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Original Article

Hub Number of Total Transformation Graphs

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Abstract *- For a graph G, the hub set S is defined to be the subset of vertices of G with the property that for any pair of vertices in $V\backslash S$, there exists a path with all intermediate vertices which belongs to S. The hub number of a graph G is defined to be the smallest size of hub set. In this paper, we develop a method to find the hub number of total transformation graphs in terms of order and size of the graph considered.*

Keywords – *Hub set, Hub number, Total transformation graphs.*

1. Introduction

In 2006, Walsh came across the following problem [17]: Imagine that we have a graph G which represents a large industrial complex, with edge between two buildings if it is an easy walk from one to other. The corporation of the city is considering implementation of a rapid-transit system (RTS) and wants to place its stations in buildings so that to travel between two nonadjacent buildings, one need only to walk to an adjacent station, take the RTS, and walk to the desired buildings. The corporation would like to implement this plan as cheaply as possible, which involves converting as few buildings as possible into transit stations. Walsh [17], translated the above problem to graph theoretical model and defined *hub number of a graph* in this context follows:

Let $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where all intermediate vertices are from S. A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G)\backslash S$ there is an S -path in G between x and y. The minimum cardinality of hub set is called hub number and is denoted by $h(G)$. A connected hub set, and the size of a smallest connected hub set will be the connected hub number $h_c(G)$ of the graph G.

The problem in the first paragraph can be rephrased as: *what is the smallest size of a hub set in G* ? It is clear that $h(G)$ is well defined for any G, as $V(G)$ is itself a hub set.

This inspired many graph theorists to study the hub number of graphs. To mention few, Grauman et al. [7] obtained the relationship between hub number, connected hub number and connected domination number of a graph. Cauresma et al. [5] obtained hub number of join, corona and cartesian product of two connected graphs. In [17] the relation between domination number $\gamma(G)$ and hub number of a graph has been obtained as $\gamma(G) \leq h(G) + 1$. In the same paper, the author has conjectured that, for any connected non-tree graph G,

$$h(G) \geq g(G) - 3.$$

The relation between the hub sets and cut sets was noticed by Vandell and Walsh in [17]. Goddard and Walsh [17], proved that the recognition of a hub set can be done in polynomial time, since vertex contraction and clique recognition are both polynomial-time operations. The hub number of Sierpiński graph is given in [12] and the connected hub number of Mycielski Graph in [13]. Hub numbers of grid graphs were obtained in [9] and the hub number of co-comparability graphs in [14]. The relation between connected hub number and the connected domination number and the structural characterization of graphs which satisfy $\gamma_c(G) = h_c(G) + 1$ are studied in [11] where, $\gamma_c(G)$ is the connected domination number. Basavanagoud et al. [1, 2] studied hub number of some wheel related graphs and hub number of generalized middle graphs. The concept of hub number was extended to total hub number of a graph in [15]. Inspired by these, in this paper, we obtain hub number of total transformation graphs.

The present paper is organized as follows: In section 2, we discuss the preliminaries and definition of total transformation graphs denoted by G^{xyz}. In section 3, we obtain the main results on the hub number of total transformation graphs G^{xyz}.

2. Preliminaries

In this paper, we consider only nontrivial, connected, simple and undirected graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Thus $|V(G)| = n$ and $|E(G)| = m$ where, n and m are called *order* and *size* of graph G respectively. The pendant vertices are called leaf vertices of graph G, it is denoted by l. An edge incident to a pendant vertex l of a graph G is called pendant edge and denoted by l_e. The girth of a graph G defined as the length of the shortest cycle in G and denoted by $g(G)$. The *complement* of a graph G [8] is denoted by $\overline{G}$ whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* $L(G)$ of a graph G [16] is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The *subdivision graph* $S(G)$ of a graph G [8] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. The *partial complement of subdivision graph* $\overline{S}(G)$ of a graph G [10] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G not incident with it. In this paper, graph G means only some standard class of graphs which are path, cycle, star, tree and complete graph. A path, cycle, star, tree and complete graph are denoted by P_n, C_n, $K_{1,n}$, T and K_n respectively. Here we have shown selection of hub set in respective class of graphs. For undefined terminology and notations refer [6, 8].

