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Research Article

Discreetness of Spectrum of Schrödinger Operator on Riemannian Manifold by using Lebesgue Measure

Farah Diyab[1], B. Surender Reddy[2]

[1,2]*Department of Mathematics, Osmania University, Hyderabad, India.*

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Abstract - *We formulate the conditions for discreteness Dirichlet spectrum of Schrödinger operator $H = -\Delta + V(x)$ on Riemannian Manifold. Its formulated by using Lebesgue measure instead of harmonic capacity. We also provide the recent related results.*

Keywords - *Dirichlet problem, Spectrum, Manifolds, Schrödinger, Spectral Geometry.*

1. Introduction

Assume that $M$is connected manifold, let $g = g_{ij}$is Riemannian metric structure on$M$and$dim\, M = n$.We introduce self-adjoint Schrödinger operator$H$ in $L^2(M, dM)$.

$$H = -\Delta + V(x). \tag{1}$$

Let $u$be a scalar functions on$M$, $\Delta$is the Laplace-Beltrami operator .
By using usual summation, we have

$$\Delta u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x^j}\right). \tag{2}$$

where$g^{ij}$is the inverse matrix to g_{ij},$g = det(g_{ij})$ and $(x^1, x^2, \dots, x^n)$be a local system of coordinates in$M$.
The potential$V(x)$in equation (1)is real measurable function in L^2,semi-bounded below such that for $x \in M$ and $c \in \mathbb{R}$

$$V(x) \geq -c. \tag{3}$$

Let $B(x, r)$ denote the open ball with radius $r$and center at$x$in$M$, $\overline{B}(x, r)$the closure of$B(x, r)$.
In order to perform boundary geometry on a Riemannian manifold, the following conditions must be met:

1) The radius of injectiuity of M is positive i.e.$r_{inj}(M) > 0$.

2) $|\Delta^m R| < C_m$where $\Delta^m R$ is a m-covariant derivative of curvature tensor.

It is clear that the first condition reads as

$$r_{inj}(M) = inf\{r_{inj}(x) | x \in M\}.$$

$r_{inj}(x)$ can be defined as the biggest $r > 0$ such that $exp: B\,(x, r) \to M$ is diffeomorphism. References [3,7] provides further information on Riemannian manifolds.

Now, for Dirichlet spectral problem : Suppose a Riemannian manifold (M, g)has a boundary. Find all real numbersλsuch that there exists a function $u \in C^\infty(M)$for which

$$\begin{cases} Hu = \lambda u \\ u = 0 \text{on boundary of } M \end{cases}$$

In spectral terms, we are interested in finding the spectrum of Hon the set of smooth functions with compact support on M. Indeed, the discrete spectrum $\sigma_d(H)$ is comprised of all eigenvalues with finite multiplicity. For a pure algebraic approach see [16].

In this paper, we define the Sobolev space

$$H^1(M) = \overline{C^\infty(M)}.$$

with respect to the norm

$$\|u\|_1 = \sqrt{\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2}.$$

$H^1(M)$ is a Hilbert space induced by the inner product

$$\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{L^2}.$$

In addition, we include the following

$$H_0^1(M) = \overline{C_c^\infty(M)}.$$

As a result, we have

$$\overline{C_c^\infty(M)} \subset H_0^1(M) \subset H^1(M) \subset L^2(M).$$

In order to measure the smallness part A of manifold M we need to define the concept of capacity :

$$cap(A) = inf\left\{\int_M |\nabla u|^2 dM \mid u \in H^1(M), \int_M u\, dM = 0, u - 1 \in H_0^1(M - A)\right\}.$$

Where $H_0^1(M - A) = \overline{C_c^\infty(M - A)}$.

In this paper, the second section states the main theorem. In the third section ,we give a general review of literature, and we focus on recent results. The capacity condition is replaced by the Lebesgue measure in the fourth section, resulting in discrete spectrum.

