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Review Article

A Survey on Power Graphs of Semigroups

Miji Philip, P.U. Anusha

Department of Mathematics, TKM College of Engineering, APJ Abdul Kalam Technological University, Kerala, India

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Abstract - *The study of the graphical representation of semigroups and groups has become an exciting research area in the past few decades, leading to many captivating results and questions. This article reviews the current state of knowledge on the power graphs of semigroups by presenting all the results recorded in the literature.*

Keywords - *Semigroup, power graph, Undirected power graph, k- power graph, Enhanced power graph.*

1. Introduction

The concept of power graphs of a semigroup is a very recent development in the domain of graphs from semigroups. In 1964, Bosak [3] studied certain graphs over semigroups. The notation of the power graph of a semigroup S is introduced by Kelarev and Quinn [7]. The power graph Pow(S) of a semigroup S has all elements as vertices and has edges (u, v) ∀u, v ∈ S such that u ≠v and v is a power of u. Motivated by this, Chakrabarty et al. [4] defined the undirected power graph of a semigroup. The undirected power graph G (S) of a semigroup S is an undirected graph whose vertex set is S and, two vertices a, b ∈ S are adjacent if and only if a ≠ b and $a^m = b$ and $b^m = a$ for some positive integer m. In [5], S. Chattopadhyay et al. defined the k- power graph. For a semigroup S and a fixed positive integer k with k ≥ 2, we define k-power graph G (S, k) as a graph whose vertex set is S and two distinct vertices u, v are adjacent if and only if $u^k = v$ or $v^k = u$. Recently, Sandeep Dalal et al. introduced an enhanced power graph of semigroups [8]. The enhanced power graph Pe(S) of a semigroup S is a simple graph whose vertex set is S and two vertices x, y ∈ S are adjacent if and only if x, y ∈ ⟨z⟩ for some z ∈ S, where ⟨z⟩ is the subsemigroup generated by z.

There are various graphs defined on semigroups and groups[9 – 28]. Survey papers are available for power graphs of groups [1], [2]. In this paper, the power graph of semigroup is surveyed.

2. Preliminaries

Definition 2.1. *[6] A semigroup S is called monogenic if there exists a ∈ S such that S =* ⟨a⟩

Definition 2.2. *[6] An element a of a semigroup S is said to be idempotent if* $a^2 = e$.

Definition 2.3. *[4] Let* $n = p_1^{n_1} p_2^{n_2} \ldots p^{n_k} k.PFS[n] = \{ p_1, p_2, \ldots, p_k \}$ *and*
$\pi[n] = p_1 p_2 \ldots p_k$. *Assume* $PFS[1] = \varphi$ *and* $\varphi[1] = 1$.

3. Results

In [4], Chakrabarty et al. characterize the class of semigroup S for which G(S) is connected or complete. They show that G(S) (where S is finite) is connected if and only if S contains a single idempotent. Otherwise, G (S) is consisting of several components, each of which contains a single idempotent. Moreover, when S is commutative then the set of vertices of each component forms an Archimedean subsemigroup of S.

Here, E(S) denote the set of all idempotents of S and C_e denotes the equivalence class of e ∈ E(S) under ρ, i.e. C_e = {a ∈ S|aρe} = {a ∈ S|a^m = e} for some m ∈ N

Lemma 3.1. [4] Let S be a finite semigroup and ρ be the binary relation definedby aρb ⇔ $a^m = b^m$ for some m ∈ N, then for any a, b ∈ S, aρb if and only if for some m_1, m_2 ∈ N, e∈ E(S).

Corollary 3.2. [4] If S is a finite commutative semigroup then ρ is a congruence on S and S/ρ is a semilattice which is isomorphic to E(S). Each equivalence class is an Archimedean subsemigroup of S.

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Theorem 3.3. [4] Let S be a finite semigroup and $a, b \in S$ such that $a \neq b$ then a and b are connected by a path in the graph G(S) if and only if $a\rho b$.