The *total transformation graphs* G^{xyz}, introduced by Wu and Meng [3] are defined as follows: Let $G = (V(G), E(G))$ be a graph, and x, y, z be three variables taking values $+$ or $-$. The transformation graph G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set, and for $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds:

(i) $\alpha, \beta \in V(G)$. α and β are adjacent in G if $x = +$; α and β are nonadjacent in G if $x = -$.
(ii) $\alpha, \beta \in E(G)$. α and β are adjacent in G if $y = +$; α and β are nonadjacent in G if $y = -$.
(iii) $\alpha \in V(G), \beta \in E(G)$. α and β are incident in G if $z = +$; α and β are nonincident in G if $z = -$.

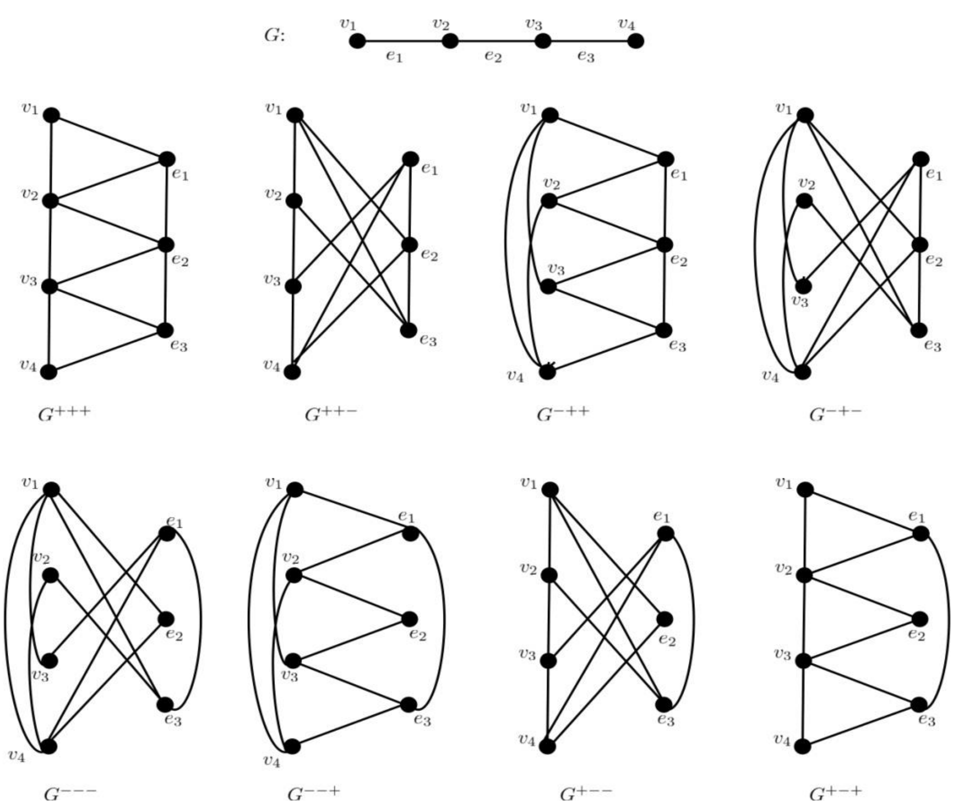

Fig. 1 Graph G and its total transformation graphs G^{xyz}.

Thus, one can obtain eight kinds of transformation graphs, in which G^{+++} is the total graph [4] of G, and G^{---} is its complement. Also, G^{--+}, G^{-+-}, and G^{-++} are the complements of G^{++-}, G^{+-+}, and G^{+--} respectively, are depicted in Figure 1. The vertex v of G^{xyz} corresponding to a vertex v of G is referred to as a *point vertex*. The vertex e of G^{xyz} corresponding to an edge e of G is referred to as a *line vertex*.