2. Main theorem

Theorem 2.1. Assume that (M, g) is a Riemannian manifold of bounded geometry, H is Schrödinger operator with potential satisfying equation (3) .There exists $c > 0$, such that spectrum of H is discrete if and only if : For any $(x_k) \subset M, k = 1,2,\ldots$ such that $x_k \to \infty$ as $k \to \infty$ and for compact subset $F_k \subset \overline{B}(x_k, r)$
where

$$cap(F_k) \le c r^{n-2} \; if\; n \ge 3,$$
$$cap(F_k) \le c\left(log\frac{1}{r}\right)^{-1} if\; n = 2.$$

Then the following condition holds

$$\int_{B(x_k,r)/F_k} V(x) d\mu(x) \to \infty \; as\; k \to \infty \tag{4}$$

The proof is announced in [13].

3. Literature surveys

For one-dimensional Schrödinger operator in $L^2(0,\infty)$, Weyl [5] gave the condition for the discreteness of H. He proved that if $V(t)$ is monotone i.e. $V(x) \to \infty$ as $x \to \infty$ then the spectrum of H is discrete.

For $L^2(\mathbb{R})$ space ,the condition (4) of the main theorem is sufficient for discrete spectrum you can follow [8] by I.M. Glazman. After that Molchanov results in $L^2(\mathbb{R})$ and I.B rink [7] replaced the semi-bounded condition (3) by the condition

$$\int_J V(x) dx \ge -c$$

Where J the interval of length ≤ 1.
Further improvements in $\mathbb{R}^n$ of discreteness spectrum for multidimensional operator is established by K.Friedrich [10] who used the condition $V(x) \to \infty$ as $x \to \infty$ whereas A.M. Molchanov [2] revealed two cases in $\mathbb{R}^n$ $(n \ge 3)$ and $(n = 2)$, he used the capacity in spectral theory.
Also V. G Mazya worked for same result [14] and other examples of discreteness of spectrum of Multidimsional Schrödinger operator by using concept of capacity in [9] .

The credited of studding spectrum of Schrödinger operator on Riemannian manifold due to G.Courtois [4] who treated the particular case when he applied perturbation type of Laplace $-\Delta$ on closed compact Riemannian manifold $M-A$, he proved that

$$0 \le \lambda_k(M-A) - \lambda_k \le C_k\sqrt{cap(A)}.$$

where$A$is a compact subset of M and $\lambda_k$the k^{th}eigenvalue of $-\Delta$on M.

For more details in spectrum of Schrödinger operator$H$on Riemannian manifold see [1] . In addition,[11,12] R.Brooks expressed in geometric terms the discreteness of spectrum of$H$by boundary estimates for the bottom of essential spectrum of Laplace .

4. Main Result

In this section, we will replace the capacity in theorem2.1by Lebesgue measure, we will prove that using Lebesgue measure lead us to equivalent condition for the spectrum of $H$to be discrete.

Theorem 4.1. Assume that$(M,g)$is Riemannian manifold of bounded geometry. Consider Has in equation (1) with semi-bounded potential$V(x)$.

(1) In the case of$n \ge 3$ there exists $c>0$ for any sequence $\{x_k | k = 1,2,\dots\} \subset Mandx_k \to \infty$ as $k \to \infty$for any$r < \frac{r_0}{2}$ and compact subsets $F_k \subset B(x_k, r) \subset M$such that

$$mesF_k \le cr^n. \tag{5}$$

where $mesF_k$is a Lebesgue measure in normal geodesic coordinates center at x_k.If the condition

$$\int_{B(x_k,r)\backslash F_k} V(x)d\mu \to \infty \text{ as} k \to \infty. \tag{6}$$

is satisfied then the spectrum of Schrödinger operator $\sigma_d(H)$ is discrete .

(2) In the case of$n=2$,there exist$N>0$,$c>0$ and $r_1>0$ such that the condition (6) holds for $r \in (0, r_1)$ and

$$mesF_k \le cr^N. \tag{7}$$

Then the spectrum of H is discrete.

