\sin\lambda$ and $\Delta ADC = \frac{1}{2}cd\sin\sigma$. Furthermore, let $AB = AE = EF = BF = a$, $BC = BG = CH = GH = b$, $DC = CI = DJ = IJ = c$ dan $AD = AL = DK = KL = d$.
By considering ΔAEL Figure 4 on obtained

$$L\Delta AEL = \frac{1}{2}\, ad\, \sin\angle LAE$$

$$L\Delta AEL = \frac{1}{2}\, ad\, \sin\,(360° - 90° - 90° - \alpha)$$

$$L\Delta AEL = \frac{1}{2}\, ad\, \sin\,(180° - \alpha)$$

$$L\Delta AEL = \frac{1}{2}\, ad\, \sin\,\alpha$$

$$L\Delta AEL = L\Delta BAD \quad (1)$$

In the same way, by considering ΔBFG, ΔCHI and ΔDKJ obtained

$$L\Delta BFG = L\Delta ABC, \quad (2)$$
$$L\Delta CHI = L\Delta BCD, \quad (3)$$
$$L\Delta DKJ = L\Delta ADC \quad (4)$$

Based on the equation (1) and (3) will be obtained $L\square ABCD$ is

$$L\Delta BAD - L\Delta BCD = L\square ABCD,$$
$$L\Delta AEL - L\Delta CHI = L\square ABCD \quad (5)$$

Based on the equation (2) and (4) will be obtained $L\square ABCD$ is

$$L\Delta ABC + L\Delta ADC = L\square ABCD,$$
$$L\Delta BFG + L\Delta DKJ = L\square ABCD \quad (6)$$

Based on the equation (5) and (6) then Theorem 4 is proven. ■

Furthermore, the modification of the Cross Theorem on a non-convex quadrilateral is carried out by expanding twice by constructing a square leading to each side of the non-convex quadrilateral and connecting the closest outer square points to form four new two-dimentional figures. This article explained what if the Cross theorem on a convex quadrilateral is expanded once again by constructing a square on each side of the triangle formed from the Cross theorem on the original convex quadrilateral and connecting the nearest outer square points. The proof of the modification of the Cross Theorem in on-convex quadrilaterals is done using the sine and cosine rules. For more details about this modification of the Cross Theorem on a non-convex quadrilateral in Theorem 5.

Theorem 5. Let ΔAEL, ΔBFG, ΔCHI and ΔDJK are triangles formed from $ABCD$ Cross Theorem on a non-convex quadrilateral. Furthermore, $\square ELMN$, $\square GFOP$, $\square JKTS$ and $\square ADKL$ are constructed as rectangles on each side of the triangle outward *EL*, *FG*, *HI* and *JK*. if vertices squares are connected then it will form a two-dimentional figure and have area $L\square EFON - L\square IJSR = 4\, L\square ABCD$ and $L\square KLMT - LGHQP = 4\, L\square ABCD$, shown in Figure 5.