Proposition 2.1 [17] *The hub number of*
i) The path P_n, $h(P_n) = n - 2$,
ii) The cycle C_n, $h(C_n) = n - 3$,
iii) The complete graph K_n, $h(K_n) = 0$,
iv) The tree T, $h(T) = n - l$.

3. Hub number of total transformation graphs

The main purpose of this section is to provide hub number of total transformation graphs through a most convenient method. We also present the hub number of these graphs in terms of the order and and size of the graph considered and wherever generalization is not possible, we have considered case by case. We begin with the theorem on hub number of total graph of a tree.

Theorem 3.1 *Let G be tree of order n and l be leaf vertices. Then*

$$h(G^{+++}) = n - l.$$

Proof. Let $v_1, v_2, \dots, v_n$ be point vertices of G^{+++} and $e_1, e_2, \dots, e_m$ be line vertices of G^{+++}. Consider $v_i \in V(G^{+++})$ with maximum degree then $d_{G^{+++}}(v_i) = 2d_G(u)$. Now $S = \{v_i | v_i \in V(G^{+++})\}$ such that v_i has maximum degree in G^{+++}. Then choose a vertices in G which has degree of a vertex ≥ 2, which is adjacent to all other line vertices in G^{+++}. i.e., nonleaf vertices $(n - l)$ of G, which gives the minimum hub set in G^{+++}. Thus, $S = \{v_i | v_i \in V(G), d_G(v) \geq 2\}$. Therefore, $|S| = n - l$. Hence, $h(G^{+++}) = n - l$.

Theorem 3.2 *Let G be cycle or complete graph of order n. Then*

$$h(G^{+++}) = \begin{cases} 1 & if \;\; n = 3, \\ n - 1 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be point vertices of G^{+++} and $e_1, e_2, \dots, e_m$ be line vertices of G^{+++}. We consider the following cases:
Case 1: If $n = 3$, then G^{+++} is a 4-regular graph with 6 vertices. Choose a set $S = \{v_i\}$ such that it contains a point vertex v_i in G^{+++}. So $|S| = 1$. Therefore, $h(G^{+++}) = 1$.
Case 2: If $n \geq 4$, then choose a set S such that it contains $n - 1$ vertices of G such that either they are mutually adjacent in G or they should form a path of length $n - 1$ in G. So, $|S| = n - 1$. For any two vertices $x, y \in G^{+++} \backslash S$ have S-path between them. Therefore S is a hub set of G^{+++} minimum cardinality. Hence, $h(G^{+++}) = n - 1$.

Theorem 3.3 *Let G be tree of order $n \geq 3$. Then*

$$h(G^{++-}) = \begin{cases} 2 & if \;\; n = 3, \; 4, \\ 3 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be point vertices of G^{++-} and $e_1, e_2, \dots, e_m$ be line vertices of G^{++-}. We consider the following cases:
Case 1: If $n = 3$, then $P_3^{++-} \cong C_5$. By Proposition 2.1, $h(P_3^{++-}) = 2$.
Case 2: If $n = 4$, then we get two graphs, P_4 and $K_{1,3}$. If $G = P_4$, then choose nonpendant vertices, which forms a S-path in P_4^{++-}. Similarly, if $G = K_{1,3}$, choose two pendant vertices which gives the minimum hub set. Therefore, $h(G^{++-}) = 2$.
Case 3: If $n \geq 5$, then choose a set S such that it contains two pendant edges $\{e_1, e_2\}$ and a vertices v_i which is nonincident to pendant edges in G. Thus, $S = \{e_1, e_2, v_i\}$ gives the minimum hub set. Therefore, $|S| = 3$. Hence, $h(G^{++-}) = 3$.

