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An Extended Kumaraswamy-Gull Alpha Power Exponential Distribution: Properties and Application to Real Data

Mohamed Kpangay [1], Leo O. Odongo [2], George O. Orwa [3]

[1] *Department of Mathematics, Pan African University Institute for Basic Sciences, Technology and Innovation, Nairobi, Kenya.*

[2] *Department of Mathematics and Actuarial Science, Kenyatta University, Nairobi, Kenya.*

[3] *Department of Statistics and Actuarial Science, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya.*

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Abstract−*This paper proposes a four-parameterized distribution, namely an extended Kumaraswamy-Gull Alpha Power Exponential distribution. The proposed distribution gives rise to some well-known sub-models. Some basic properties of the distribution are derived. The method of maximum likelihood estimation was employed to estimate the parameters of the distribution. A Monte Carlo simulation study was conducted to evaluate the performance of the MLE estimates. From the simulation results, it is observed that with an increase in sample size the average estimates approach the true value of the parameters, and the average bias, MSE, and RMSE decrease, in general. The proposed K-GAPE distribution is fitted to two real data sets and compared to its sub-models. A conclusion can be made that the purposed distribution performs better than its underlying sub-models.*

Keywords−*Exponential distribution, Gull Alpha Power Family, Kumaraswamy distribution, Maximum likelihood estimation*

1. INTRODUCTION

Probability distributions are a fundamental statistical concept that is mostly used in theory as well as practice. Data from various fields of study may show characteristics such as skewness, kurtosis, monotonic hazard rates such as increase or decrease hazard rates, and non-monotonic failure rates such as bathtub or modified bathtub hazard rates, such data are handled with distributions that are more flexible. Most of the classical distributions and some of the family of distributions cannot handle two or more of these data characteristics at the same time due to the single shape parameter of their cumulative density function.

To offer additional location, scale, and shape parameters to distributions variety of techniques have been used such as the variable transformation, exponentiation method, Quantile method, the combination of two or more distributions/models method, etc.

Therefore due to the availability of the above methods several new distributions and family of distributions have been developed and extended which include the exponentiated exponential family by [1], the Generalized exponentiated exponential family by [2], the Lomax exponential distribution by [3], the Modi exponential distribution by [4], The exponentiated generalized class of distributions by [5], The generalized odd half-Cauchy family of distributions: properties and applications by [6], On some further properties and application of Weibull-R family of distributions by [7], the odd generalized exponential family of distributions with applications by [8], the Kumaraswamy Marshall-Olkin family by [9], the Kumaraswamy Transmute-G family of distributions by [10], the generalized Marshall-Olkin Kumaraswamy-G family by [11], the Kumaraswamy alpha power-G family by [12], the Gull Alpha power Weibull distribution by [13], a variant of the Gull Alpha Power distribution by modifying the Chen-G type by [14], the Gull Alpha Power Ampadu-G type by [15] the exponentiated generalized gull alpha power Rayleigh (EGGAPR) distribution by [16], the exponentiated generalized gull alpha power exponential (EGGAPE) distribution by [17], the exponentiated gull alpha power exponential (EGAPE) distribution by [18] among others.

A four-parameterized model called K-GAPE has been introduced in this paper.

Two years ago, [13] defined the Gull Alpha Power family of distributions with CDF, $F(x)$ and PDF, $f(x)$ respectively as:

$$F(x) = \begin{cases} \frac{\alpha F(x)}{\alpha^{F(x)}}, & if \quad \alpha > 0, \alpha \neq 1 \\ F(x), & if \qquad \alpha = 1 \end{cases} \tag{1}$$

$$f(x) = log(\alpha)\alpha^{1-F(x)}(-f(x)F(x)) + f(x)\alpha^{1-F(x)}, \alpha > 0, \alpha \neq 1 \tag{2}$$

The CDF and PDF of GAPF are used as a baseline in the Kumaraswamy-G family of distributions introduced by [19] in which the new family called Kamuraswamy-Gull Alpha Power Family is developed. The main aim of extending this family is to develop a distribution that is characterized by different shapes (including hazard rate shapes), have heavy-tailed in order to model different real data sets, and provides a distribution that offers a better fit compared to other competing models with the same baseline distribution. The exponential distribution is used as a baseline in the new family of distributions to developed the Kumaraswamy-Gull Alpha Power Exponential distribution abbreviated as K-GAPE.

The proposed distribution has some special sub-models shown in Table 1.

The rest of the article is structured as follows: the new family and its proposed distribution is presented in Section “Proposed distribution”, some statistical properties are presented in Section “Statistical properties”, Parameters estimation, and the Monte Carlo simulation study results are presented in Section “Parameters Estimation”. The proposed distribution is then applied to real data sets with its competing sub-models in Section "Application to real data", and in Section "conclusion" the concluding remarks are presented.

Table 1: **Summary of special sub-models of K-GAPE distribution.**

α	a	b	**Sub-model**	**comment**
1	1	1	The E distribution	Exist
1	a	1	Th EE distribution	Exist
α	1	1	The GAPE distribution	New
α	a	1	The EGAPE distribution	Exist
1	a	b	The KE distribution	Exist

2. PROPOSED DISTRIBUTION

2.1. *Kumaraswamy-Gull Alpha Power Family of Distributions*

According to [19] the CDF, F(x) and PDF, f(x) of Kumaraswamy-G family distributions, respectively, are given by

$$F_{K-G}(x) = 1 - \{1 - [F(x)]^a\}^b, a > 0, b > 0 \tag{3}$$

and

$$f_{K-G}(x) = abf(x)[F(x)]^{a-1}\{1 - [F(x)]^a\}^{b-1} \tag{4}$$

where $x > 0$, $a > 0$, and $b > 0$

In order to develop the new family of distributions called K-GAPF, we insert Eqs. (1) and (2) of GAPF into Eqs. (3) and (4). Then the new family will have a CDF defined by

$$F_{K-GAPF}(x) = 1 - \left\{1 - \left[\frac{\alpha F(x)}{\alpha^{F(x)}}\right]^a\right\}^b \tag{5}$$

and PDF as

$$f_{K-GAPF}(x) = ab(log(\alpha)\alpha^{1-F(x)}(-f(x)F(x)) + f(x)\alpha^{1-F(x)}) \times \left[\frac{\alpha F(x)}{\alpha^{F(x)}}\right]^{a-1} \times \left\{1 - \left[\frac{\alpha F(x)}{\alpha^{F(x)}}\right]^{a}\right\}^{b-1} \quad (6)$$

for $a > 0, b > 0$.

The survival function and hazard rate function of the new family, respectively, are given by

$$SF(x) = 1 - F(x) = \left\{1 - \left[\frac{\alpha F(x)}{\alpha^{F(x)}}\right]^{a}\right\}^{b} \quad (7)$$

and

$$HRF(x) = \frac{f(x)}{1 - F(x)} = \frac{f_{K-GAPF}(x)}{SF(x)} \quad (8)$$

2.2. ***K-GAP Exponential distribution***

To exhaustively define the K-GAPE distribution, CDF, $F(x)$ and PDF, $f(x)$ of the exponential distribution are first defined

$$F(x) = 1 - e^{-\beta x}$$

and

$$f(x) = \beta e^{-\beta x}$$

for $x > 0$, and $\beta > 0$ is scale parameter.

