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Original Article

Fixed Point Theorems for Expansive Mappings Via Tri-Simulation Function In Metric Space

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Abstract *- In this paper, we prove some fixed point theorems for expansive mappings in metric spaces with the aid of tri-simulation function.*

Keywords *- Tri-simulation function, Simulation function, α-permissible mapping.*

1. Introduction

Banach[3] in 1922, manifested unique fixed point of complete metric space. Various contraction conditions were introduced and there generalization was done by Banach principle(see [9], [14]). And it is obvious that from each new contraction condition a new theorem rises. Popa [12] in 1997, integrated some such conditions. To support this, the implicit function was introduced and it was utilized in ([1], [13]).

Wardowski [17], extended the principle introduced by Banach. He also inaugurated F-contraction. Imdad [7] and Piri [11] explored the f- contraction. Khojasteh *et al.* [8] integrated some contractions conditions and established the concept of simulation function.

Theorem 1.1 [17] Let (M, d) be a complete metric space. If h is a mapping of M into itself and if there exists a constant $q > 1$ such that

$$d(h(\bar{x}), h(\bar{y})) \geq q d(\bar{x}, \bar{y})$$

for each $\bar{x}, \bar{y} \in M$ and h is onto, then h has a unique fixed point in M.

Definition 1.2 [14] Let Ψ be the family of all functions $\psi: [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

1. $\Sigma_{j=1}^{+\infty} \psi^j(t) < +\infty$ for every $t > 0$, where ψ^j is the pth iterate of ψ;
2. ψ is nondecreasing.

Definition 1.3 [16] Let (M, d) be a metric space and $R: M \to M$ be a given self mapping. R is said to be an (ξ, γ)- expansive mapping if there exist two functions $\xi \in M$ and $\gamma: M \times M \to [0, +\infty)$ such that

$$\xi(d(Mx, My)) \geq \gamma(x, y) d(x, y) \quad (1)$$

for all $x, y \in M$.

Definition 1.4 [14] Let (M, d) be a metric space and $R: M \to M$ be a given self mapping. R is said to be an $\alpha - \psi$-contractive mapping if there exist two functions $\alpha: M \times M \to [0, \infty)$ and $\psi \in \varphi$ such that
$\alpha(\bar{x}, \bar{y}) d(R\bar{x}, R\bar{y}) \leq \psi\big(d(\bar{x}, \bar{y})\big)$, for all $\bar{x}, \bar{y} \in M$.

Definition 1.5 [14] Let $R: M \to M$ and $\alpha: M \times M \to [0, \infty)$. R is said to be α-admissible if
$\bar{x}, \bar{y} \in M$, $\alpha(\bar{x}, \bar{y}) \geq 1 \Rightarrow \alpha(R\bar{x}, R\bar{y}) \geq 1$.

Theorem 1.6 [14] Let (M, d) be a complete metric space and $R: M \to M$ be an $\alpha - \psi$-contractive mapping satisfying the following conditions:

1. R is α-admissible;
2. there exists $p_0 \in M$ such that $\alpha(p_0, Rp_0) \geq 1$;
3. R is continuous.

Then R has a fixed point, that is, there exists $\ddot{x} \in M$ such that $R\ddot{x} = \ddot{x}$.

Theorem 1.7 [14] Let (M, d) be a complete metric space and $R: M \to M$ be an $\alpha - \psi$-contractive mapping satisfying the following conditions:

1. R is α-admissible;
2. there exists $p_0 \in M$ such that $\alpha(p_0, Rp_0) \geq 1$;
3. if p_n is a sequence in M such that $\alpha(p_n, p_{n+1}) \geq 1$ for all n and $p_n \to p \in M$ as $n \to \infty$, then $\alpha(p_n, p) \geq 1$ for all n.

Then R has a fixed point.