Theorem 3.4 *Let G be cycle or complete graph of order n. Then*

$$h(G^{++-}) = \begin{cases} 2 & if \ \ n = 3, \\ 3 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be point vertices of G^{++-} and $e_1, e_2, \dots, e_m$ be line vertices of G^{++-}. We consider the following cases:
Case 1: If $n = 3$, then choose a vertex v_1 and edge e_1 which is nonincident to v_1 in C_3 gives the hub set. Therefore, $h(C_3^{++-}) = 2$.
Case 2: If $n = 4$, then choose three vertices $\{v_1, v_2, v_3\}$ in G, which forms a S-path in G^{++-}. Therefore, $h(G^{++-}) = 3$.
Case 3: If $n \geq 5$, then $G = C_n, K_n$. If $G = C_n$, then choose a set S such that it contains any two nonadjacent vertices v_1, v_2 and edge e_1 which is nonincident to vertices v_1 and v_2 in C_n. If $G = K_n$, then choose a set S such that it contains any two nonadjacent edges e_1, e_2 and vertex v_1 which is nonincident to edges e_1 and e_2 in K_n. Thus, S gives the minimum hub set. Therefore, $|S| = 3$. Hence, $h(G^{++-}) = 3$.

Theorem 3.5 *Let P_n be path graph of order n. Then*

$$h(P_n^{-++}) = \begin{cases} 1 & if \ \ n = 2, \\ 2 & if \ \ n = 3, \\ \left\lfloor \frac{n}{2} \right\rfloor & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be point vertices of P_n^{-++} and $e_1, e_2, \dots, e_m$ be line vertices of P_n^{-++}. We consider the following cases:
Case 1: If $n = 2$, then $P_2^{-++} \cong P_3$. By Proposition 2.1, $h(G^{-++}(P_2)) = 1$.
Case 2: If $n = 3$, then choose pendant vertices $\{v_1, v_2\}$ in P_3, which forms a S-path in P_3^{-++}. Therefore, $h(P_3^{-++}) = 2$.
Case 3: If $n \geq 4$, then choose a set S such that it contains $\left\lfloor \frac{n}{2} \right\rfloor$ vertices of P_n, in such way that these vertices are incident to at most two edges of P_n. So, $|S| = \left\lfloor \frac{n}{2} \right\rfloor$. For any two vertices $x, y \in P_n^{-++} \backslash S$, there exist an S-path between them in P_n^{-++}. Therefore S is a hub set of P_n^{-++} minimum cardinality. Hence, $h(P_n^{-++}) = \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 3.6 *Let C_n be cycle graph of order n. Then*

$$h(C_n^{-++}) = \begin{cases} 2 & if \ \ n = 3, \\ 3 & if \ \ n = 4, \\ \left\lceil \frac{n}{2} \right\rceil & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be point vertices of C_n^{-++} and $e_1, e_2, \dots, e_m$ be line vertices of C_n^{-++}. We consider the following cases:
Case 1: If $n = 3$, then choose a vertex v_1 and edge e_1 which is nonincident to v_1 in C_3 gives the hub set. Therefore, $h(C_3^{-++}) = 2$.
Case 2: If $n = 4$, then choose any three adjacent edges $\{e_1, e_2, e_3\}$ in C_4, which forms a S-path in C_4^{-++}. Therefore, $h(C_4^{-++}) = 3$.
Case 3: If $n \geq 5$, then choose a set S such that it contains $\left\lceil \frac{n}{2} \right\rceil$ vertices of C_n, in such way that these vertices are incident to at most two edges of C_n. So, $|S| = \left\lceil \frac{n}{2} \right\rceil$. For any two vertices $x, y \in C_n^{-++} \backslash S$, there exist an S-path between them in C_n^{-++}. Therefore S is a hub set of C_n^{-++} minimum cardinality. Hence, $h(C_n^{-++}) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 3.7 *Let $K_{1,n}$ be star graph of order $n \geq 2$. Then*

$$h(K_{1,n}^{-++}) = 2.$$

Proof. Let $v_1, v_2, \dots, v_n$ be point vertices of $K_{1,n}^{-++}$ and $e_1, e_2, \dots, e_m$ be line vertices of $K_{1,n}^{-++}$. Let $S \subset V(K_{1,n}^{-++})$. Choose a set S containing vertex v_1 and edge e_1 which is incident to vertex v_1 in $K_{1,n}$. Therefore, $|S| = 2$. Hence, $h(K_{1,n}^{-++}) = 2$.