Therefore, the proposed distribution, K-GAPE (a, b, α, β) distribution has the CDF, PDF, and HRF defined as:

$$F(x) = 1 - \left\{1 - \left[\frac{\alpha(1 - e^{-\beta x})}{\alpha^{1-e^{-\beta x}}}\right]^{a}\right\}^{b} \quad (9)$$

$$f(x) = ab(\beta\alpha^{e^{-\beta x}}e^{-\beta x} - \alpha^{e^{-\beta x}}log(\alpha)\beta(1-e^{-\beta x})e^{-\beta x})\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a-1} \left\{1-\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a}\right\}^{b-1} \quad (10)$$

$$HRF(x) = \frac{J\left[\frac{\alpha(1-e^{-\beta x})}{\alpha^{e^{-\beta x}}}\right]^{a-1}\left\{1-\left[\frac{\alpha(1-e^{-\beta x})}{\alpha^{1-e^{-\beta x}}}\right]^{a}\right\}^{b-1}}{\left\{1-\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a}\right\}^{b}} \quad (11)$$

for $J = ab(\beta\alpha^{e^{-\beta x}}e^{-\beta x} - \alpha^{e^{-\beta x}}log(\alpha)\beta(1-e^{-\beta x})e^{-\beta x}), x > 0.$

Figure 1 shows the plots of an extended K-GAPE PDF which can be unimodal, reversed J-shaped, right-skewed, almost symmetric (approximately symmetric), left-skewed, with a fat tail, and a highly flexible kurtosis, making it suitable for a wide range of data.

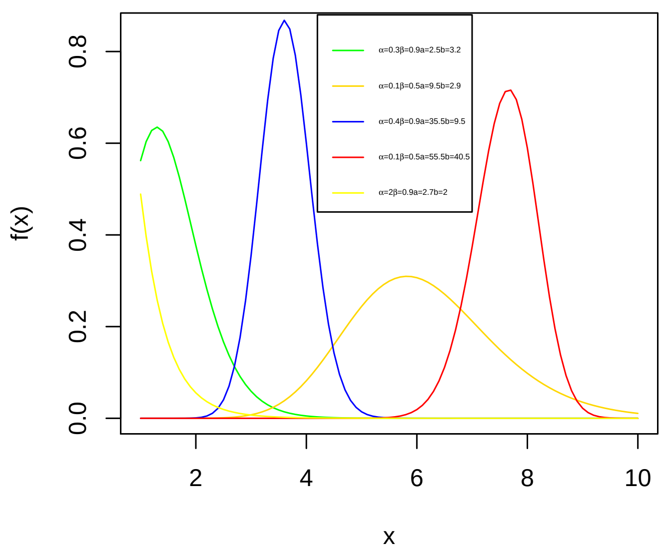


Figure 1: **K-GAPE density function for some parameters values**

As observed in **Figure 2** the plot of the hazard rate function for different parameter values shows a variety of shapes, including bathtub shapes, increasing, and decreasing shapes.

These are very appealing characteristics that make the extended K-GAPE distribution suitable for modeling monotonic and non-monotonic hazard behaviors that are more likely to be encountered in practical situations such as reliability analysis, human mortality, and biomedical applications, thereby increasing its adaptability to fit diverse survival data.

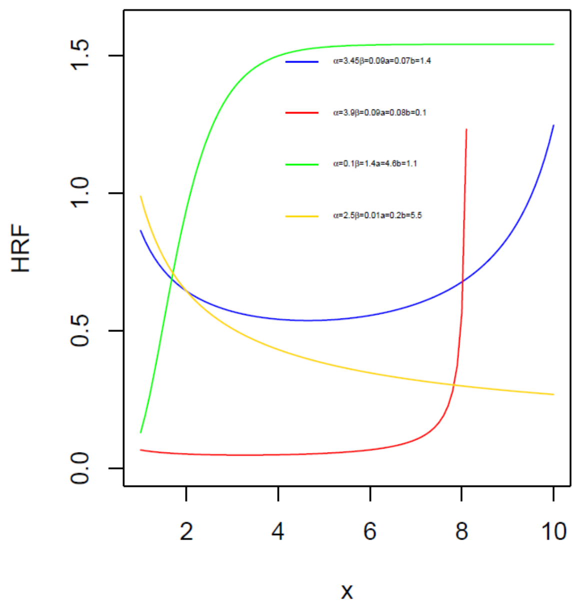


Figure 2: **K-GAPE hazard rate function shapes**

3. STATISTICAL PROPERTIES

This section investigate and gives the shapes/plots and numerical values of some of the basic statistical properties of K-GAPE distribution.

*3.1. **Quantile function***

The quantile function has been used to calculate median, skewness, kurtosis, conducting simulation study etc. Using Mathematica software, the quantile function is obtained as:

$$QF_x(\mu) = \frac{log\left[\frac{log(\alpha)}{W(z)[-G]+log(\alpha)}\right]}{\beta}, \alpha > 0, \neq 1 \tag{12}$$

where $G = \frac{\left(1-(1-\mu)^{\frac{1}{b}}\right)^{\frac{1}{a}} log(\alpha)}{\alpha}$ and the $productLog$ function $W(z)$ is defined as

$$W(z) = \frac{\sum_{n=1}^{\infty}(-1)^{n-1}n^{n-2}}{(n-1)!}z^n$$

For the median, put $\mu = 0.5$ in Eq. (12).
Table 2 shows some values for the quantile function for different parameter values.

Table 2: **Summary of some Quantile values of K-GAPE distribution.**

μ	$\alpha = 0.6$, $\beta = 0.5$, $a = 3.5$, $b = 2.5$	$\alpha = 0.6$, $\beta = 0.5$, $a = 4.5$, $b = 2.7$
0.1	1.448188	1.809884
0.2	1.860773	2.235246
0.3	2.194995	2.573243
0.4	2.505562	2.88346
0.5	2.817362	3.191967
0.6	3.150815	3.519253
0.7	3.532745	3.891365
0.8	4.014819	4.357621
0.9	4.751236	5.06406

3.2. *The r^{th} Moments*

Given the r^{th} Moments as

$$\omega_r^{'} = \int_0^{\infty} x^r f_{K-GAPE}(x)dx \tag{13}$$

where $f(x)$ is the PDF and $r = 1, 2, 3, ..., n$

by substituting the K-GAPE PDF as in Eq. (10) into Eq. (13) we have:

$$\omega_r^{'} = \int_0^{\infty} x^r ab(\beta\alpha^{e^{-\beta x}} e^{-\beta x} - \alpha^{e^{-\beta x}} log(\alpha)\beta(1-e^{-\beta x})e^{-\beta x}) \left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a-1} \left\{1-\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^a\right\}^{b-1} dx \quad (14)$$

Useful binomial expansion representations

$$(m-n)^{n-1} = \sum_{t=0}^{\infty} \binom{n-1}{t} m^{n-1-t}(-n)^t = \sum_{t=0}^{\infty}(-1)^t \binom{n-1}{t} (m)^{n-1-t}(n)^t$$

$$(1-x)^{n-1} = \sum_{t=0}^{\infty}(-1)^t \binom{n-1}{t} (1)^{n-1-t}x^t = \sum_{t=0}^{\infty}(-1)^t \binom{n-1}{t} x^t$$

Using the binomial expansion representations in Eq. (14),
then

$$\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a-1} = \left(\alpha^{e^{-\beta x}} - \alpha^{e^{-\beta x}}e^{-\beta x}\right)^{a-1} = \sum_{c=0}^{\infty}(-1)^c \binom{a-1}{c} \left(\alpha^{e^{-\beta x}}\right)^{a-c-1} \left(\alpha^{e^{-\beta x}}e^{-\beta x}\right)^c$$

$$\left\{1-\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^a\right\}^{b-1} = \sum_{d=0}^{\infty}(-1)^d \binom{b-1}{d} \left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{ad}$$

$$\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{ad} = \left(\alpha^{e^{-\beta x}} - \alpha^{e^{-\beta x}}e^{-\beta x}\right)^{ad} = \sum_{g=0}^{\infty}(-1)^g \binom{ad}{g} \left(\alpha^{e^{-\beta x}}\right)^{ad-g} \left(\alpha^{e^{-\beta x}}e^{-\beta x}\right)^g$$

By using the three relations above, Eq. (14) can be written as;

$$\omega_r^{'} = ab\beta \sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\int_0^{\infty} x^r(-1)^{c+d+g}\binom{a-1}{c}\binom{b-1}{d}\binom{ad}{g} \left(\alpha^{e^{-\beta x}}\right)^{a-c-g+ad-1}\left(\alpha^{e^{-\beta x}}e^{-\beta x}\right)^{c+g+1} \left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{ad}\left(1-log(\alpha)(1-e^{-\beta x})\right)dx \quad (15)$$

From observation, we notice that Eq. (15) is not in closed form, hence, the moments are obtained by numerical integration.
Table 3 shows the first five values for the moments of the K-GAPE distribution for different parameter values. **I:** $\alpha = 0.1$, $\beta = 1.9$, $a = 10.7$, $b = 15.9$; **II:** $\alpha = 0.7$, $\beta = 1.9$, $a = 10.7$, $b = 15.9$; **III:** $\alpha = 0.1$, $\beta = 2.5$, $a = 10.7$, $b = 15.9$; **VI:** $\alpha = 0.1$, $\beta = 1.9$, $a = 14.5$, $b = 15.9$; **V:** $\alpha = 0.1$, $\beta = 1.9$, $a = 10.7$, $b = 2.10$.

