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Original Article

Concircular ϕ-Symmetric (ε)-Lorentzian Para-Sasakian Manifold Admitting Quarter-Symmetric Metric Connection

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Abstract - *In this paper we consider a quarter-symmetric metric connection in a (ε)-Lorentzian Para-Sasakian manifold and study Locally ϕ-symmetric, Locally concircular ϕ-symmetric and ξ-concircularly flat (ε)-Lorentzian Para-Sasakian manifold with respect to a quarter-symmetric metric connection.*

Keywords - *(ε)-Lorentzian Para-Sasakian manifold, ϕ-symmetry, concircular curvature tensor, quarter-symmetric metric connection.*

AMS Subject Classification - *53C15, 53C25, 53D10*

1. Introduction

A linear connection $\tilde{\nabla}$ in an n-dimensional differentiable manifold is said to be a quarter-symmetric connection [6] if its torsion tensor T is of the form

$$\begin{aligned} T(X, Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \\ &= \eta(Y)\,\phi X - \eta(X)\,\phi Y, \qquad (1.1) \end{aligned}$$

where η is a 1-form and ϕ is a tensor of type (1, 1). In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [5]. Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection. And if quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y, Z) = 0,$$

for all X, Y, Z ∈ X(M), where X(M) is the Lie algebra of vector fields on the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection.

In this paper, we study some results on a quarter-symmetric metric connection in an (ε) - Lorentzian Para-Sasakian manifold. This paper is organized is as follows: In Section 2, we give a brief introduction of an (ε) - Lorentzian Para-Sasakian manifold and define quarter-symmetric metric connection. In Section 3, we study the Locally ϕ - symmetric (ε)-Lorentzian Para-Sasakian manifold with respect to a quarter-symmetric metric connection. In Section 4, we study the locally concircular ϕ - symmetric (ε) - Lorentzian Para-Sasakian manifold with respect to a quarter-symmetric metric connection and Section 5 is devoted ξ -concircularly flat (ε) - Lorentzian Para-Sasakian manifold with respect to the quarter-symmetric metric connection.

2. Preliminaries

An n-dimensional smooth manifold (M, g) is (ε)-Lorentzian Para-Sasakian manifold if it admits a (1, 1)-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\xi, \xi) = -\varepsilon,\ \phi\xi = 0,\ \eta(\phi X) = 0, \tag{2.3}$$

$$\eta(X) = \varepsilon g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \tag{2.5}$$

for all vector fields X, Y on M, where ε is 1 or -1 according to which is either ξ is space like or time like vector field. If an ε - contact metric manifold satisfies

$$(\nabla_X \phi)Y = g(X, Y)\xi + \varepsilon \eta(Y)X + 2\varepsilon \eta(X)\eta(Y)\xi, \tag{2.6}$$

where ∇ denotes the Levi-Civita connection with respect to g, then M is called an (ε)-LP-Sasakian manifold. An ε-contact metric manifold is an (ε)-LP-Sasakian manifold if and only if

$$\nabla_X \xi = \varepsilon \phi X, \tag{2.7}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in an (ε)-LPSasakian manifold M with respect to the Levi-Civita connection satisfy [11]

$$(\nabla_X \eta)Y = g(\phi X, Y), \tag{2.8}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

$$R(\xi, X)Y = \varepsilon g(X, Y)\xi - \eta(Y)X, \tag{2.10}$$

$$\eta(R(X, Y)Z) = \varepsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.11}$$

$$S(X, \xi) = (n-1)\eta(X),\ Q\xi = \varepsilon(n-1)\xi, \tag{2.12}$$

where X, Y,Z ϵ X(M) and g(QX, Y) = S(X, Y).