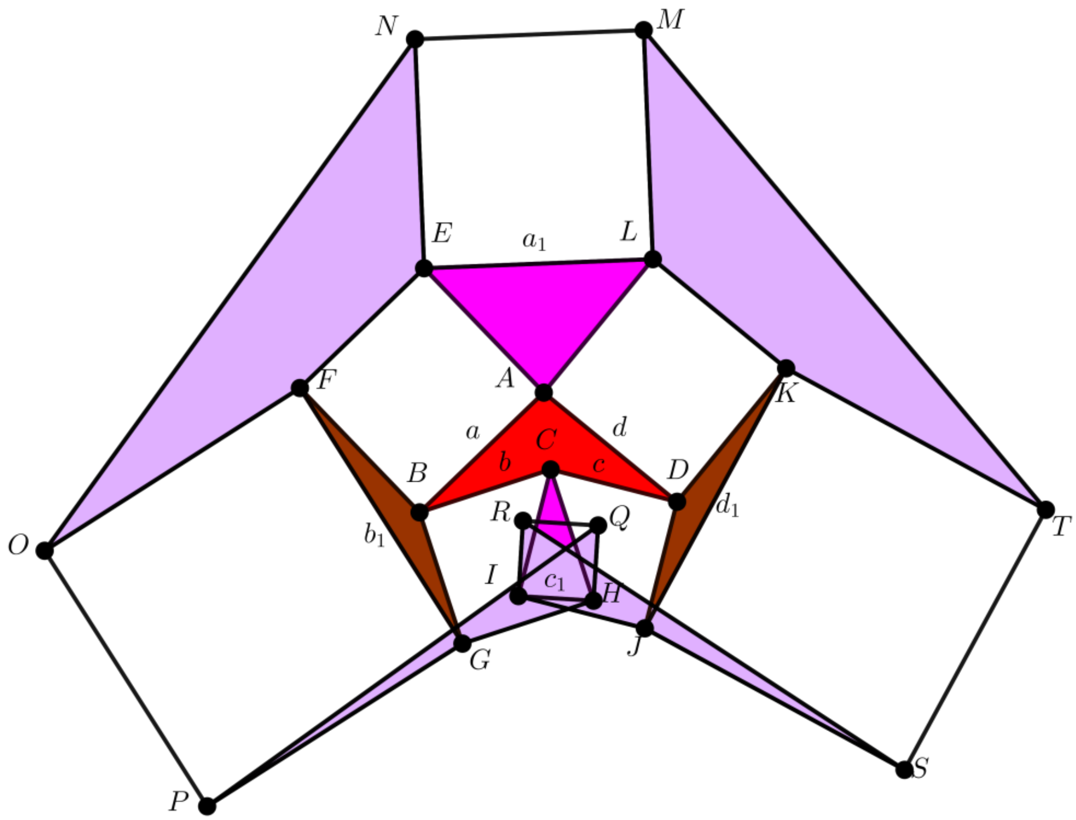

Fig. 5 Modification of the Cross Theorem in a Non-Convex Quadrilateral.

Proof. Let sides $AB = AE = BF = EF = a, BC = CH = BG = GH = b$, $CD = CI = DJ = IJ = c$ and $AD = AL = DK = KL = d$. Next, let the sides $EL = EN = LM = MN = a_1$, $FG = FO = GP = OP = b_1$, $HI = HQ = IR = QR = c_1$ and $JK = JS = KT = ST = d_1$. Obtained $L\square ABCD = L\Delta BAD - L\Delta BCD$ and $L\square ABCD = L\Delta ABC + L\Delta ADC$. Using the sine and cosine rules to calculate the area of the quadrilateral $ABCD$, it is obtained that the area $\Delta BAD = 1/2\, ad \sin \alpha$, $\Delta ABC = 1/2\, ab \sin \beta$, $\Delta BCD = 1/2\, bc \sin \lambda$ and $\Delta ADC = 1/2\, cd \sin \sigma$. By considering ΔAEL on Figure 5, so by using the cosine rule obtatained

$$a_1{}^2 = a^2 + d^2 + 2ad \cos \alpha \quad (7)$$
$$b_1{}^2 = a^2 + b^2 + 2ab \cos \beta \quad (8)$$

$$c_1^{\,2} = b^2 + c^2 + 2bc\cos\lambda \tag{9}$$
$$d_1^{\,2} = c^2 + d^2 + 2cd\cos\sigma \tag{10}$$

Look for $L\square EFON$, divide the quadrilateral into two triangles namely ΔENO and ΔEFO, so we will find the area of the two triangles, shown in Figure 6.