Theorem 3.8 *Let T be tree of order $n \geq 4$ and l be leaf vertices. Then*

$$h(T^{-++}) = n + m - l - l_e.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of T^{-++} and $e_1, e_2, \ldots, e_m$ be line vertices of T^{-++}. Let $S \subset V(T^{-++})$. Choose a set S containing nonpendant vertices and nonpendant edges which are incident to vertices in T. Therefore, $|S| = n + m - l - l_e$. Hence, $h(T^{-++}) = n + m - l - l_e$.

Theorem 3.9 *Let G be complete graph of order $n \geq 4$. Then*

$$h(K_n^{-++}) = n - 1.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of K_n^{-++} and $e_1, e_2, \ldots, e_m$ be line vertices of K_n^{-++}. Let $S \subset V(K_n^{-++})$. Choose a set S containing $n - 1$ adjacent edges in such way that it forms path of length $n - 1$ in K_n. Therefore, $|S| = n - 1$. Hence, $h(K_n^{-++}) = n - 1$.

Theorem 3.10 *Let G be tree of order $n \geq 4$. Then*

$$h(G^{-+-}) = 2.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of G^{-+-} and $e_1, e_2, \ldots, e_m$ be line vertices of G^{-+-}. Choose a set $S \subset V(G^{-+-})$ of vertices of G^{-+-} corresponding to any two pendant vertices of G. So, $|S| = 2$. For any two vertices $x, y \in G^{-+-} \backslash S$, there exist an S-path between them in G^{-+-}. Therefore S is a hub set of G^{-+-} minimum cardinality. Hence, $h(G^{-+-}) = 2$.

Theorem 3.11 *Let C_n be cycle graph of order n. Then*

$$h(C_n^{-+-}) = \begin{cases} 3 & if \;\; n = 3,\; 4,\; 5, \\ 2 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of C_n^{-+-} and $e_1, e_2, \ldots, e_m$ be line vertices of C_n^{-+-}. We consider the following cases:
Case 1: If $n = 3$, then C_3^{-+-} is isomorphic to crown graph CW_4. Therefore, $h(C_3^{-+-}) = 3$ (since $h(CW_n) = n - 1$ [1]).
Case 2: If $n = 4{,}5$, then $G = C_4, C_5$. Choose any three adjacent edges $\{e_1, e_2, e_3\}$ in G, which forms an S-path in G^{-+-}. Therefore, $h(G^{-+-}) = 3$.
Case 3: If $n \geq 6$, then choose a set S such that it contains any two nonadjacent vertices of C_n, in such way that distance between these two vertices are at least three. Then for any two vertices $x, y \in C_n^{-+-} \backslash S$, there exist an S-path between them in C_n^{-+-}. Therefore S is a hub set of C_n^{-+-} minimum cardinality. Hence, $h(C_n^{-+-}) = 2$.

Theorem 3.12 *Let K_n be complete graph of order $n \geq 4$. Then*

$$h(K_n^{-+-}) = 3.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of K_n^{-+-} and $e_1, e_2, \ldots, e_m$ be line vertices of K_n^{-+-}. We consider the following cases:
Case 1: If $n = 4$, then choose any three mutually adjacent edges $\{e_1, e_2, e_3\}$ in K_4, which forms a S-path in K_4^{-+-}. Therefore, $h(K_4^{-+-}) = 3$.
Case 2: If $n \geq 5$, then choose a set S such that it contains any two nonadjacent edges e_1, e_2 and a vertex v_1, which is nonincident to edges e_1 and e_2 in K_n. Then for any two vertices $x, y \in K_n^{-+-} \backslash S$, there exist an S-path between them in K_n^{-+-}. Therefore S is a hub set of K_n^{-+-} minimum cardinality. Hence, $h(K_n^{-+-}) = 3$.