The concircular curvature tensor $\bar{C}$ is given by

$$\bar{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{2.13}$$

For (ε)-LP-Sasakian manifold the relation between the quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\text{-}X - g(\phi X, Y)\xi. \tag{2.14}$$

By the virtue of equations (2.1), (2.7) and (2.8), equation (2.14) reduces to

$$\tilde{R}(X, Y)Z = R(X, Y)Z + \varepsilon[\eta(Y)X - \eta(X)Y]\eta(Z) \tag{2.15}$$

$$+ \varepsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi$$
$$+ \varepsilon[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X],$$

where X, Y, Z ϵ X(M) and $\tilde{R}$ is the Riemannian curvature of the connection $\tilde{\nabla}$.
From (2.15) it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) + \varepsilon(n-1)\eta(Y)\eta(Z) + \varepsilon g(\phi Y,Z)\phi, \tag{2.16}$$

where $\tilde{S}$ and S are the Ricci tensors of connection $\tilde{\nabla}$ and ∇, respectively and φ = traceϕ.

Contracting the above equation, we get

(2.17) $\tilde{r} = r - \varepsilon(n-1) - \varepsilon\,\varphi^2,$

where $\tilde{r}$ and r are the scalar curvature of the connection $\tilde{\nabla}$ and ∇, respectively.

3. Locally ϕ-Symmetric (ε)-Lorentzian Para-Sasakian Manifold with respect to a Quarter-Symmetric Metric Connection

A locally ϕ-symmetric (ε)-Lorentzian Para-Sasakian manifold with respect to a quarter-symmetric metric connection is given by

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X, Y)Z) = 0, \quad (3.1)$$

for any vector fields X, Y,Z and W orthogonal to ξ .

Using (2.14) we get

$$(\tilde{\nabla}_W \tilde{R})(X, Y)Z = (\nabla_W \tilde{R})(X, Y)Z - \eta(X)\tilde{R}(\phi W, Y)Z + g(\phi W, X)\tilde{R}(\xi, Y)Z - \eta(Y)\tilde{R}(X, \phi W)Z + g(\phi W, Y)\tilde{R}(X, \xi)Z. \quad (3.2)$$

Now differentiating (2.15), with respect to W and using (2.6), we obtain

$$\begin{aligned}(\nabla_W \tilde{R})(X, Y)Z = (\nabla_W R)(X, Y)Z &+ \varepsilon\, g(Y,Z)g(\phi W,X)\,\xi - g(X,Z)g(\phi W, Y)\,\xi \\ &+ \varepsilon\, g(Y,Z)\eta(X)\phi W - \varepsilon\, g(X,Z)\eta(Y)\phi W \\ &+ [\tfrac{1}{\varepsilon} g(W,X)\eta(Z) + \varepsilon\, g(W,Z)\eta(X) + 2\eta(W)\eta(X)\eta(Z)]\phi Y \\ &- [\tfrac{1}{\varepsilon} g(W, Y)\eta(Z) + \varepsilon\, g(W,Z)\eta(Y) + 2\eta(W)\eta(Y)\eta(Z)]\phi X \\ &+ g(\phi X,Z)[g(W, Y)\,\xi + \varepsilon\,\eta(Y)W + 2\varepsilon\,\eta(W)\eta(Y)\,\xi] \\ &- g(\phi Y,Z)[g(W,X)\,\xi + \varepsilon\,\eta(X)W + 2\varepsilon\,\eta(W)\eta(X)\,\xi] \\ &- g(\phi W,Z)[\eta(Y)X + \eta(X)Y]\}. \end{aligned} \quad (3.3)$$