For uniqueness the following condition $\breve{H}$ is added by Samet *et al.* [2] to the hypotheses of Theorem 1.5 and Theorem 1.6 : $(\breve{H})$ For all $\bar{x}, \bar{y} \in M$, there exists $\dot{z} \in M$ such that $\alpha(\bar{x}, \dot{z}) \geq 1$ and $\alpha(\bar{y}, \dot{z}) \geq 1$.

Shahi et al. [15] introduced new notion of (ξ, α)expansive mappings. And established different fixed point hypotheses for such mappings in complete metric spaces. On getting motivation from this we formulated the following result.

Theorem 1.8 [14] Let (M, d) be a complete metric space and $F: M \times M \to M$ be a given mapping. Suppose that there exists $\psi \in \Psi$ and a function $\gamma : M^2 \times M^2 \to [0, +\infty)$ such that

$\gamma((\bar{x}, \bar{y}), (\bar{u}, \bar{v}))d(F(\bar{x}, \bar{y}), F(\bar{u}, \bar{v})) \leq \frac{1}{2}\psi(d(\bar{x}, \bar{u}) + d(\bar{y}, \bar{v}))$, for all $(\bar{x}, \bar{y}), (\bar{u}, \bar{v}) \in M \times M$.

Suppose also that

(1) for all $(\bar{x}, \bar{y}), (\bar{u}, \bar{v}) \in M \times M$, we have

$$\gamma((\bar{x}, \bar{y}), (\bar{u}, \bar{v})) \geq 1$$
$$\Rightarrow \gamma((F(\bar{x}, \bar{y}), F(\bar{y}, \bar{x})), (F(\bar{u}, \bar{v}), F(\bar{v}, \bar{u}))) \geq 1;$$

(2) there exists $(\bar{x}_0, \bar{y}_0) \in M \times M$ such that

$\gamma((\bar{x}_0, \bar{y}_0), (F(\bar{y}_0, \bar{x}_0), F(\bar{y}_0, \bar{x}_0))) \geq 1$ and $\gamma((F(\bar{y}_0, \bar{x}_0), F(\bar{x}_0, \bar{y}_0)), (\bar{y}_0, \bar{x}_0)) \geq 1$;

(3) F is continuous. Then F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Definition 1.9 [8] Let $\xi: R_+ \times R_+ \to R$ be a mapping. Then ξ is called a simulation function if it satisfies the following conditions:

(ξ1): $\xi(0,0) = 0$;

(ξ2): $\xi(y, x) < x - y$, for all $x, y > 0$;

(ξ3): if y_n and x_n are sequences in $(0, \infty)$ with $\lim_{n\to\infty}\sup\xi(y_n, x_n) < 0$.

Argoubi *et al.* [2] and $\text{Lo'pez} - \text{de} - \text{Hierro}$ *et al.* [6] sharpened this definition as follows:

Definition 1.10 Argoubi et al. [2] Let $\xi: R_+ \times R_+ \to R$ be a mapping. Then ξ is called a stimulation function if it satisfies the following conditions:

(ξ2): $\xi(y, x) < x - y$, for all $x, y > 0$;

(ξ3): if y_n and x_n are sequences in $(0, \infty)$ with $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$, then $\lim_{n\to\infty}\sup\xi(y_n, x_n) < 0$.

Definition 1.11 $\text{Lo'pez} - \text{de} - \text{Hierro}$ et al. [8] sharpened this definition.

Let $\xi: R_+ \times R_+ \to R$ be a mapping. Then ξ is called a stimulation function if it satisfies the following conditions:

(ξ1): $\xi(0,0) = 0$;

(ξ2): $\xi(y, x) < x - y$, for all $x, y > 0$;

(ξ3): if y_n and x_n are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$ and $y_n < x_n$, for all $n \in N$, then $\lim_{n\to\infty}\sup\xi(y_n, x_n) < 0$.

Definition 1.12 [8] A self-mapping S on a metric space (M, d) is said to be $\mathcal{Z}$-contraction with respect to a stimulation function ξ if the following condition is satisfied:

$$\xi(d(Su, Sv), d(u, v)) \geq 0, \tag{2}$$

for all $u, v \in M$.