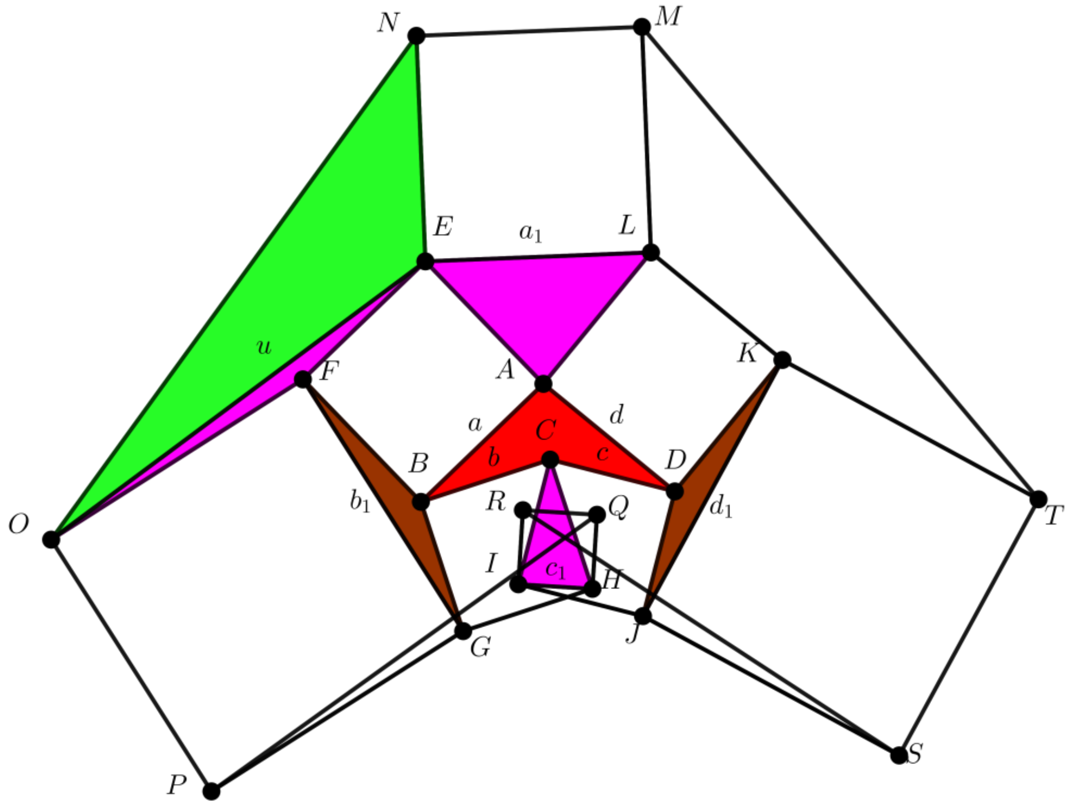

Fig. 6 The Illustration on $\square EFON$ is Divided Into Two Triangles.

By considering ΔBFG, ΔAEL and ΔEFO in Figure 6, using the sine rule obtained

$$\sin\angle GFB = \frac{b\sin\beta}{b_1} \tag{11}$$
$$\sin\angle AEL = \frac{d\sin\alpha}{a_1} \tag{12}$$
$$\sin\angle OEF = \frac{b\sin\beta}{u} \tag{13}$$

By considering ΔBFG, ΔAEL and ΔEFO on Figure 6, using the cosine rule obtained

$$\cos\angle GFB = \frac{a^2+b_1^{\,2}-b^2}{2ab_1} \tag{14}$$
$$\cos\angle AEL = \frac{a^2+a_1^{\,2}-d^2}{2aa_1} \tag{15}$$
$$\cos\angle OEF = \frac{a^2+u^2-b_1^{\,2}}{2au} \tag{16}$$

Next, to find the value of u, using the cosine rule obtained

$$u^2 = 2a^2 + 2b_1^{\,2} - b^2 \tag{17}$$

Using the sine rule on ΔEFO obtained

$$L\Delta EFO = \frac{1}{2}ab_1\sin(360° - 90° - 90° - \angle GFB) \tag{18}$$
$$L\Delta EFO = \frac{1}{2}ab_1\sin(180° - \angle GFB) \tag{19}$$

$$L\Delta EFO \ = \frac{1}{2} ab_1 \sin \angle GFB \tag{20}$$

By substituting the value $\sin \angle GFB$ in equation (11) into equation (20) obtained.

$$L\Delta EFO \ = \frac{1}{2} ab_1 \frac{b \sin \beta}{b_1} \tag{21}$$

$$L\Delta EFO \ = \frac{1}{2} ab \sin \beta \tag{22}$$

$$L\Delta EFO \ = L\Delta BFG \tag{23}$$

In the same way in finding the area ΔEON, obtained

$$L\Delta EON = \frac{1}{2} a_1 u \sin(360° - 90° - 90° - \angle AEL - \angle OEF)$$

$$L\Delta EON = \frac{1}{2} a_1 u \sin(180° - (\angle AEL + \angle OEF))$$

$$L\Delta EON = \frac{1}{2} a_1 u \sin(\angle AEL + \angle OEF)$$

$$L\Delta EON = \frac{1}{2} a_1 u \sin \angle AEL \cos \angle OEF + \sin \angle OEF \cos \angle AEL \tag{24}$$

By substituting equations (7), (12), (13), (15), (16) and (17) into the equation (24) obtained

$$L\Delta EON \ = 2L\Delta EAL \ + \ L\Delta FBG + \frac{bd \sin(\alpha+\beta)}{2} \tag{25}$$

Look for $L\square IJSR$, divide the quadrilateral into two triangles namely ΔIJS and ΔRIS, so we will find the area of the two triangles, shown in Figure 7.