Proof. We will demonstrate that Lebesgue condition in equation (5) is equivalent to capacity condition in theorem2.1with different constants, and we use the reference [15] to compare capacity with the use of Lebesgue measure in$\mathbb{R}^n$.

$$cap(F_k) \ge w_n^{2/n} n^{(n-2)/n}(n-2)(mesF_k)^{(n-2)/n} if n \ge 3, \tag{8}$$

$$cap(F_k) \ge 4\pi \left(log\frac{mesw_n}{mesF_k}\right)^{-1} if n = 2 \tag{9}$$

By using equation (8) , it is clear that

$$mesF_k \le A_n\big(cap(F_k)\big)^{n/(n-2)}$$

Therefore,

$$B_n\frac{1}{mesF_k} \ge \big(cap(F_k)\big)^{(n-2)/n}$$

Where $B_n = 1/A_n$ we get,

$$cap(F_k) \le c_1(mesF_k)^{(n-2)/n} \le c_1(r^n c)^{(n-2)/n}$$

This implies that for$C>0$

$$cap(F_k) \le Cr^{(n-2)}$$

It is obvious by using the main theorem2.1that spectrum of $H$is discrete .Similarly, for the case $n=2$

showing that condition$cap(F_k) \le \tilde{c}\left(log\frac{1}{r}\right)^{-1}$satisfies the condition $mesF_k \le cr^N$

where $r \in (0, r_1)$ and the constants r_1, c. In order to establish this, we can use the equation (9) we note that

$$mesF_k \leq mes\Omega.exp\left(-4\pi\left(cap(F_k)\right)^{-1}\right) \leq mes\Omega.exp\left(-4\tilde{c}^{-1}\,log\frac{1}{r}\right)$$
$$= mes\Omega.r^{4\pi\tilde{c}^{-1}} \leq c.r^N$$

Provided $\tilde{c}, r_1$ are sufficiently small.

Proposition 4.1. Assume that for any $A > 0$ and any $r \in (0, r_0/2)$

$$mes\{y|y \in B(x,r), V(y) \leq A\} \to 0 \text{as } x \to \infty \qquad (10)$$

Prove that under the conditions of theorem 2.1, the spectrum of H is discrete.

Proof. we denote mes the Lebesgue measure in geodesic coordinates. We have to prove that (10) implies $mesF_k \leq cr^n$ Indeed, we can obviously obtain

$$mes(B(x_k,r)\backslash F_k) \geq \frac{1}{2}mesB(x_k,r)$$

for c to be arbitrarily small.

we get

$$mes\big(\{y|V(y) \geq A\} \cap (B(x_k,r)\backslash F_k)\big) \geq \frac{1}{4}mesB(x_k,r)$$

hence

$$\int_{B(x_k,r)\backslash F_k} V(x)dx \geq \frac{1}{4}A.mesB(x_k,r)$$

which implies

$$\int_{B(x_k,r)\backslash F_k} V(x)dx \to \infty \text{ as } x \to \infty$$

Therefore by theorem 4.1 we have $\sigma(H) = \sigma_d$.

Example 4.2. Consider the Schrödinger operator $H = -\Delta + 4x^2y^2$ in $\mathbb{R}^2$ with the standard metric.
We will note a short supply in the set $\{(x,y)|4x^2y^2 \geq A\}$ at infinity. Therefore the condition (10) is satisfied , where the intersections with a ball of a fixed radius r have zero measure as the ball tends to infinity so $\sigma(H) = \sigma_d$.

Example 4.3. The spectrum of $H = -\Delta + x^2y^2 + x^2z^2 + y^2z^2$ in $\mathbb{R}^3$, is discrete spectrum.
Similarly the proof of this example can be followed immediately by use (10) .

5. Conclusion

We investigated the discrete spectrum in theorem 4.1 employing the Lebesgue condition instead of capacity. We also reviewed all previous results in the study of discrete spectrum .Finally application examples were given to illustrate the main results.

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