Corollary 3.4. [4] The components of the graph G (S) are precisely $\{C_e | e \in E(S)\}$. Each component C_e contains the unique idempotent e. If S is commutative then C_e is a subsemigroup of S for each $e \in E(S)$.

Corollary 3.5. [4] Let S be a finite semigroup, then G (S) is connected if and only if S contains a single idempotent.

Proposition 3.6. [4] Let S be a semigroup such that G(S) is connected. Then S contains at most one idempotent.

Corollary 3.7. [4] Let S be a regular semigroup. If G (S) is connected, then S is a group.

Proposition 3.8. [4] Let S be a semigroup. Then G (S) is complete if and only if the cyclic subsemigroups of S are linearly ordered with respect to the usual containment relation (i.e., for any two cyclic subsemigroups S_1, S_2 of $S, S_1 \subseteq S_2$ or $S_2 \subseteq S_1$).

Next consider the graph G (Z_n) for the multiplicative semigroup Z_n of integers modulo n. They completely characterize the graph G(Z_n) along with all its components.

Corollary 3.9. [4] The components of the graph of G (Z_n) are precisely the collection of subsemigroups $\{C_S | S \subseteq PFS[n]\}$ of Z_n, where $C_S = \{\overline{a} \in Z_n | PFS[(n, a)] = S\}$. In particular C_φ is the group (U_n,.).

Theorem 3.10. [4] The graph G (Z_n) is disconnected for n > 1 and its components are determined by the congruence classes under ρ. There are $2^{|PFS[n]|}$ components in G (Z_n) and the number of vertices of the component C_a containing a vertex $\overline{a} \in Z_n$ is $\frac{\varphi(n)}{\varphi(\pi[(n,a)])} = l(a)$ (say). The vertex $\overline{a^{l(a)}}$ (which is an idempotent in Z_n) of C_a is adjacent to every other vertices of C_a and hence its degree is $l(a)-1$. In particular, the component C_1 is the subgraph G (U_n) with $l(1) = \varphi(n)$ and so $d(\overline{1}) = \varphi(n) - 1$.

Proposition 3.11. [4] G(Z_n) is Eulerian if and only if n = 2.

In [5], authors studied the structures of the components of G(Z_n) for the multiplicative semigroup Z_n. They observe that for every positive integer $k \geq 2$, G(S, k) is a spanning subgraph of G(S). Also, observe that the k-power graph of the multiplicative semigroup Zn of integers modulo n is the underlying simplegraph of the digraph G(n, k).

Theorem 3.12. [5] Let $n = n_1 n_2$, where $n_1 > 1$, $n_2 \geq 1$, and $\gcd(n_1, n_2) = 1$.

- **(i)** Suppose that $n_1 = p^\alpha$, where p is an odd prime and $\alpha \geq 1$. Suppose further that $k \equiv 1 (\mathrm{mod}\, p-1)$ and $p^{\alpha-1} | k$. Then G(Z_n, k) is symmetric of order p.
- **(ii)** Suppose that $n_1 = q_1 q_2 ... q_s$, where q_i's are distinct primes and $s \geq 2$. Suppose that $k \equiv 1 (\mathrm{mod}\, \lambda(n_1))$. Then G($Z_n$, k) is symmetric of order n_1.
- **(iii)** Suppose that $n_1 = p^\alpha q_1 q_2 ... q_s$ $\alpha \geq 2$, $s \geq 1$ and the q_i's are distinct primes such that $p \neq q_i$, and $p \nmid q_i - 1$ for $i = 1, 2, .. s$. Suppose further that $k \equiv 1 (\mathrm{mod}\, \lambda(p q_1 q_2 ... q_s))$ and $p^{\alpha-1} | k$. Then G(Z_n, k) is symmetric of order $p q_1 q_2 ... q_s$.