Table 3: **Summary of r^{th} moments, skewness(B), and kurtosis(M)**

$\omega_r^{'}$	**I**	**II**	**III**	**IV**	**V**
$\omega_1^{'}$	1.27813	0.84823	0.97138	1.43591	1.79874
$\omega_2^{'}$	1.66815	0.74810	0.96352	2.09669	3.41301
$\omega_3^{'}$	2.22030	0.68578	0.97466	3.11060	6.83049
$\omega_4^{'}$	3.01033	0.64901	1.00431	4.68523	14.42011
$\omega_5^{'}$	4.15356	0.63322	1.05315	7.15977	32.12425
Var.	0.034523	0.029505	0.019941	0.034840	0.177531
SD	0.18580	0.17177	0.14121	0.18665	0.42134
CV	0.14537	0.20251	0.14537	0.12999	0.23424
B	-0.01170	0.08003	-0.01170	-0.01757	0.70453
M	3.01186	2.97860	3.01186	3.01560	4.00557

From **Table 3** it is observed that K-GAPE distribution is versatile in terms of means and variance. Also from **Table 3**, based on the values of the skewness, the K-GAPE distribution can be left-skewed (i.e. $B < 0$) and right-skewed (i.e. $B > 0$) when compared to the normal distribution (i.e.

$B = 0$). The K-GAPE distribution can also be platykurtic (i.e. $M < 3$), almost mesokurtic (i.e. almost $M = 3$), and leptokurtic (i.e $M > 3$) based on the values of the kurtosis.

3.3. Skewness and Kurtosis

To show the influence or effect of the additional shape parameters on the measure of shape skewness and kurtosis, the Bowley skewness coefficients and the Moors kurtosis are computed using quartiles and octiles, respectively. Both Bowley skewness and Moors kurtosis are insensitive to outliers, and their existence is guaranteed even for distributions with no moment. [20] defined the Bowley skewness based on quartiles as:

$$B = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

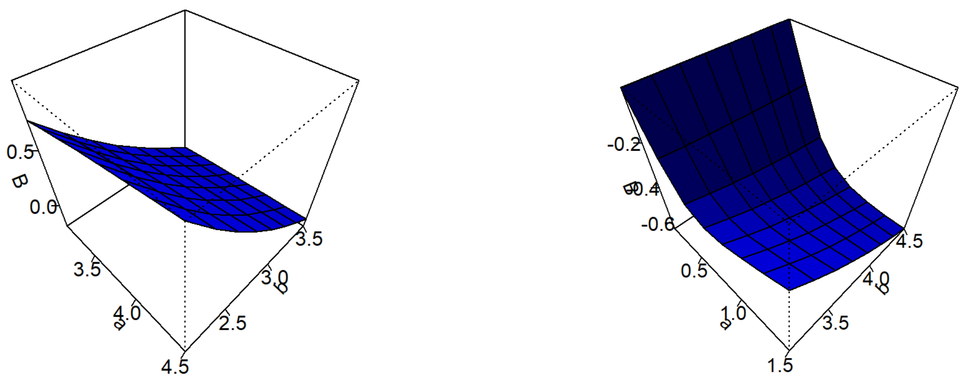
and [21] defined the Moors kurtosis based on octiles as

$$M = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

where Q is the quantile function.
With Eq. (12) the Bowley skewness and the Moors kurtosis are obtained.

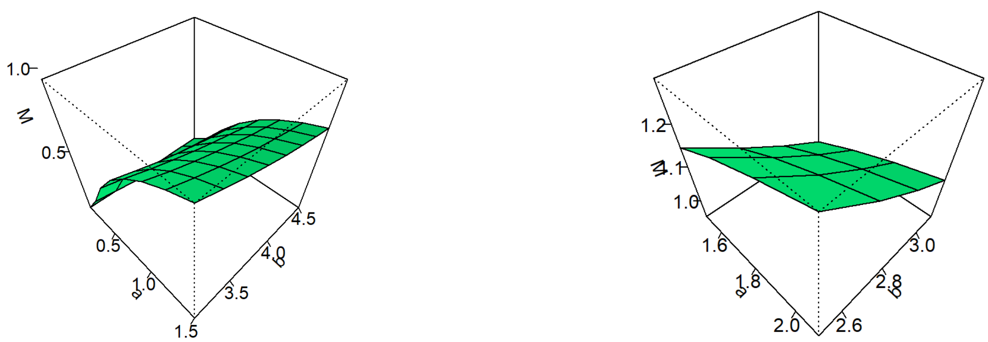
Figure 3 shows a 3D visualization of Bowley's skewness for fixed baseline parameter values of $\alpha = 0.5$ and $\beta = 0.8$. The shapes of the skewness coefficient are affected by the additional parameters, as observed. This enhances the K-GAPE distribution's flexibility and reinforces the importance of the additional parameters.

Figure 4 shows a 3D visualization of Moors's kurtosis for fixed baseline parameter values of $\alpha = 0.5$ and $\beta = 0.8$. The shapes of the kurtosis coefficient are affected by the additional parameters, as observed. This enhances the K-GAPE distribution's flexibility and reinforces the importance of the additional parameters.



(a) (b)

Figure 3: **Plots of Bowley's skewness for K-GAPE distribution for fixed baseline parameter values of** $\alpha = 0.5$ **and** $\beta = 0.8$.



(a) (b)

Figure 4: **Plots of Moors kurtosis for K-GAPE distribution for fixed baseline parameter values of** $\alpha = 0.5$ **and** $\beta = 0.8$.

3.4. Entropy of K-GAPE Distribution

Entropy has been used as a measure of variation or uncertainty of a random variable in various circumstances or situations in science, engineering, and probability theory.

[22] defined the R$\acute{e}$nyi Entropy for a random variable X with any distribution and order δ as

$$R_\delta(X) = \frac{1}{1-\delta} log \left\{ \int_0^\infty [f(x)]^\delta \, dx \right\} \quad (16)$$

for $\delta > 0$, $\delta \neq 1$.
Substituting PDF of K-GAPE distribution as in Eq. (10) into Eq. (16), we have

$$R_\delta(X) = \frac{1}{1-\delta} log\{\int_0^\infty (ab(\beta\alpha^{e^{-\beta x}} e^{-\beta x} - \alpha^{e^{-\beta x}} log(\alpha)\beta(1-e^{-\beta x})e^{-\beta x}) \times \left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a-1} \left[1 - \left((1-e^{-\beta x})\alpha^{e^{-\beta x}}\right)^a\right]^{b-1})^\delta dx\} \quad (17)$$

Employing the useful binomial expansions in Eq. (17), we can now write the Rényi entropy as

$$R_\delta(X) = \frac{1}{1-\delta} \times log\{(ab\beta)^\delta \sum_{c=0}^\infty \sum_{d=0}^\infty \sum_{g=0}^\infty \int_0^\infty [(-1)^{c+d+g} \binom{a-1}{c} \binom{b-1}{d} \binom{ad}{g} \left(\alpha^{e^{-\beta x}}\right)^{a-c-g+ad-1} \left(\alpha^{e^{-\beta x}} e^{-\beta x}\right)^{c+g+1} \left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{ad} \left(1 - log(\alpha)(1-e^{-\beta x})\right)]^\delta dx\} \quad (18)$$

The K-GAPE distribution entropy values are obtained through numerical integration and are shown in **Table 4** for various parameter values
I: $\alpha = 0.6$, $\beta = 1.3$, $a = 1.5$, $b = 1.6$;
II: $\alpha = 0.9$, $\beta = 1.7$, $a = 2.5$, $b = 1.3$;
III: $\alpha = 2.4$, $\beta = 0.7$, $a = 2.1$, $b = 1.7$;
IV: $\alpha = 2.5$, $\beta = 0.5$, $a = 0.9$, $b = 1.1$;
V: $\alpha = 1.1$, $\beta = 2.0$, $a = 2.2$, $b = 1.1$.