Theorem 1.13 [14] Let (M, d) be a complete metric space and $S: M \to M$ a continuous mapping satisfying $\alpha(u, v)d(Su, Sv) \leq \varphi(d(u, v))$, for all $u, v \in M$, where $\psi: R_+ \to R_+$ is non-decreasing function such that $\Sigma_{n=1}^{+\infty}\varphi^n(t) < \infty$, for all $t > 0$ and $\alpha: M \times M \to R_+$. Assume that the following two conditions hold:
(i) there exists $u_o \in M$ such that $\alpha(u_o, Su_o) \geq 1$;
(ii) S is a α −admissible i.e.,
$\alpha(u, v) \geq 1 \Rightarrow \alpha(Su, Sv) \geq 1$, for all $u, v \in M$.
Then S has a fixed point.

Karapinar[10] and Gubran[4] introduced admissible function in stimulation function, without this above theorem cannot be hold by a stimulation function. Gubran *et al.*[5] presented a new type of stimulation function.

Definition 1.14 [4] Let $T: {R_+}^3 \to R$ be a mapping. Then T is called a tri-stimulation function if it satisfies the following conditions:
(T1) $T(z, y, x) < x - yz$, for all $x, y > 0, z \geq 0$;
(T2) if z_n, y_n and x_n are sequences in $(0, \infty)$ such that $y_n < x_n$, for all $n \in N$, $\lim_{n\to\infty} z_n \geq 1$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$, then

$$\lim_{n\to\infty} \sup T(z_n, y_n, x_n) < 0.$$

$\daleth$ denotes the set of all tri-stimulation functions.

Definition 1.15 [4] A self-mapping S on a metric space (M, d) is said to a $\alpha\daleth$-contraction with respect to $T \in \daleth$ if

$$T(\alpha(u, v), d(Su, Sv), d(u, v)) \geq 0, \tag{3}$$

for all $u, v \in M$, $\alpha : M \times M \to R_+$.

Definition 1.16 [15] Let M be a non-empty set. A self-mapping S is called α-orbital admissible if for all $u, v \in M$,
$\alpha(u, Su) \geq 1 \Rightarrow \alpha(Su, S^2v) \geq 1$.

Definition 1.17 [14] S is said to be triangular α-admissible if for all u, v and $w \in M$,
(i) $\alpha(u, v) \geq 1 \Rightarrow \alpha(Su, Sv) \geq 1$;
(ii) $\alpha(u, v) \geq 1 \text{and} \alpha(w, v) \geq 1 \Rightarrow \alpha(u, v) \geq 1$.

Definition 1.18 [15] S is said to be triangular α-orbital admissible if for all $u, v \in M$,
(i) $\alpha(u, Su) \geq 1 \Rightarrow \alpha(Su, S^2u) \geq 1$;
(ii) $\alpha(u, v) \geq 1 \text{and} \alpha(v, Sv) \geq 1 \Rightarrow \alpha(u, Sv) \geq 1$.

Definition 1.19 [4] S is said to be α-permissible if for all $m \geq n \geq 1$ and $u, v \in M$,

$$\alpha(u, v) \geq 1 \Rightarrow \alpha(S^n u, S^m v) \geq 1.$$

Definition 1.20 [4] S is said to be α- orbital permissible if for all $m \geq n \geq 1$ and for all $m \geq n \geq n \geq 1$ and $u \in M$,

$$\alpha(u, Su) \geq 1 \Rightarrow \alpha(S^n u, S^m u) \geq 1.$$

Definition 1.21 [4]Let (X, d) be a metric space and $\alpha: X \times X \to [0, \infty)$ be a mapping then any map $S: X \to X$ is said to be α −permissible if $\forall\ m \geq n \geq 1$ and $u, v \in X$.