Theorem 3.13. [5] Let $n = p^m$, where p is an odd prime and $m \in N$. Then G(Zn) is not symmetric.

Theorem 3.14. [5] Let $n = 2p^m$, where p is an odd prime and $m \in N$. Then G (Z_n) is symmetric if and only if m = 1.

In [8] Sandeep Dalal et al. described the structure of $P_e(S)$ for an arbitrary semigroup S and discussed the connectedness of C. Further, they characterized the semigroup S such that $P_e(S)$ is complete, bipartite, regular, tree and null graph, respectively and had investigated the planarity together with the minimum degree and independence number of $P_e(S)$.

Proposition 3.15. [8] The set C(x) is a connected component of $P_e(S)$. Moreover, the components of the graph $P_e(S)$ are precisely $\{C(x)|x \in S\}$.

Corollary 3.16. [8] Let S be a semigroup. Then $P_e(S)$ is connected if and only if $\langle x\rangle\langle y\rangle$ for all $x, y \in S$. In this case, $diam(P_e(S)) \leq 2$.

Theorem 3.17. [8] Let S be a semigroup of bounded exponent. Then S_f is a connected component of $P_e(S)$ with unique idempotent f. Moreover, the con- nected components of C are precisely $S_f : f \in E(S)$ and the number of connectedcomponents of $P_e(S)$ is equal to $|E(S)|$.

Corollary 3.18. *[8] A semigroup S is a band if and only if* $P_e(S)$ *is a null graph.*

Theorem 3.19. [8] Let S be a semigroup with exponent n. Then $P_e(S)$ is complete if and only if S is a monogenic semigroup.

Theorem 3.20. [8] Let S be a semigroup. Then the following statements are equivalent: (i) The set $\pi(S) \subseteq \{1, 2\}$; (ii) $P_e(S)$ is acyclic graph; (iii) $P_e(S)$ is bipartite.

Corollary 3.21. *[8] The enhanced power graph* $P_e(S)$ *is a tree if and only if*
$|E(S)| = 1$ *and* $\pi(S) \leq \{1, 2\}$.

Theorem 3.22. *[8] The enhanced power graph* $P_e(S)$ *is k-regular if and only if*
S is the union of mutually disjoint monogenic subsemigroups of S of size $k + 1$.

Theorem 3.23. [8] Let S be a semigroup of bounded exponent. Then the connected components of the enhanced power graph $P_e(S)$ are complete if and only if S is the union of mutually disjoint monogenic subsemigroup of S.

Theorem 3.24. [8] A semigroup S with exponent n is completely regular if andonly if each connected component of $P_e(S)$ forms a group.

Proposition 3.25. [8] An element a of an arbitrary semigroup S is an isolated vertex in $P_e(S)$ if and only if

- ***(i)*** a is an idempotent in S.
- ***(ii)*** $H_a = \{a\}$.
- ***(iii)*** $m_x = 1$ for each $x \in S_a$.

Proposition 3.26. [8] Let $P_e(S)$ be a planar graph. Then $o(a) < 5$ for all $a \in S$.

Theorem 3.27. [8] Let S be a semigroup such that the index of every element of order four is either one or two. Then $P_e(S)$ is planar if and only if the following condition holds

(i) For $a \in S$, we have $o(a) \leq 4$

(ii) S does not contain $a, b, c \in S$ such that $o(a) = o(b) = o(c) = 4$, $m_a = m_b = m_c = 2$ and $|\langle a\rangle \cap \langle b\rangle \cap \langle c\rangle| = 3$.

Corollary 3.28. [8] Let S be a completely regular semigroup. Then $P_e(S)$ isplanar if and only if $o(a) \leq 4$ for all $a \in S$.

Theorem 3.29. [8] Let S be a semigroup with exponent n. Then
(i) $\delta(P_e(S)) = m - 1$, where $m = \min\{o(x) : x \in M\}$.

$\alpha(P_e(S))$ is the number of maximal monogenic subsemigroup of S.

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