Table 4: **Some values of the Rényi entropy for K-GAPE distribution.**

R_δ	**I**	**II**	**III**	**IV**	**V**
$R_{(0.2)}$	1.46917	1.40774	1.57952	2.17575	1.34036
$R_{(0.5)}$	0.99666	0.93090	0.89879	1.26013	0.82013
$R_{(0.7)}$	0.86412	0.79490	0.71454	0.97610	0.66969
$R_{(1.5)}$	0.63281	0.55439	0.41099	0.45783	0.40279
$R_{(1.7)}$	0.60261	0.52257	0.37301	0.38635	0.36768
$R_{(2.5)}$	0.52091	0.43610	0.27207	0.18433	0.27278

*3.5. **Order Statistic***

If the general i^{th} order statistic is given by

$$f_{(i:n)}(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1 - F(x)]^{n-i}f(x) \quad (19)$$

Then by substituting Eqs. (9) and (10) into Eq. (19), the PDF of the i^{th} order statistic of the K-GAPE distribution is given by:

$$\begin{aligned} f_{(i:n)}(x) = \frac{n!}{(i-1)!(n-i)!} ab(\beta\alpha^{e^{-\beta x}}e^{-\beta x} - \alpha^{e^{-\beta x}}log(\alpha)\beta(1-e^{-\beta x})e^{-\beta x}) \\ \times \left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a-1}\left\{1-\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a}\right\}^{b-1} \\ \times\left\{1-\left[1-\left((1-e^{-\beta x})\alpha^{e^{-\beta x}}\right)^{a}\right]^{b}\right\}^{i-1}\left\{\left[1-\left((1-e^{-\beta x})\alpha^{e^{-\beta x}}\right)^{a}\right]^{b}\right\}^{n-i} \end{aligned} \quad (20)$$

while the PDF of the smallest order statistic, $f_{(1:n)}(x)$ is given by

$$f_{(1:n)}(x) = n{\cdot}ab\beta e^{-\beta x}\alpha^{e^{-\beta x}}\left(1-log(\alpha)(1-e^{-\beta x})\right)\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a-1}\times$$
$$\left\{1-\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a}\right\}^{b-1}\left\{\left[1-\left((1-e^{-\beta x})\alpha^{e^{-\beta x}}\right)^{a}\right]^{b}\right\}^{n-1} \quad (21)$$

and the PDF of the largest order statistic, $f_{(n:n)}(x)$ is given by

$$f_{(n:n)}(x) = n{\cdot}ab\beta e^{-\beta x}\alpha^{e^{-\beta x}}\left(1-log(\alpha)(1-e^{-\beta x})\right)\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a-1}\times$$
$$\left\{1-\left[(1-e^{-\beta x})\alpha^{e^{-\beta x}}\right]^{a}\right\}^{b-1}\left\{1-\left[1-\left((1-e^{-\beta x})\alpha^{e^{-\beta x}}\right)^{a}\right]^{b}\right\}^{n-1} \quad (22)$$

4. PARAMETERS ESTIMATION

In order to determine the MLEs of the given parameter estimation, we utilize the log-likelihood function, for the model parameters a, b, α, β, which can be written as

$$log(\ell) = nlog(ab)+2nlog(\beta)-2\beta\sum_{i=1}^{n}x_i+log(log(\alpha))+\sum_{i=1}^{n}log(1-e^{-\beta x_i})+$$
$$(a-1)\sum_{i=1}^{n}log\left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)+(b-1)\sum_{i=1}^{n}log\left[1-\left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)^{a}\right] \quad (23)$$

For the parameters a, b, α, and β, the maximum likelihood estimates $\hat{a}$, $\hat{b}$, $\hat{\alpha}$, $\hat{\beta}$ are the values that maximize the log-likelihood function given in Eq. (23). The first partial derivatives of the log-likelihood function in Eq. (23) with respect to a, b, α, β are given by:

For easy differentiation,
first, let

$$t = nlog(ab) + 2nlog(\beta) - 2\beta \sum_{i=1}^{n} x_i + log(log(\alpha)) + \sum_{i=1}^{n} log(1 - e^{-\beta x_i}),$$

$$v = (a-1) \sum_{i=1}^{n} log\left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)$$

and

$$z = (b-1) \sum_{i=1}^{n} log\left[1 - \left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)^{a}\right]$$

Eq. (23) is now reduced to

$$log(\ell) = t + v + z \tag{24}$$

Then we express the log-likelihood in terms of t, v, and z and perform their derivatives w.r.t. a, b, α, β as shown.

$$\frac{\partial t}{\partial a} = \frac{n}{a}, \frac{\partial t}{\partial b} = \frac{n}{b}$$

$$\frac{\partial t}{\partial \alpha} = \frac{1}{\alpha \log \alpha},$$

$$\frac{\partial t}{\partial \beta} = \frac{2n}{\beta} - 2\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{e^{-\beta x_i} x_i}{(1 - e^{-\beta x_i})}$$

$$\frac{\partial v}{\partial a} = \sum_{i=1}^{n} log\left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)$$

$$\frac{\partial v}{\partial b} = 0$$

$$\frac{\partial v}{\partial \alpha} = (a-1)\sum_{i=1}^{n} e^{-\beta x_i}\alpha^{-1}$$

$$\frac{\partial v}{\partial \beta} = (a-1)\sum_{i=1}^{n} e^{-\beta x_i} x_i \left[1 - log(1-e^{-\beta x_i})\right]$$

$$\frac{\partial z}{\partial a} = (b-1)\sum_{i=1}^{n} \frac{\left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)^a log\left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)}{\left[\left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)^a - 1\right]}$$

$$\frac{\partial z}{\partial b} = \sum_{i=1}^{n} log\left[1 - \left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)^a\right]$$

$$\frac{\partial z}{\partial \alpha} = a(b-1)\sum_{i=1}^{n} \frac{e^{-\beta x_i}\left((1-e^{-\beta x_i})\alpha^{e^{-\beta_i}}\right)^a}{\alpha\left[\left(\alpha^{e^{-\beta x_i}}(1-e^{-\beta x_i})\right)^a - 1\right]}$$

$$\frac{\partial z}{\partial \beta} = a(b-1)\sum_{i=1}^{n} \frac{e^{-\beta x_i} x_i \left(\frac{\alpha(1-e^{-\beta x_i})}{\alpha^{1-e^{-\beta x_i}}}\right)^a \left(e^{-\beta x_i}(log(\alpha)-1) - log(\alpha)\right)}{\left[\left(\alpha^{e^{-\beta x_i}}(1-e^{-\beta x_i})\right)^a - 1\right](e^{\beta x_i}-1)} = z_\beta^{'}$$

Hence, the partial derivatives of log(ℓ) with respect to each parameter and equating each to zero are as follows:

$$\frac{\partial log(\ell)}{\partial a} = \frac{n}{a} + \frac{\partial v}{\partial a} + \frac{\partial z}{\partial a} = 0 \tag{25}$$

$$\frac{\partial log(\ell)}{\partial b} = \frac{n}{b} + \frac{\partial z}{\partial b} = 0 \tag{26}$$

$$\frac{\partial log(\ell)}{\partial \alpha} = \frac{1}{\alpha \log \alpha} + \frac{\partial v}{\partial \alpha} + \frac{\partial z}{\partial \alpha} = 0 \tag{27}$$

$$\frac{\partial log(\ell)}{\partial \beta} = \frac{2n}{\beta} - 2\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{e^{-\beta x_i} x_i}{(1-e^{-\beta x_i})} + \frac{\partial v}{\partial \beta} + \frac{\partial z}{\partial \beta} = 0 \quad (28)$$

From observation, we notice that Eqs. (25)-(28) do not have a closed form solution and hence, a numerical optimization method is required to find their solutions.
In this paper, the Broyden-Fletcher-Goldfarb-Shannon (BFGS) algorithm was used to estimate the parameters of K-GAPE distribution, and both the log-likelihood function gradient vector and Hessian matrix are required. The Hessian Matrix is a square matrix of second-ordered partial derivatives of the log-likelihood function with respect to the parameters.