$$\alpha(u, v) \geq 1 \Rightarrow \alpha(S^n u, S^m v) \geq 1.$$

Definition 1.22 [4] Let $T: {\mathbb{R}_+}^3 \to \mathbb{R}$be a mapping. Then T is called Trisimulation function if it satisfies the following conditions: (T_1) $T(z, y, x) < x - yz$ for all $x, y > 0$, $z \geq 0$; (T_2)If z_n, y_n, x_n are sequences in $(0, \infty)$ such that $y_n < x_n$, for all $n \in \mathbb{N}$
$\lim_{n\to\infty} z_n \geq 1$ and $\lim\limits_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$, then

$$\lim_{n\to\infty} \limsup T(z_n, y_n, x_n) < 0.$$

2. Main Results

In this section, we shall prove some fixed point theorems in metric spaces with the aid of tri-simulation function for expansive mappings.

Definition 2.1 Let S be a self mapping on a metric space (M, d). Then S is said to be $\alpha\mathrm{T}$ expansive mapping with respect to $\mathrm{T} \in \tau$ if

$$T(\alpha(x,y), d(x,y), d(Sx,Sy)) \geq 0. \tag{4}$$

Remark 2.2 Using $(\mathrm{T1})$ one can observe that $\mathrm{d}(\mathrm{Sx},\mathrm{Sy}) > \alpha(\mathrm{x},\mathrm{y})\mathrm{d}(\mathrm{x},\mathrm{y})$.

Theorem 2.3 Let(M, d) be a complete metric space and $\mathrm{S}: \mathrm{M} \to \mathrm{M}$ be an $\alpha\mathrm{T}$- expansive bijective with respect to $\mathrm{T} \in \tau$. Suppose that

(a) S^{-1} is α-permissible;

(b) there exists $u_0 \in M$ such that $\alpha(u_0, S^{-1}u_0) \geq 1$;

(c) S is continuous.

Then S has a fixed point, that is, there exists $u \in S$ such that $Su = u$.

Proof. Let $u_0 \in M$ such that $\alpha(u_0, S^{-1}u_0) \geq 1$.

And define $u_n = Su_{n+1}$ $\forall n$,

then $\alpha(u_0, u_1) \geq 1$.

From condition (a) one can say that

$$\alpha(u_n, u_m) \geq 1 \forall n \geq m \geq 1. \tag{5}$$

Now, if $u_n = u_{n+1}$ for some $n \in \mathbb{N}$. Then we are done.

So, let $u_n \neq u_n + 1$ $\forall$ n.

Now, taking $x = u_n$ and $y = u_{n+1}$ in equation (4), we get

$$T(\alpha(u_n, u_{n+1}), d(u_n, u_{n+1}), d(Su_n, Su_{n+1}) \geq 0$$
$$\Rightarrow T(\alpha(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n-1}, u_n) \geq 0.$$

Using condition $(\mathrm{T1})$, we get

$$0 < d(u_{n-1}, u_n) - \alpha(u_n, u_{n+1}) d(u_n, u_{n+1}), \tag{6}$$

this implies

$$d(u_n, u_n + 1) \leq \alpha(u_n, u_{n+1}) d(u_n, u_{n+1}) < d(u_{n-1}, u_n).$$

This implies,

$d(u_n, u_{n+1})$ is a monotonically decreasing sequence of positive real numbers.

So, it is convergent to $r \geq 0$(say).

Suppose if possible $r > 0$. Taking $n \to \infty$ in (6) and using equation (5) we can say that

$$\lim_{n\to\infty} \alpha(u_n, u_{n+1}) = 1. \tag{7}$$

Moreover, by using $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n-1}, u_n)) < 0.$$

Which is a contradiction.

Therefore $r = 0$.

So,

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{8}$$

Now, we will prove that $\{u_n\}$ is a bounded sequence.

So, if possible, suppose that $\{u_n\}$ is unbounded. Then there exists a subsequence $\{u_{n_k}\}$ such that n_1 and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(u_{n_k}, u_{n_{k+1}}) > 1 \tag{9}$$

and $d(u_{n_k}, u_m) \leq 1$ for $n_k \leq m \leq n_{k+1} - 1$.