Theorem 3.13 *Let T be tree of order $n \geq 4$. Then*

$$h(T^{---}) = 2.$$

Proof. The proof is similar to that of Theorem 3.10.

Theorem 3.14 *Let C_n be cycle graph of order $n \geq 4$. Then*

$$h(C_n^{---}) = \begin{cases} 3 & if\ \ n = 4, \\ 2 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of C_n^{---} and $e_1, e_2, \ldots, e_m$ be line vertices of C_n^{---}. We consider the following cases:
Case 1: If $n = 4$, then choose any two adjacent vertices v_1, v_2 and an edge e_1 which is nonincident to vertices v_1 and v_2 in C_4, which forms a S-path in C_4^{---}. Therefore, $h(C_4^{---}) = 3$.
Case 2: If $n = 5$, then choose a vertex v_1 and an edge e_1, which are nonincident in C_5 and also whose length is two in C_5. Therefore $h(C_4^{---}) = 2$. For $n \geq 6$, then choose a set S such that it contains any two nonadjacent vertices of C_n, in such way that distance between whose length is at least three. Then for any two vertices $x, y \in C_n^{---} \backslash S$, there exist an S-path between them in C_n^{---}. Therefore S is a hub set of C_n^{---} minimum cardinality. Hence, $h(C_n^{---}) = 2$.

Theorem 3.15 *Let K_n be complete graph of order $n \geq 4$. Then*

$$h(K_n^{---}) = 3.$$

Proof. The proof is similar to that of Theorem 3.12.

Theorem 3.16 *Let P_n be path graph of order n. Then*

$$h(P_n^{--+}) = \begin{cases} 1 & if\ \ n = 2, \\ 2 & if\ \ n = 3,4, \\ 3 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of P_n^{--+} and $e_1, e_2, \ldots, e_m$ be line vertices of P_n^{--+}. We consider the following cases:
Case 1: If $n = 2$, then $P_2^{--+} \cong P_3$. By Proposition 2.1, $h(P_2^{--+}) = 1$.
Case 2: If $n = 3$, then $P_3^{--+} \cong C_5$. By Proposition 2.1, $h(P_3^{--+}) = 2$.
Case 3: If $n = 4$, then choose nonpendant vertices, which forms an S-path in P_4^{--+}. Therefore, $h(P_4^{--+}) = 2$.
Case 4: If $n \geq 5$, then choose a set S such that it contains a pendant vertex v_1 and any two adjacent edges e_1, e_2, in which one of edge is incident to a vertex v_1 in P_n. Then for any two vertices $x, y \in P_n^{--+} \backslash S$, there exist an S-path between them in P_n^{--+}. Therefore S is a hub set of P_n^{--+} minimum cardinality. Hence, $h(P_n^{--+}) = 3$.

Theorem 3.17 *Let C_n be cycle graph of order n. Then*

$$h(C_n^{--+}) = \begin{cases} 3 & if\ \ n = 3,4,5, \\ 4 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of C_n^{--+} and $e_1, e_2, \ldots, e_m$ be line vertices of C_n^{--+}. We consider the following cases:
Case 1: If $n = 3$, then $C_3^{--+} \cong C_6$. By Proposition 2.1, $h(C_3^{--+}) = 3$.
Case 2: If $n = 4$, then choose any three adjacent vertices in C_4 gives the minimum hub set. Therefore, $h(C_4^{--+}) = 3$.
Case 3: If $n = 5$, then choose any two adjacent vertices v_1, v_2 and next choose another vertex v_3 which in nonadjacent in C_5, which forms an S-path in C_5^{--+}. Therefore, $h(C_5^{--+}) = 3$.
Case 4: If $n \geq 6$, then choose a set S such that it contains two nonadjacent vertices v_1, v_2 and two nonadjacent edges e_1, e_2, which are incident to each vertices in C_n. Then for any two vertices $x, y \in C_n^{--+} \backslash S$, there exist an S-path between them in C_n^{--+}. Therefore S is a hub set of C_n^{--+} minimum cardinality. Hence, $h(C_n^{--+}) = 3$.