The observed information matrix of the K-GAPE distribution is given by

$$J^{-1}(\theta) = \begin{bmatrix} \frac{\partial^2 log(\ell)}{\partial \alpha^2} & \frac{\partial^2 log(\ell)}{\partial \alpha \partial \beta} & \frac{\partial^2 log(\ell)}{\partial \alpha \partial a} & \frac{\partial^2 log(\ell)}{\partial \alpha \partial b} \\ & \frac{\partial^2 log(\ell)}{\partial \beta^2} & \frac{\partial^2 log(\ell)}{\partial \beta \partial a} & \frac{\partial^2 log(\ell)}{\partial \beta \partial b} \\ & & \frac{\partial^2 log(\ell)}{\partial a^2} & \frac{\partial^2 log(\ell)}{\partial a \partial b} \\ & & & \frac{\partial^2 log(\ell)}{\partial b^2} \end{bmatrix}$$

evaluated at $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})'$. On request, the expressions of the observed information matrix will be provided.

4.1. *Monte Carlo simulation study*

A Monte Carlo simulation study was conducted to investigate the average biases (ABs), mean square errors (MSEs), root mean square errors (RMSEs), and the average estimates (AEs) of the maximum likelihood estimators for the parameters of the K-GAPE distribution.
A random sample of sizes **n**=50, 100, 150, 200, ..., 500 with 1000 iterations in each **n** value was generated from the K-GAPE distribution using the quantile function given in Eq. (12) and selected initial values for pa-

rameters set **I**: (α, β, a, b)=(0.6, 0.3, 1.6, 1.3) and set **II**: (α, β, a, b)=(0.4, 0.5, 1.4, 1.5). In each set, the average estimates (AEs), average biases (ABs), mean square errors (MSEs), and root mean square errors (RMSEs) are recorded shown in Tables 5 and 6.

To compute the Average Biases (ABs), the Mean Squared Error (MSEs), and the Root Mean Squared Error (RMSEs) the following equations were used

$$AB_{(\Phi)} = \frac{1}{G}\sum_{i=1}^{G}(\widehat{\Phi}_i - \Phi) \tag{29}$$

$$MSE_{(\Phi)} = \frac{1}{G}\sum_{i=1}^{G}(\widehat{\Phi}_i - \Phi)^2 \tag{30}$$

and

$$RMSE_{(\Phi)} = \sqrt{\frac{1}{G}\sum_{i=1}^{G}(\widehat{\Phi}_i - \Phi)^2} \tag{31}$$

Where G is the number of iterations, and $\widehat{\Phi}_i$ is an estimator of Φ.

Simulation results for AEs, ABs, MSEs, and RMSEs are displayed in Table 5 for the values of parameters in set **I** and Table 6 for the values of parameters in set **II**.

From Tables 5 and 6 the simulation results for the K-GAPE distribution show that the average estimates (AEs) approach the true values of the parameters as the sample size increases. The ABs, MSEs, and RMSEs for the estimators of the parameters decrease, in general, as the sample size presented increases.

Summary remarks on simulation results

i. As sample size increases, parameter estimates' values get closer/converge to the true values.

ii. With an increase in sample size, the AB, MSE, and RMSE of the parameters decrease generally.

Table 5: **Monte Carlo simulation study results for set I parameters**

(α, β, a, b)=(0.6, 0.3, 1.6, 1.3)								
	AEs				**ABs**			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
50	0.83159	0.39242	2.19337	3.36111	0.23159	0.09242	0.59337	2.06111
100	0.73711	0.34228	1.77443	3.37362	0.13711	0.04228	0.17443	2.07362
150	0.69701	0.33233	1.63980	2.87373	0.09701	0.03233	0.03980	1.57373
200	0.67916	0.30865	1.58639	2.95123	0.07916	0.00865	-0.01361	1.65123
250	0.65111	0.31824	1.57744	2.94126	0.05111	0.01824	-0.02246	1.64126
300	0.63047	0.31713	1.55362	2.61595	0.03047	0.01713	-0.04638	1.31595
350	0.64710	0.31208	1.57508	2.54004	0.04710	0.01208	-0.02492	1.24004
400	0.65016	0.30820	1.56076	2.54372	0.05016	0.00820	-0.03924	1.24372
450	0.62282	0.30421	1.54136	2.61883	0.02282	0.00421	-0.05864	1.31883
500	0.62464	0.30094	1.54391	2.64077	0.02464	0.00094	-0.05609	1.34077
	MSEs				**RMSEs**			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
50	0.53013	0.07608	3.64306	142.10708	0.72810	0.27583	1.90868	11.92087
100	0.33974	0.05077	0.85825	125.91265	0.58287	0.22532	0.92642	11.22108
150	0.28972	0.04552	0.46360	25.21725	0.53826	0.21336	0.68088	5.02168
200	0.23480	0.03750	0.27228	31.15187	0.48456	0.19365	0.52180	5.58129
250	0.18723	0.04062	0.23190	25.50512	0.43270	0.20154	0.48156	5.05026
300	0.17550	0.03542	0.18945	14.65920	0.41893	0.18820	0.43526	3.82674
350	0.16388	0.03277	0.16762	13.49265	0.40482	0.18104	0.40941	3.67323
400	0.15025	0.03498	0.14657	11.58580	0.38761	0.18704	0.38284	3.40379
450	0.12603	0.03144	0.11614	15.09647	0.35501	0.17730	0.34079	3.88542
500	0.12060	0.03198	0.11375	13.48112	0.34728	0.17883	0.33727	3.67166

Table 6: **Monte Carlo simulation study results for set II parameters**

(α, β, a, b)=(0.4, 0.5, 1.4, 1.5)								
	AEs				**ABs**			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
50	0.65771	0.62402	1.94034	4.15786	0.25771	0.12402	0.54034	2.67786
100	0.56063	0.59321	1.57915	3.83829	0.16063	0.09321	0.17915	2.33829
150	0.50655	0.55663	1.44440	3.14680	0.10655	0.05663	0.04440	1.64680
200	0.47637	0.54061	1.39799	3.20796	0.07637	0.04061	-0.00201	1.70796
250	0.48086	0.52579	1.41101	3.19389	0.08086	0.02579	0.01101	1.69389
300	0.46230	0.50107	1.37346	3.27715	0.06230	0.00107	-0.02654	1.77715
350	0.45955	0.49930	1.39433	3.20008	0.05955	-0.00070	-0.00567	1.70008
400	0.44027	0.50815	1.37191	3.11285	0.04027	0.00815	-0.02809	1.61285
450	0.43960	0.50981	1.36331	2.72725	0.03960	0.00981	-0.03559	1.22725
500	0.43455	0.49888	1.36823	2.97168	0.03455	-0.00112	-0.03177	1.47168
	MSEs				**RMSEs**			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
50	0.48051	0.17978	3.78942	112.82177	0.69319	0.42401	1.94664	10.62176
100	0.31261	0.15631	0.81812	79.80760	0.55912	0.39536	0.90450	8.93351
150	0.22665	0.11500	0.41197	26.30607	0.47680	0.33911	0.64185	5.12894
200	0.17371	0.10813	0.24718	35.52417	0.41679	0.32883	0.49717	5.96022
250	0.16238	0.09592	0.20755	28.50664	0.40296	0.30972	0.45558	5.33916
300	0.13829	0.08324	0.16169	24.03832	0.37187	0.28852	0.40210	4.90289
350	0.11485	0.07386	0.13184	26.41472	0.33890	0.27178	0.36310	5.13953
400	0.09951	0.08003	0.11480	20.30341	0.31545	0.28290	0.33883	4.50593
450	0.09799	0.06965	0.10193	13.51624	0.31303	0.26392	0.31926	3.67644
500	0.08911	0.06861	0.09176	17.26409	0.29851	0.26194	0.30292	4.15501

iii. The performance of each estimator is good, and they all yield low AB, MSE, and RMSE values, this makes it very evident that the method of parameter estimation using maximum likelihood method works well.