Now by using triangular inequality and equation (9), we get

$$1 < d\big(u_{n_k}, u_{n_{k+1}}\big) \leq d\big(u_{n_k}, u_{n_{k+1}-1}\big) + d\big(u_{n_{k+1}-1}, u_{n_{k+1}}\big) < 1 + d\big(u_{n_{k+1}-1}, u_{n_{k+1}}\big).$$

Taking $k \to \infty$, we get

$$\lim_{n\to\infty} d(u_{n_k}, u_{n_{k+1}}) = 1. \quad (10)$$

By Remark (1), Taking $x = u_{n_{k-1}}$ and $y = u_{n_{k-1}-1}$. We have

$$\alpha(u_{n_k}, u_{n_{k+1}})d(u_{n_k}, u_{n_{k+1}}) \lneqq d(u_{n_{k-1}}, u_{n_{k+1}-1}).$$

Using equation (5) and (9), we get

$$\begin{aligned} 1 &< \alpha(u_{n_k}, u_{n_{k+1}})d(u_{n_k}, u_{n_{k+1}}) \\ &< d(u_{n_{k-1}}, u_{n_{k+1}-1}) \\ &< d(u_{n_{k-1}}, u_{n_k}) + d(u_{n_k}, u_{n_{k+1}-1}) \\ &< d(u_{n_{k-1}}, u_{n_k}) + 1. \end{aligned}$$

Taking $k \to \infty$, we get

$$\lim_{n\to\infty} d(u_{n_{k-1}}, u_{n_k}) = 1 \quad (11)$$

$$\lim_{n\to\infty} \alpha(u_{n_k}, u_{n_{k+1}}) = 1. \quad (12)$$

Hence, on using $(T2)$,equations (10), (11) and (12), we have

$0 \le \limsup_{n\to\infty} T(\alpha(u_{n_k}, u_{n_{k+1}}), d(u_{n_k}, u_{n_{k+1}}), d(u_{n_{k-1}}, u_{n_{k+1}-1}) < 0.$

Which is a contradiction.

So,$\{u_n\}$ is a bounded sequence.

Now, let $c_n := \sup\{d(u_i, u_j): i, j \ge n\}$.

By the above discussion, we conclude that $\{c_n\}$ is decreasing sequence of non-negative real numbers which is bounded. Therefore, there exists $c \ge 0$ such that $\lim_{n\to\infty} c_n = c$. If $c \ne 0$, then by definition of $\{c_n\}$, for all $k \in \mathbb{N}$, there exists m_k and n_k with $m_k > n_k \ge k$ such that

$$c_k - \frac{1}{k} \le d(u_{m_k}, u_{n_k}) \le c_k,$$

by taking $k \to \infty$, we get

$$\lim_{k\to\infty} d(u_{m_k}, u_{n_k}) = c. \quad (13)$$

From equations (4) and (5), we get

$$\begin{aligned} d(u_{n_k}, u_{m_k}) &\le \alpha(u_{n_k}, u_{m_k})d(u_{n_k}, u_{m_k}) \\ &\le d(u_{n_{k-1}}, u_{m_{k-1}}) \\ &\le d(u_{n_{k-1}}, u_{n_k}) + d(u_{n_k}, u_{m_k}) + d(u_{m_k}, u_{m_{k-1}}). \end{aligned}$$

Taking $k \to \infty$ and using equation (8), we get

$$\lim_{n\to\infty} d(u_{n_{k-1}}, u_{m_{k-1}}) = c \quad (14)$$

and

$$\lim_{n\to\infty} \alpha(u_{n_k}, u_{m_k}) = 1. \quad (15)$$

As S is $\alpha T-$contraction, so using equations (13), (14) and (15), we get

$$0 \le \limsup_{n\to\infty} T(\alpha(u_{n_k}, u_{m_k}), d(u_{n_k}, u_{m_k}), d(u_{n_{k-1}}, u_{m_{k-1}}) < 0.$$

Which is a contradiction.