Using (2.1) and (3.3) in (3.2) and applying ϕ^2, we get

$$\begin{aligned}\phi^2((\tilde{\nabla}_W \tilde{R})(X, Y)Z) = \phi^2((\nabla_W \tilde{R})(X, Y)Z) &+ \varepsilon\{g(Y,Z)g(\phi W,X)\phi^2\xi - g(X,Z)g(\phi W, Y)\phi^2\xi \\ &+ \varepsilon\, g(Y,Z)\eta(X)\phi^2(\phi W) - \varepsilon\, g(X,Z)\eta(Y)\phi^2(\phi W) \\ &+ [\tfrac{1}{\varepsilon} g(W,X)\eta(Z) + \varepsilon\, g(W,Z)\eta(X) + 2\eta(W)\eta(X)\eta(Z)]\phi^2(\phi Y) \\ &- [\tfrac{1}{\varepsilon} g(W, Y)\eta(Z) + \varepsilon\, g(W,Z)\eta(Y) + 2\eta(W)\eta(Y)\eta(Z)]\phi^2(\phi X) \\ &+ g(\phi X,Z)[g(W, Y)\phi^2\xi + \varepsilon\,\eta(Y)\phi^2 W + 2\varepsilon\,\eta(W)\eta(Y)\phi^2\xi] \\ &- g(\phi Y,Z)[g(W,X)\phi^2\xi + \varepsilon\,\eta(X)\phi^2 W + 2\varepsilon\,\eta(W)\eta(X)\phi^2\xi] \\ &- g(\phi W,Z)[\eta(Y)\phi^2 X + \eta(X)\phi^2 Y]\}. \\ &- \eta(X)\phi^2(\tilde{R}(\phi W, Y)Z) + g(\phi W,X)\phi^2(\tilde{R}(\xi, Y)Z) \\ &- \eta(Y)\phi^2(\tilde{R}(X, \phi W)Z) + g(\phi W, Y)\phi^2(\tilde{R}(X, \xi)Z). \end{aligned} \quad (3.4)$$

If X, Y,Z and W orthogonal to ξ , then (3.4) then becomes

(3.5) $\phi^2((\tilde{\nabla}_W \tilde{R})(X, Y)Z) = \phi^2((\nabla_W R)(X, Y)Z).$

Thus we can state

Theorem 3.1. A (ε)-Lorentzian Para-Sasakian manifold is locally ϕ-symmetric with respect to a quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection ∇.

4. Locally Concircular ϕ-Symmetric (ε)-Lorentzian Para-Sasakian Manifold with respect to a Quarter-Symmetric Metric Connection

A (ε)-Lorentzian Para-Sasakian manifold M is said to be locally concircular ϕ-symmetric with respect to a quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W \tilde{\tilde{C}})(X, Y)Z) = 0, \quad (4.1)$$

for any vector fields X, Y,Z and W orthogonal to ξ, where $\tilde{\tilde{C}}$ is the concircular curvature tensor with respect to quarter-symmetric metric connection given by

$$\tilde{\tilde{C}}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{R}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \quad (4.2)$$

Using (2.14), we can write

$$(\tilde{\nabla}_W \tilde{\tilde{C}})(X,Y)Z = (\nabla_W \tilde{\tilde{C}})(X,Y)Z + \eta\tilde{\tilde{C}}(X,Y)Z)\phi W - g(\phi W, \tilde{\tilde{C}}(X,Y)Z)\xi$$
$$- \eta(X)\tilde{\tilde{C}}(\phi W,Y)Z - \eta(Y)\tilde{\tilde{C}}(X,\phi W)Z - \eta(Z)\tilde{\tilde{C}}(X,Y)\phi W \quad (4.3)$$
$$+ g(\phi W,X)\tilde{\tilde{C}}(\xi,Y)Z + g(\phi W,Y)\tilde{\tilde{C}}(X,\xi)Z + g(\phi W,Z)\tilde{\tilde{C}}(X,Y)\xi.$$

Differentiating (4.2) w.r.t W, we obtain

$$(\nabla_W \tilde{\tilde{C}})(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{\nabla_W \tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \quad (4.4)$$

Using (3.3) and (2.16) in (4.4), we can write

$$(\nabla_W \tilde{\tilde{C}})(X,Y)Z = (\nabla_W R)(X,Y)Z + \varepsilon[g(\phi W,Y)\eta(Z) + g(\phi W,Z)\eta(Y)]X$$
$$- \varepsilon[g(\phi W,X)\eta(Z) - g(\phi W,Z)\eta(X)]Y + \varepsilon[g(Y,Z)g(\phi W,X)]\xi$$
$$- \varepsilon[g(X,Z)g(\phi W,Y)]\xi + \varepsilon^2[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\phi W \quad (4.5)$$
$$- \varepsilon^2[\eta(Y)g(W,Z) - 2\eta(Y)\eta(Z)\eta(W)]\phi X - g(W,Y)(Z)\phi X$$
$$- \frac{\nabla_W \tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y].$$