5. APPLICATION TO REAL DATA

In this section, the importance and flexibility of K-GAPE distribution are studied by analyzing two real data sets. By employing the goodness-of-fit test and information criteria techniques, the data sets are used to compare the fits of the K-GAPE distribution and other competing models (i.e. its sub-models).

These sub-models include:

(a) The Exponential (E) distribution

When $\alpha = a = b = 1$ the CDF of the K-GAPE is reduced to the Exponential (E) distribution with CDF given as

$$F_E(x) = (1 - e^{-\beta x}), \text{ for } \beta > 0, x > 0$$

(b) The Exponentiated Exponential (EE) distribution

When $\alpha = b = 1$ the CDF of the K-GAPE is reduced to the Exponentiated Exponential (EE) distribution introduced by [23] with CDF given as

$$F_{EE}(x) = \left(1 - e^{-\beta x}\right)^a, \text{ for } a > 0, \neq 1, \beta > 0, x > 0$$

(c) The Kumaraswamy Exponential (KE) distribution

When $\alpha = 1$ the CDF of the K-GAPE is reduced to the Kumaraswamy Exponential (KE) distribution introduced by [24] with CDF given as

$$F_{KE}(x) = 1 - \left[1 - \left(1 - e^{-\beta x}\right)^a\right]^b, \text{ for } a, b > 0, \neq 1, \beta, x > 0$$

(d) The Exponentiated Gull Alpha Power Exponential (EGAPE) distribution

When $b = 1$ the CDF of the K-GAPE is reduced to the Exponentiated Gull Alpha Power Exponential (EGAPE) distribution introduced by [18] with CDF given as

$$F_{EGAPE}(x) = \left[\frac{\alpha(1-e^{-\beta x})}{\alpha^{1-e^{-\beta x}}}\right]^{a}, \text{for } \alpha, a > 0, \neq 1, \beta > 0, x > 0$$

(e) The Gull Alpha Power Exponential (GAPE) distribution

When $a = b = 1$ the CDF of the K-GAPE is reduced to the Gull Alpha Power Exponential (GAPE) distribution with CDF given as

$$F_{GAPE}(x) = \left[\frac{\alpha(1-e^{-\beta x})}{\alpha^{1-e^{-\beta x}}}\right], \text{for } \alpha > 0, \neq 1, \beta > 0, x > 0$$

Table 1 displays a summary of these sub-models.

5.1. *Data set 1: Sierra Leone COVID-19 daily confirmed cases data.*

The first data set consists of 62 COVID-19 daily confirmed cases in Sierra Leone ranging from 7^{th} December 2021 to 24^{th} February 2022, which are shown in **Table 7** . Sierra Leone COVID-19 daily confirmed cases data was retrieved from **`https://covid19.who.int`**.

Table 7: **Sierra Leone COVID-19 daily confirmed cases data.**

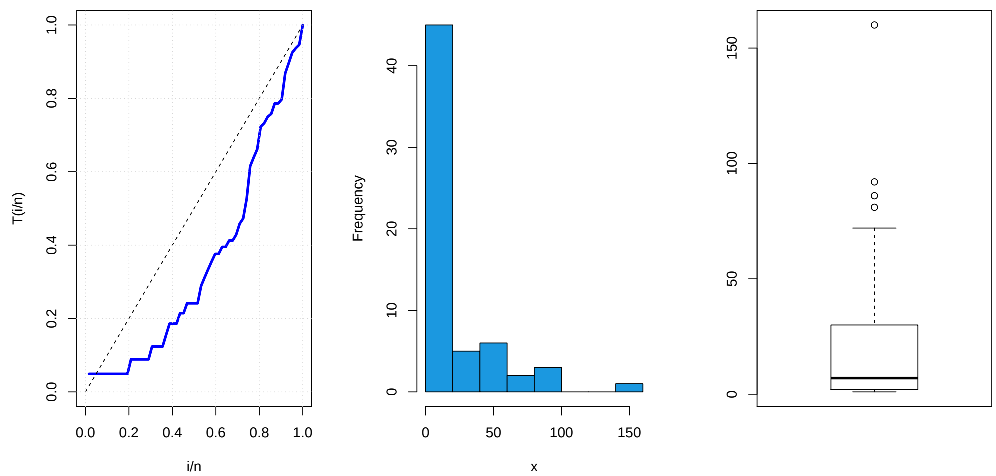
3	15	1	1	1	11	18	19	34	5	15	48	86
160	72	92	81	50	65	41	48	40	44	30	23	43
6	7	32	7	13	16	13	2	14	4	5	5	12
6	3	14	1	3	1	3	7	1	2	2	1	9
1	10	2	2	7	2	1	1	1	1			

Table 8 provides a summary of the most important descriptive statistics of Sierra Leone COVID-19 daily confirmed cases data. The value of the kurtosis is greater than 3 which implies that the data is leptokurtosis (i.e. greater than that of a normal distribution) and right-skewed due to the positive sign in the skewness value.

From **Figure 5** the TTT plot shows that the data has a monotonic decreasing (or convex) hazard rate shape defined by [25] and with a minimum amount of outliers.

Table 8: **Descriptive statistics for Sierra Leone COVID-19 data.**

Statistic	**min.**	**max.**	**mean**	**median**	**mode**	**var.**	**sd.**	**skewness**	**kurtosis**
Value	**1**	**160**	**20.37097**	**7**	**1**	**860.3683**	**29.33204**	**2.458649**	**10.25769**



(a) (b) (c)

Figure 5: **(a) TTT plot, (b) Histogram, and (c) Boxplot for data set 1**

In **Table 9**, the estimates for the parameters of the fitted models are presented along with their standard errors in parentheses. At the 5% level of significance, most of the parameters of the fitted models are significant according to the standard error test which state that a parameter is said to be significant at a 5% level of significance if the standard error is less than half the parameter value. Hence, the K-GAPE distribution offers a better fit to the Sierra Leone COVID-19 daily confirmed cases data than its sub-models.

As observed in **Tables 10** and **11**, the K-GAPE distribution has the smallest values of negative log-likelihood, A^*, $K-S$, W^*, and the information criteria statistics compare to its sub-models. This shows that

Table 9: **Summary of the estimates and SEs(in parentheses) for data set 1**

$Model$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
K-GAPE	2.6208(0.7899)	0.2039(0.0021)	0.8915(0.2783)	0.1628(0.0238)
EGAPE	1.7929(0.5529)	0.0023(0.0087)	0.7403(0.1293)	–
GAPE	1.5450(0.4581)	0.0377(0.0099)	–	–
KE	–	0.0255(0.0217)	0.6688(0.0994)	1.3696(0.9808)
EE	–	0.0355(0.0068)	0.6272(0.0977)	–
E	–	0.0490(0.0062)	–	–

the K-GAPE distribution provides a better fit in modeling the Sierra Leone COVID-19 daily confirmed cases data than its sub-models, although its sub-models also fit the data except for exponential distribution which has a $p - value < 0.05$.