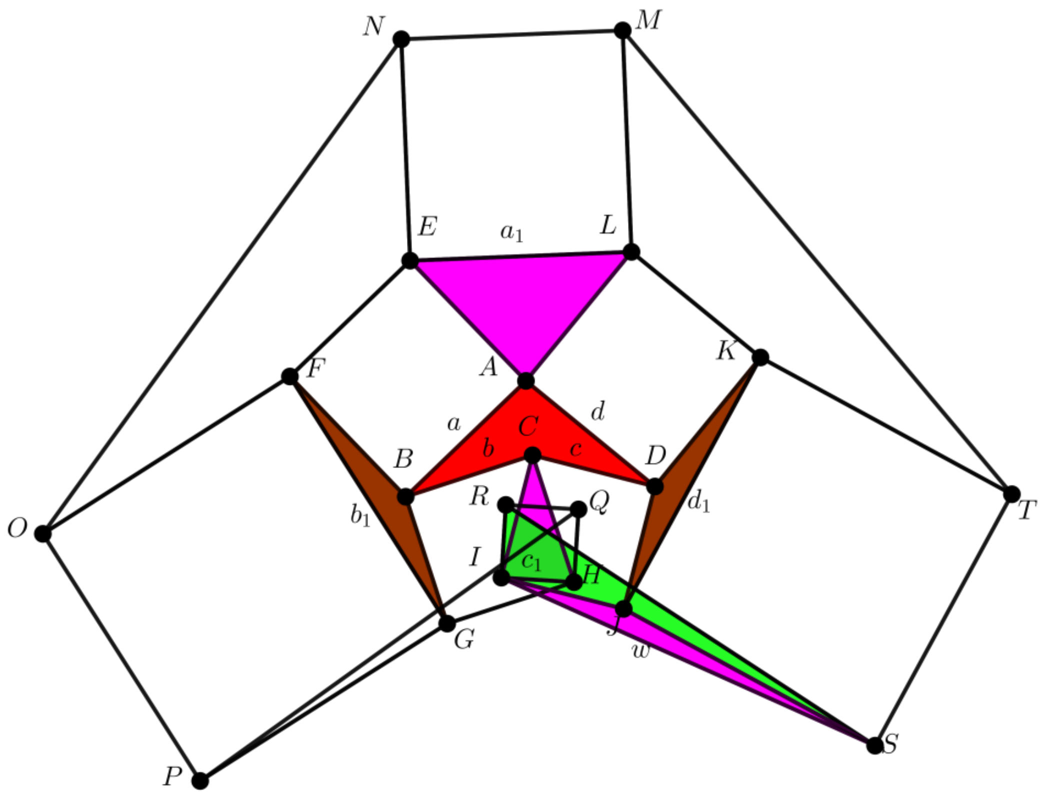

Fig. 7 The illustration on $\square IJSR$ is divided into two triangles.

In the same way will be obtained $L\Delta IJS$ and $L\Delta RIS$

$$L\Delta IJS \ = L\Delta DKJ \tag{25}$$

$$L\Delta RIS \ = 2L\Delta CHI \ - \ L\Delta DKJ + \frac{bd \sin(\lambda-\sigma)}{2} \tag{26}$$

After obtaining the value of $L\Delta IJS$ and $L\Delta RIS$, to get the relationship of the area of the two-dimentional figure formed with the area of the original non-convex quadrilateral, the next step is to substitute equations (23) and (25) into equation (6) obtained

$$L\Delta BFG + L\Delta DKJ = L\square ABCD$$
$$L\Delta EFO + L\Delta IJS = L\square ABCD \quad (27)$$

Subtract ΔEON on equation (24) with ΔRIS in equation (26), obtained

$$L\Delta EON - L\Delta RIS = 3\, L\square ABCD \quad (28)$$

The results obtained from equations (27) and (28) it is obtained

$$L\square EFON - L\square IJSR = 4L\square ABCD \quad (29)$$

In the same way for $\square KLMT$ and $\square GHQP$, each is divided into two triangular, as shown in Figure 8.