Theorem 3.18 *Let $K_{1,n}$ be star graph of order $n \geq 3$. Then*

$$h(K_{1,n}^{--+}) = 3.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of $K_{1,n}^{--+}$ and $e_1, e_2, \ldots, e_m$ be line vertices of $K_{1,n}^{--+}$. If $n \geq 3$, then choose a set S such that it contains central vertex v_i and pendant vertex v_1. Next choose an edge which is incident to v_1 and v_i in $K_{1,n}$. Then for any two vertices $x, y \in K_{1,n}^{--+} \backslash S$, there exist an S-path between them in $K_{1,n}^{--+}$. Therefore S is a hub set of $K_{1,n}^{--+}$ minimum cardinality. Hence, $h(K_{1,n}^{--+}) = 3$.

Theorem 3.19 *Let T be tree of order $n \geq 3$. Then*

$$h(T^{--+}) = 3.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of T^{--+} and $e_1, e_2, \ldots, e_m$ be line vertices of T^{--+}. If $n \geq 3$, then choose two pendant edges e_1, e_2 and a vertex v_1 which is incident to one of pendant edge in T. Then for any two vertices $x, y \in T^{--+} \backslash S$, there exist an S-path between them in T^{--+}. Therefore S is a hub set of T^{--+} minimum cardinality. Hence, $h(T^{--+}) = 3$.

Theorem 3.20 *Let K_n be complete graph of order $n \geq 4$. Then*

$$h(K_n^{--+}) = \begin{cases} 4 & if \;\; n = 4, \\ n-1 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of K_n^{--+} and $e_1, e_2, \ldots, e_m$ be line vertices of K_n^{--+}. We consider the following cases:
Case 1: If $n = 4$, then choose all vertices in K_4 gives the minimum hub set. Therefore, $h(K_4^{--+}) = 4$.
Case 2: The proof is similar to that of Theorem 3.9.

Theorem 3.21 *Let P_n be path graph of order n. Then*

$$h(P_n^{+-+}) = \begin{cases} 0 & if \;\; n = 2, \\ 1 & if \;\; n = 3, \\ 2 & if \;\; n = 4, \\ \left\lceil \frac{n}{2} \right\rceil & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of P_n^{+-+} and $e_1, e_2, \ldots, e_m$ be line vertices of P_n^{+-+}. We consider the following cases:
Case 1: If $n = 2$, then $P_2^{+-+} \cong C_3$. By Proposition 2.1, $h(G^{+-+}(P_2)) = 0$.
Case 2: If $n = 3$, then P_3^{+-+} is isomorphic to friendship graph f_2. Therefore, $h(G^{+-+}(P_3)) = 1$ (since $h(f_n) = 1$ [1]).
Case 3: If $n = 4$, then choose nonpendant vertices $\{v_1, v_2\}$ in P_4, which forms an S-path in P_4^{+-+}. Therefore, $h(P_4^{+-+}) = 2$.
Case 4: If $n \geq 5$, then choose a set S such that it contains $\left\lceil \frac{n}{2} \right\rceil$ edges of P_n, in such way that these edges are incident to at most two vertices of P_n. So, $|S| = \left\lceil \frac{n}{2} \right\rceil$. Then for any two vertices $x, y \in P_n^{+-+} \backslash S$, there exist an S-path between them in P_n^{+-+}. Therefore S is a hub set of P_n^{+-+} minimum cardinality. Hence, $h(P_n^{+-+}) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 3.22 *Let G be cycle or complete graph of order n. Then*