Table 10: **The $-$log(ℓ) and goodness-of-fit results for data set 1.**

$Model$	$-log(\ell)$	A^*	$K-S$	$p-value$	W^*
K-GAPE(proposed)	**240.126**	**1.022**	**0.128**	**0.2597**	**0.131**
EGAPE(**exist**)	242.761	1.288	0.129	0.2571	0.180
GAPE(**new**)	244.837	1.286	0.137	0.2009	0.192
KE(**exist**)	243.632	1.410	0.133	0.2253	0.206
EE(**exist**)	243.945	1.450	0.129	0.2540	0.213
E(**exist**)	248.875	1.443	0.226	0.0036<.05	0.212

Table 11: **Summary of information criteria results for data set 1**

$Model$	AIC	BIC	$CAIC$	$HQIC$
K-GAPE(proposed)	**488.252**	**496.761**	**488.954**	**491.593**
EGAPE(**exist**)	491.522	497.904	491.936	494.028
GAPE(**new**)	493.674	497.896	493.881	495.329
KE(**exist**)	493.263	499.645	493.677	495.769
EE(**exist**)	491.889	497.144	492.093	493.560
E(**exist**)	499.750	501.877	499.817	500.585

Figure 6 displays a fitted densities plot for the K-GAPE distribution and its sub-models for Sierra Leone COVID-19 daily confirmed cases data

and it is seen that the K-GAPE distribution shows a promising fit over its sub-models.

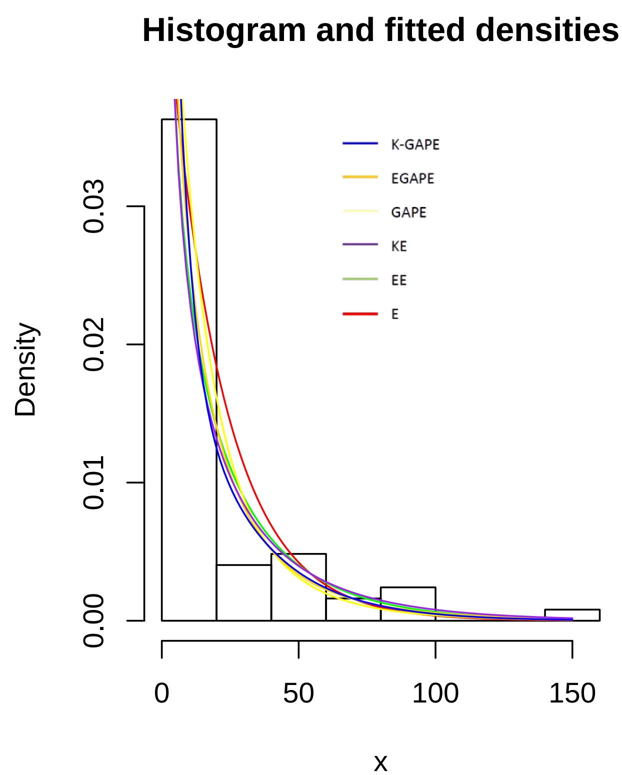


Figure 6: **The fitted densities for Sierra Leone COVID-19 daily confirmed cases data**

5.2. ***Data set 2: Tunisia COVID-19 daily death cases data.***

The second data set consists of 61 COVID-19 daily death cases in Tunisia ranging from 1^{st} September 2021 to 31^{st} October 2021, which are shown in **Table 12**. Tunisia COVID-19 daily death cases data was retrieved from **https://covid19.who.int**.

Table 13 provides a summary of the most important descriptive statistics of Tunisia COVID-19 daily death cases data. The value of the kurtosis is

Table 12: **Tunisia COVID-19 daily death cases data.**

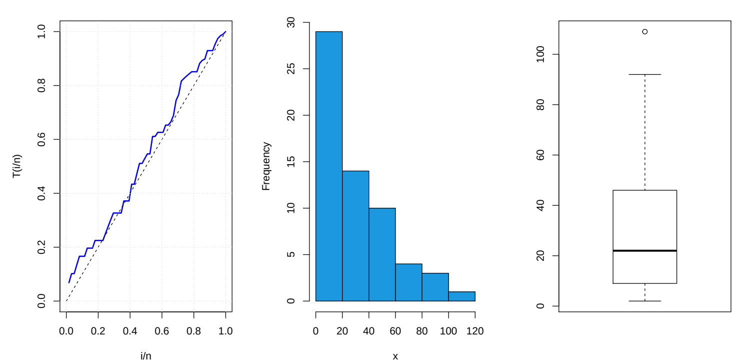
87	109	63	63	44	29	55	92	48	45	72	47	39
30	63	46	32	27	48	37	26	81	20	22	29	27
56	6	48	26	22	11	20	13	5	27	5	16	9
21	11	11	7	7	13	16	3	13	2	3	18	11
7	10	6	5	53	8	6	4	7				

greater than 3 which implies that the data is leptokurtosis (i.e. greater than that of a normal distribution), bimodal (i.e. having two modes), and right-skewed due to the positive sign in the skewness value.

Table 13: **Descriptive statistics for Tunisia COVID-19 data.**

Statistic	**min.**	**max.**	**mean**	**median**	**mode**	**var.**	**sd.**	**skewness**	**kurtosis**
Value	**2**	**109**	**29.29508**	**22**	**7&11**	**635.2781**	**25.20472**	**1.147799**	**3.748942**

From **Figure 7** the TTT plot shows that the data has a non-monotonic or modified bathtub hazard rate shape defined by [25] and with a single outlier.



(a) (b) (c)

Figure 7: **(a) TTT plot, (b) Histogram, (c) Boxplot for data set 2**

Table 14 displays the estimates for the parameters of the fitted models along with their standard errors in parentheses. At the 5% significance level, the parameters of the fitted models are significant according to the standard error test (i.e the standard error is less than half the parameter value).

Table 14: **Summary of the estimates and SEs(in parentheses) for data set 2**

$Model$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
K-GAPE	2.4248(0.0305)	0.3098(0.0019)	6.7446(0.1139)	0.1099(0.0141)
EGAPE	1.3854(0.6031)	0.0379(0.0091)	1.5902(0.4072)	-
GAPE	0.5945(0.2731)	0.6418(0.0079)	-	-
KE	-	0.3256(0.0039)	2.6928(0.5275)	0.1098(0.0142)
EE	-	0.0419(0.0065)	1.3685(0.2425)	-
E	-	0.0341(0.0044)	-	-

In **Tables 15** and **16**, the K-GAPE distribution has the smallest values of negative log-likelihood, A^*, $K-S$, W^*, and the information criteria statistics compare to its sub-models. This implies that the K-GAPE distribution also provides a better fit in modeling the Tunisia COVID-19 daily death cases data than its sub-models, although its sub-models also fit the data set i.e. their $p-values > 0.05$. **Figure 8** displays a fitted densities plot for

Table 15: **The $-\log(\ell)$ and goodness-of-fit results for data set 2.**

$Model$	$-log(\ell)$	A^*	$K-S$	$p-value$	W^*
K-GAPE(proposed)	**262.855**	**0.251**	**0.078**	**0.8541**	**0.041**
EGAPE(**exist**)	265.253	0.485	0.090	0.7083	0.076
GAPE(**new**)	266.393	0.588	0.083	0.7925	0.091
KE(**exist**)	264.283	0.391	0.095	0.6456	0.063
EE(**exist**)	265.480	0.509	0.089	0.7224	0.079
E(**exist**)	267.023	0.508	0.091	0.6895	0.079

Table 16: **Summary of information criteria results for data set 2**

$Model$	AIC	BIC	$CAIC$	$HQIC$
K-GAPE(proposed)	**533.710**	**538.153**	**534.424**	**535.019**
EGAPE(**exist**)	536.507	542.839	536.928	538.989
GAPE(**new**)	536.786	541.007	536.992	538.440
KE(**exist**)	534.566	540.899	534.987	537.048
EE(**exist**)	534.961	539.153	535.168	536.615
E(**exist**)	536.045	538.156	536.113	536.873

Tunisia COVID-19 daily death cases data and it is also observed that the K-GAPE distribution is the best-fitted compared to its sub-models.