Taking account of concircular curvature tensor in (4.5), then use of (4.3) and applying -2, we get

$$\phi^2((\tilde{\nabla}_W \tilde{\tilde{C}})(X,Y)Z = \phi^2(\nabla_W \tilde{\tilde{C}})(X,Y)Z + \varepsilon[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\phi^2(\phi W)$$
$$- \varepsilon[g(\phi W,X)\eta(Z)\phi^2(Y) - g(\phi W,Y)\eta(Z)\phi^2(X)]$$
$$+ \varepsilon[g(X,Z)g(\phi W,Y)]\phi^2\xi - \varepsilon[g(Y,Z)g(\phi W,X)]\phi^2\xi$$
$$- \eta(X)\phi^2(R(\phi W,Y)Z) - \eta(Y)\phi^2(R(X,\phi W)Z) - \eta(Z)\phi^2(R(X,Y)\phi W) \quad (4.6)$$
$$+ g(\phi W,X)\phi^2((R(\xi,Y)Z)) + g(\phi W,Y)\phi^2((R(X,\xi)Z))$$
$$+ g(\phi W,Z)\phi^2((R(X,Y)\xi)).$$

If we consider X, Y, Z and W orthogonal to ξ, then (4.6) becomes

$$\phi^2((\tilde{\nabla}_W \tilde{\tilde{C}})(X,Y)Z) = \phi^2(\nabla_W \bar{C})(X,Y)Z. \quad (4.7)$$

Hence we can state the following theorem

Theorem 4.2. A (ε)-Lorentzian Para-Sasakian manifold is locally concircular ϕ-symmetric with respect to $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection ∇.

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Next using (4.5) in (4.3) and considering X, Y,Z and W orthogonal to , we get

$$\phi^2((\tilde{\nabla}_W \tilde{\tilde{C}})(X,Y)Z) = \phi^2(\nabla_W R)(X,Y)Z. \quad (4.8)$$

Thus we can state:

Theorem 4.3. If M is ϕ-symmetric with respect to a quarter-symmetric metric connection then a (ε)-Lorentzian Para-Sasakian manifold is locally concircular ϕ-symmetric with respect to a quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetric with respect to Levi-Civita connection ∇.

5. ξ-Concircularly Flat (ε)-Lorentzian Para-Sasakian Manifold with respect to the Quarter-Symmetric Metric Connection

A (ε)-Lorentzian Para-Sasakian manifold M with respect to the quarter-symmetric metric connection is said to be -concircularly flat if

$$\tilde{\tilde{C}}(X,Y)\xi = 0, \quad (5.1)$$

for all vector fields X, Y on M. If (5.1) holds for X, Y orthogonal to ξ, then a manifold is a horizontal ξ-concircularly flat manifold.

Using (2.15) in (2.13), we get

$$\tilde{\tilde{C}}(X, Y)Z = R(X, Y)Z + \varepsilon[\eta(Y)X - \eta(X)Y]\eta(Z) + \varepsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi + \varepsilon[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \quad (5.2)$$

Taking Z = ξ and using (2.1), (2.8) and (2.16) in (5.2), we get

$$\tilde{\tilde{C}}(X, Y)\xi = 0. \quad (5.3)$$

Hence we state:

Theorem 5.4. A (ε)-Lorentzian Para-Sasakian manifold is horizontal ξ-concircularly flat with respect to the quarter-symmetric metric connection.

Again using (2.17) in (5.2) and taking Z = ξ , it follows that

$$\tilde{\tilde{C}}(X, Y)\xi = \bar{C}(X, Y)\xi. \quad (5.4)$$

Hence we state:

Theorem 5.5. A (ε)-Lorentzian Para-Sasakian manifold is horizontal ξ-concircularly flat with respect to the quarter-symmetric metric connection if and only if the manifold is ξ-concircularly flat with respect to the Levi-Civita connection.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this paper.

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