$$h(G^{+-+}) = \begin{cases} 2 & if \;\; n = 3, \\ 3 & if \;\; n = 4, \\ \left\lceil \frac{n}{2} \right\rceil & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of G^{+-+} and $e_1, e_2, \ldots, e_m$ be line vertices of G^{+-+}. We consider the following cases:
Case 1: If $n = 3$, then choose a vertex v_1 and an edge which in nonincident to vertex v_1 in C_3, which forms an S-path in C_3^{+-+}. Therefore, $h(C_3^{+-+}) = 2$.
Case 2: If $n = 4$, then choose any three adjacent vertices in G, which forms an S-path in G^{+-+}. Therefore, $h(G^{+-+}) = 3$.
Case 3: The proof is similar to that of Theorem 3.21.

Theorem 3.23 *Let T be tree graph of order $n \geq 4$. Then*

$$h(T^{+-+}) = n - l.$$

Proof. The proof is similar to that of Theorem 3.1.

Theorem 3.24 *Let T be tree graph of order $n \geq 3$. Then*

$$h(T^{+--}) = \begin{cases} 3 & if \ \ n = 3, \\ 2 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of T^{+--} and $e_1, e_2, \ldots, e_m$ be line vertices of T^{+--}. We consider the following cases:
Case 1: If $n = 3$, then $P_3^{+--} \cong P_5$. By Proposition 2.1, $h(P_3^{+--}) = 3$.
Case 2: If $n \geq 4$, then choose a set S such that it contains any two pendant edges e_1 and e_2, which gives the minimum hub set. So, $|S| = 2$. Then for any two vertices $x, y \in T^{+--} \backslash S$, there exist an S-path between them in T^{+--}. Therefore S is a hub set of T^{+--} minimum cardinality. Hence, $h(T^{+--}) = 2$.

Theorem 3.25 *Let C_n be cycle graph of order $n \geq 3$. Then*

$$h(C_n^{+--}) = \begin{cases} 3 & if \ \ n = 3, \\ 2 & otherwise. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of C_n^{+--} and $e_1, e_2, \ldots, e_m$ be line vertices of C_n^{+--}. We consider the following cases:
Case 1: If $n = 3$, then $C_3^{+--} \cong CW_4$. Therefore, $h(C_3^{+--}) = 3$, (since $h(CW_n) = n - 1$ [1]).
Case 2: If $n \geq 4$, then choose a set S such that it contains any two nonadjacent edges e_1 and e_2 of C_n, in such way that distance between these edges are at least three. Then for any two vertices $x, y \in C_n^{+--} \backslash S$, there exist an S-path between them in C_n^{+--}. Therefore S is a hub set of C_n^{+--} minimum cardinality. Hence, $h(C_n^{+--}) = 2$.

Theorem 3.26 *Let $K_{1,n}$ be star graph of order $n \geq 3$. Then*

$$h(K_{1,n}^{+--}) = 3.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be point vertices of $K_{1,n}^{+--}$ and $e_1, e_2, \ldots, e_m$ be line vertices of $K_{1,n}^{+--}$. If $n \geq 3$, then choose a set S such that it contains any three adjacent vertices of path of length two in $K_{1,n}$. Then for any two vertices $x, y \in K_{1,n}^{+--} \backslash S$, there exist an S-path between them in $K_{1,n}^{+--}$. Therefore S is a hub set of $K_{1,n}^{+--}$ minimum cardinality. Hence, $h(K_{1,n}^{+--}) = 3$.

Theorem 3.27 *Let K_n be complete graph of order $n \geq 5$. Then*

$$h(K_n^{+--}) = 3.$$

Proof. The proof is similar to that of Theorem 3.12.

4. Conclusion

In this paper, we have seen that obtaining the hub number of total transformation graphs in general is a tough problem so we have presented the results for some class of total transformation graphs. For further research one can investigate different graphs and try to generalize the results.

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