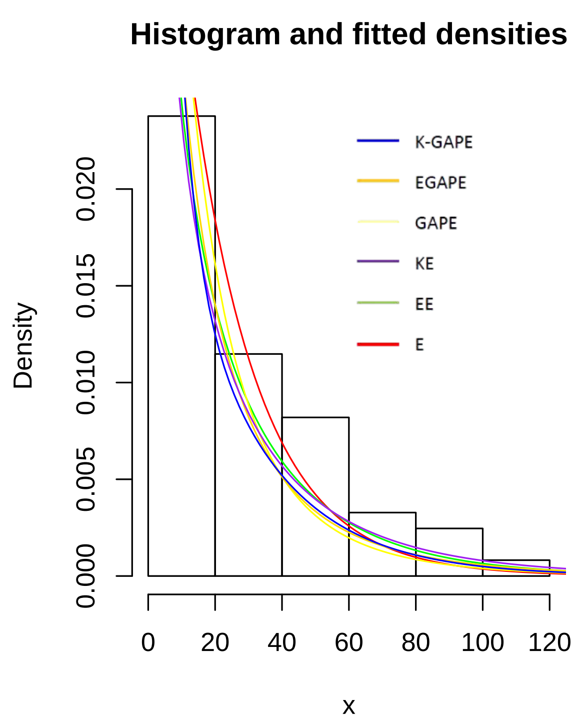


Figure 8: **The fitted densities for Tunisia COVID-19 daily death cases data**

Concluding statements on the data application

i. For the two data sets, a conclusion can be made that the K-GAPE provides the lowest values for the information criteria, $-log(\ell)$, $K-S$, the W^*, and, A^* and the highest p value compared to its competing models (i.e. sub-models).

ii. The best-fitting model for the two data sets, from Figures 6 and 8 was K-GAPE distribution.

iii. The Exponential (E) distribution offers a poor fitting for the data set I, as seen in Table 10.

iv. From the results of the two data applications, a conclusion can be made that the K-GAPE distribution gives the best fitting among its sub-models which means that the K-GAPE distribution has a greater edge in fitting this kind of data.

6. CONCLUSION

The Exponential distribution has been frequently employed in statistical research, particularly in engineering, financial science, and reliability.A new four-parameter model called an extended Kumaraswamy-Gull Alpha Power Exponential distribution, abbreviated as K-GAPE distribution has been proposed in this study. Some statistical properties such as survival function, hazard rate function, quantile function, moments, entropy, and order statistic are investigated. The parameters' maximum likelihood estimators of the K-GAPE are determined, and a Monte Carlo simulation study was conducted. The Average Estimates, the Average Biases, the Mean Square Errors, and the Root Mean Square Errors were computed. As the sample size increases the maximum likelihood estimates approach the true value, and the Average Biases, the Root Mean Square Errors, and the Root Mean Square Errors decrease with an increase in the sample size. Finally, the K-GAPE distribution was applied to two real data sets, COVID-19 data and the data show that the K-GAPE distribution gives the best fit for the data sets when compared to its competing models (sub-models).

7. CONFLICT OF INTEREST

The authors declares that there is no conflict of interest regarding the publication of this paper.

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References

[1] R. D. Gupta, D. Kundu, Exponentiated exponential family: an alternative to gamma and weibull distributions, Biometrical Journal: Journal of Mathematical Methods in Biosciences 43 (1) (2001) 117–130.

[2] R. D. Gupta, D. Kundu, Generalized exponential distribution: Existing results and some recent developments, Journal of Statistical planning and inference 137 (11) (2007) 3537–3547.

[3] M. Ijaz, S. M. Asim, Lomax exponential distribution with an application to real-life data, PloS one 14 (12) (2019) e0225827.

[4] K. Modi, D. Kumar, Y. Singh, A new family of distribution with application on two real datasets on survival problem, Science & Technology Asia 25 (1) (2020) 1–10.

[5] G. M. Cordeiro, E. M. Ortega, D. C. da Cunha, The exponentiated generalized class of distributions, Journal of data science 11 (1) (2013) 1–27.

[6] G. M. Cordeiro, M. Alizadeh, T. G. Ramires, E. M. Ortega, The generalized odd half-cauchy family of distributions: properties and applications, Communications in Statistics-Theory and Methods 46 (11) (2017) 5685–5705.

[7] I. Ghosh, S. Nadarajah, On some further properties and application of weibull-r family of distributions, Annals of Data Science 5 (3) (2018) 387–399.

[8] M. H. Tahir, G. M. Cordeiro, M. Alizadeh, M. Mansoor, M. Zubair, G. G. Hamedani, The odd generalized exponential family of distributions with applications, Journal of Statistical Distributions and Applications 2 (1) (2015) 1–28.

[9] M. Alizadeh, M. Tahir, G. M. Cordeiro, M. Mansoor, M. Zubair, G. Hamedani, The kumaraswamy marshal-olkin family of distributions, Journal of the Egyptian Mathematical Society 23 (3) (2015) 546–557.

[10] A. Z. Afifya, G. M. Cordeiro, H. M. Yousof, Z. M. Nofal, A. Alzaatreh, The kumaraswamy transmuted-g family of distributions: properties and applications, Journal of Data Science 14 (2) (2016) 245–270.

[11] S. Chakraborty, L. Handique, The generalized marshall-olkin-kumaraswamy-g family of distributions, Journal of data Science 15 (3) (2017) 391–422.

[12] M. E. Mead, A. Afify, N. S. Butt, The modified kumaraswamy weibull distribution: Properties and applications in reliability and engineering sciences, Pakistan Journal of Statistics and Operation Research 16 (3) (2020) 433–446.

[13] M. Ijaz, S. M. Asim, M. Farooq, S. A. Khan, S. Manzoor, A gull alpha power weibull distribution with applications to real and simulated data, Plos one 15 (6) (2020) e0233080.

[14] C. B. Ampadu, Gull alpha power of the chen type, Pharmacovigil and Pharmacoepi 3 (2) (2020) 16–17.

[15] C. B. Ampadu, Gull alpha power of the ampadu-type: Properties and applications, Earthline Journal of Mathematical Sciences 6 (1) (2021) 187–207.

[16] M. Kilai, G. A. Waititu, W. A. Kibira, M. Abd El-Raouf, T. A. Abushal, A new versatile modification of the rayleigh distribution for modeling covid-19 mortality rates, Results in Physics 35 (2022) 105260.

[17] M. Kilai, G. A. Waititu, W. A. Kibira, H. M. Alshanbari, M. El-Morshedy, A new generalization of gull alpha power family of distributions with application to modeling covid-19 mortality rates, Results in Physics 36 (2022) 105339.

[18] H. A. Khogeer, A. Alrumayh, M. Abd El-Raouf, M. Kilai, R. Aldallal, Exponentiated gull alpha exponential distribution with application to covid-19 data, Journal of Mathematics 2022 (2022).

[19] G. M. Cordeiro, M. de Castro, A new family of generalized distributions, Journal of statistical computation and simulation 81 (7) (2011) 883–898.

[20] J. F. Kenney, E. Keeping, Linear regression and correlation, Mathematics of statistics 1 (1962) 252–285.

[21] J. Moors, A quantile alternative for kurtosis, Journal of the Royal Statistical Society: Series D (The Statistician) 37 (1) (1988) 25–32.

[22] A. Rényi, On measures of entropy and information, in: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics, Vol. 4, University of California Press, 1961, pp. 547–562.

[23] R. D. Gupta, D. Kundu, Theory & methods: Generalized exponential distributions, Australian & New Zealand Journal of Statistics 41 (2) (1999) 173–188.

[24] K. Adepoju, O. Chukwu, Maximum likelihood estimation of the kumaraswamy exponential distribution with applications, Journal of Modern Applied Statistical Methods 14 (1) (2015) 18.

[25] M. V. Aarset, How to identify a bathtub hazard rate, IEEE transactions on reliability 36 (1) (1987) 106–108.