Which shows that $c = 0$.

So,$\{u_n\}$ is a Cauchy sequence in M and (M, d) is a complete metric space. So, $\{u_n\}$ is convergent in M i.e., there exists $w \in M$ such that

$$\lim_{n\to\infty} u_n = w. \quad (16)$$

Now, continuity of S implies that

$$\lim_{n\to\infty} u_{n-1} = \lim_{n\to\infty} Su_n \Rightarrow Sw = w.$$

So, w is the fixed point.

In the next theorem, we omit continuity.

Theorem 2.4 In Theorem 2.3, if we replace condition(c) by the following condition:
If $\{u_n\}$ is a sequence in M such that $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $u_n \to w$ as $n \to \infty$, then there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(k)}, w) \geq 1$.
Then still S has a fixed point.
Proof. Following the proof of Theorem 2.3, we obtain that $\{u_n\}$ is a convergent sequence and converge to $w \in M$.
Now, by the above new defined condition there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(k)}, w) \geq 1$ for all k.
From (4),
$0 \leq T(\alpha(u_{n(k)}, w), d(u_{n(k)}, w), d(u_{n(k)-1}, Sw)) \Rightarrow \alpha(u_{n(k)}, w)d(u_{n(k)}, w) \gneqq d(u_{n(k)-1}, Sw)$
So that

$$\begin{aligned} d(u_{n(k)}, w) &\leq \alpha(u_{n(k)}, w)d(u_{n(k)}, w) \\ &\leq d(u_{n(k)-1}, Sw). \end{aligned}$$

Taking $k \to \infty$, we get $d(w, Sw) = 0$.
$Sw = w$.

Theorem 2.5 In theorem 2.3, if we replace condition (c) by any one of the following
1. $\alpha(u, v) \geq 1$ for all $u, v \in Fix(S) := \{x \in M: Sx = x\}$
2. S is $\alpha-$ permissible and for all $u, v \in Fix(S)$ there exists $z \in M$.

such that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$.
Then,also T has a fixed point.
Proof. Let u and v be two distinct point of S. If the condition (1) satisfied, then

$$\begin{aligned} 0 &\leq T(\alpha(u, v), d(Su, v), d(Su, Sv)) \\ &< d(u, v) - \alpha(u, v)d(u, v). \end{aligned}$$

Which is a contradiction. So, $u = v$. Hence, S has unique fixed point.
If condition (2) holds, there exists $w \in M$ such that $\alpha(u, w) \geq 1$ and $\alpha(v, w) \geq 1$. If $w = u$, then processing as in condition (1), we can show that $w = v$ and we are done.
Thus we assume that $u \neq v \neq w$. Using $\alpha-$ permissible of S we can say $\alpha(u, w_n) \geq 1$ and $\alpha(v, w_n) \geq 1$ for all $n \geq 1$.
Claim : $\lim_{n\to\infty} w_n = u$.
If $w_m = u$ for some $m \in \mathbb{N}$, then claim is followed immediately. otherwise $d(u, w_n) > 0$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned} 0 &\leq T(\alpha(u, w_n), d(u, w_n), d(u, w_{n-1})) \\ &< d(u, w_{n-1}) - \alpha(u, w_n) - \alpha(u, w_n)d(u, w_n). \end{aligned}$$

This implies,
$d(u, w_n)$ is a strictly decreasing sequence of positive real numbers so convergent to $r \geq 0$(say).
If $r \neq 0$ then by $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u, w_n), d(u, w_n), d(u, w_{n-1})) < 0.$$

Which is a contradiction.
Which prove the claim.
Similarly, one can show that $lim_{n\to\infty} w_n = v$.
It proves the uniqueness of limit point i.e., $u = v$.

3. Conclusion

By using tri-simulation function in metric spaces we have proved some fixed point theorems.

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