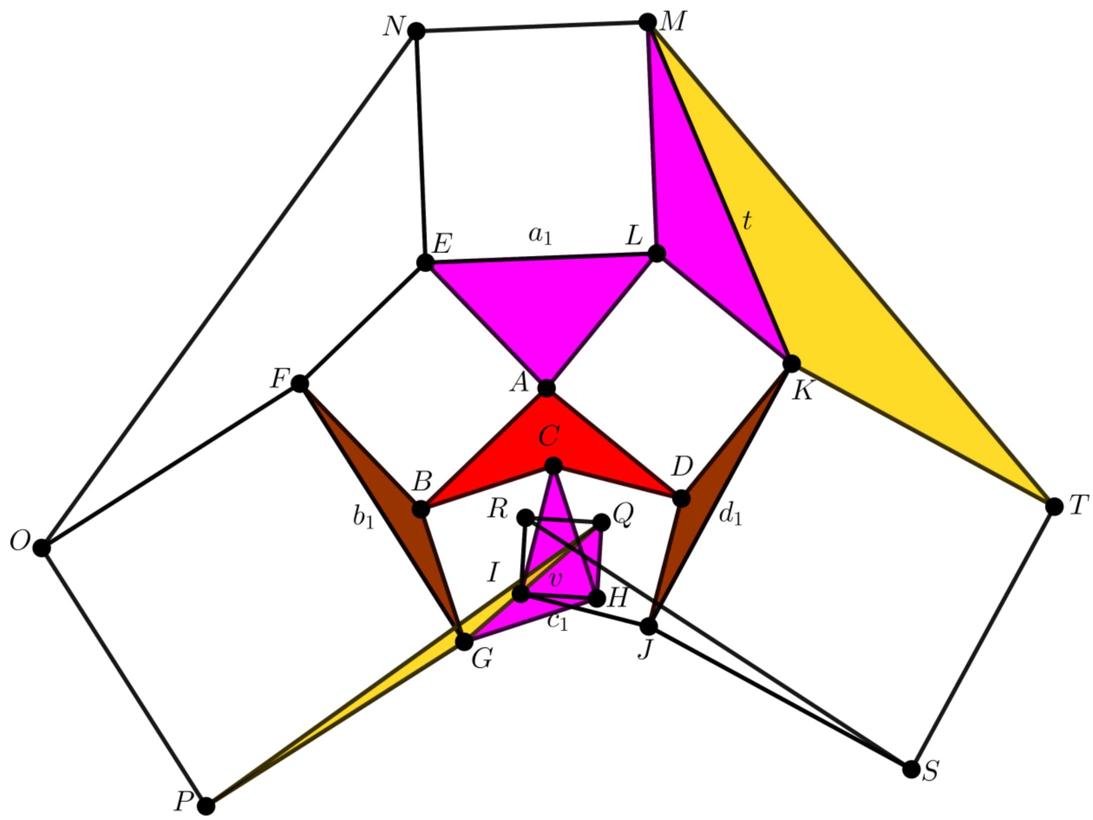

Fig. 8 Illustration of $\square KLMT$ and $\square GHQP$ divided into two triangles.

By considering Figure 8, in the same way obtained

$$L\Delta KML = L\Delta AEL \quad (30)$$
$$L\Delta KMT = 2L\Delta DKJ + L\Delta AEL + \frac{ac\sin(\alpha+\sigma)}{2} \quad (31)$$

In the same way, obtained

$$L\Delta GHQ = L\Delta CHI \quad (32)$$
$$L\Delta GPQ = -2L\Delta BFG + L\Delta CHI - \frac{ac\sin(\beta-\lambda)}{2} \quad (33)$$

After obtaining the value $L\Delta KML$ dan $L\Delta GHQ$, to get the relationship of the area of the two-dimentional figure with the area of the original non-convex quadrilateral, the next step is to substitute equations (30) and (32) into equation (5) to obtain

$$L\Delta AEL + L\Delta CHI = L\square ABCD$$
$$L\Delta KML + L\Delta GHQ = L\square ABCD \quad (34)$$

Subtract ΔKMT on equation (31) with ΔGPQ in equation (33), obtained

$$L\Delta KMT - L\Delta GPQ = 3L\square ABCD \quad (36)$$

The results obtained from equations (34) and (36) it is obtained

$$L\square EFON - L\square IJSR = 4\, L\square ABCD, \quad (37)$$

Therefore, based on the equation (29) and (37) then Theorem 5 is proven. ■

5. Conclusion

Based on the results and discussion, it can be concluded that The Cross theorem can be developed for non-convex quadrilaterals using expands twice by constructing a square pointing outward on each side of the non-convex quadrilateral and connecting the points of the adjacent quadrilateral to obtain four new two-dimentional figure. The results obtained from the proof using the sine and cosine rules are the difference between the areas of the opposite quadrilaterals is equal to four times the area of the original quadrilateral.

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