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Results Beyond Fermat's Last Theorem

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Abstract **-** In this paper, some results relating to Fermat's last theorem and beyond this theorem, have been presented. The expression of the form $(x+y)^n-(x-y)^n$, where x,y are variable positive integers and $x>y$, has been analyzed to derive some results relating to the Diophantine equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$, where $a,a_1,a_2,\cdots,a_s$ are positive integers. An attempt has been made to give a simple proof of Fermat's last theorem and further this theorem has been extended to the case of $s=3$ relative to the equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$. A result as a theorem 2.1 has been given to find the least positive integral value of s in the equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$. A solution of each of the equations $a^2=a_1^2+a_2^2+\cdots+a_n^2$ and $a^3=a_1^3+a_2^3+a_3^3+a_4^3$ has been obtained. It has been proved that the equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ can be expressed as $(u+v)^n-(u-v)^n=(2a_2)^n+\cdots+(2a_s)^n$, where $u+v=2a$, $u+v=2a_1$. It will also be shown that the Diophantine equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ is a particular case of the equation

$$(x+y)^n=(x-y)^n+2\binom{n}{1}x^{n-1}y+2\binom{n}{3}x^{n-3}y^3+\cdots+2\alpha,\quad \alpha=\begin{cases} y^n, & \text{if } n \text{ is odd} \\ \binom{n}{n-1}xy^{n-1}, & \text{if } n \text{ is even}\end{cases}$$

as it is obtained by putting some positive integral values $u,v\ (u>v)$ of x,y respectively. Finally equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ has been analyzed to conclude this paper.

Keywords*- Diophantine equation, expression, function, number of terms, positive integer*

1. Introduction

If we study carefully the expression $(x+y)^n-(x-y)^n$, where x,y are variable positive integers and $x>y$, we can derive various results relating to the Diophantine equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$, where $a,a_1,a_2,\cdots,a_s$ are positive integers. Fermat's last theorem is one of these results whose proof has been a great challenge to the mathematicians for about three centuries. As for as this theorem concerned, consider the Diophantine equation

$$a^n+b^n=c^n \tag{1}$$

where a,b,c,n are all positive integers.

Fermat's last theorem states that Equation 1 holds only when $n\le 2$ and it does not hold for $n>2$ whatever may be the values of the positive integers a,b,c. Wiles [1], and Wiles and Taylor [2] proved this theorem through two papers in 1995 by applying elliptic curves approach.

There are many studies relating to the Fermat's last theorem. Roy [3], discuses the proof of this theorem for the case of $n=4$, Rychlik[4], considered its proof for the case $n=5$ and Breusch [5], considered the cases of $n=6,10$. Adleman, Heath brown [6], discuss the first case of Fermat's Last Theorem. Edwards [7], studies this theorem in relation to number theory. Bennett, Glass, Szekely, Gabar [8], study this theorem for rational exponents. Jennifer [9], studies it in relation to Pythagorean theorem. Van der Poortan [10], gives notes on Fermat's last theorem. Ribenboim [11], delivered 13 lectures on Fermat' s last theorem, Singh [12], describes Fermat's enigma, Charles [13], describes about Fermat's Diary, Cornell, Silverman and Stevens [14], study about modular forms and Fermat's last theorem, Buzzard [15], presents the review of modular forms and Fermat's last theorem, Faltings [16], discuses about the proof of Fermat's last theorem by R. Taylor and A. Wiles and Aczel [17] gives the details of Fermat's last theorem.

Again, Fermat's last theorem states that Diophantine equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ does not hold if $s=2,n>2$ and Euler extended this conjecture to the values of $s=3,4,\cdots,n-1$.

Demjanenko [18], describes the Euler's conjecture and Lander and Parkin [19], present the counter examples to Euler's conjecture.

By Elkies [20], $20615673^4 = 2682440^4 + 15365639^4 + 18796760^4$ and similar result given by Roger Frye, $422481^4 = 95800^4 + 217519^4 + 414560^4$, these results show that Euler conjecture is false for $s = 3, n = 4$. Also from [21], $144^5 = 27^5 + 84^5 + 110^5 + 133^5$ shows that Euler conjecture is false for $s = 4, n = 5$.

There are various results on the Diophantine equations . Werebrusow [22], discuses on the equation $x^5 + y^5 = Az^5$, Frey [23], studies the links between elliptic curves and certain Diophantine equations, Michel Waldschmidt [24] discuses on open Diophantine problems , Carmichael [25], presents the study on the impossibility of certain Diophantine equations and systems of equations, Newman [26], studies about radical Diophantine equations , Dickson [27], presents the History of theory of numbers with Diophantine analysis, Roger [28], studies the integral solution of $a^{-2} + b^{-2} = d^{-2}$ and Zagier [29] studies the equation $w^4 + x^4 + y^4 = z^4$.

In this article, Fermat's last theorem and Diophantine equation $a^n = a_1^n + a_2^n + \cdots + a_s^n$ will be discussed in relation to the expression $(x + y)^n - (x - y)^n$.

2. Analysis of the Expression $(x + y)^n - (x - y)^n, x > y > 0$

If $n = 1$, then expression $(x + y)^n - (x - y)^n$ becomes $(x + y)^1 - (x - y)^1 = (x + y) - (x - y) = x + y - x + y = 2y$
Therefore, if $n = 1$, then the expression $(x + y)^n - (x - y)^n$ has one term $2y$.

If $n = 2$, then expression $(x + y)^n - (x - y)^n$ becomes $(x + y)^2 - (x - y)^2 = 4xy$, therefore if $n = 2$, then the expression $(x + y)^n - (x - y)^n$ has 1 term $4xy$. If $x = u^2$, $y = v^2$, then $(u^2 + v^2)^2 - (u^2 - v^2)^2 = 4u^2v^2 = (2uv)^2$

$$\Rightarrow \quad (u^2 + v^2)^2 = (u^2 - v^2)^2 + (2uv)^2 \Rightarrow \quad a^2 = a_1^2 + a_2^2, \text{where } a = u^2 + v^2, a_1 = u^2 - v^2, a_2 = 2uv$$

If $n = 3$, then expression $(x + y)^n - (x - y)^n$ becomes $(x + y)^3 - (x - y)^3 = 6x^2y + 2y^3$, therefore if $n = 3$, then the expression $(x + y)^n - (x - y)^n$ has 2 terms $6x^2y, 2y^3$. Expressions $2y^3$, $6x^2y + 2y^3$ cannot be expressed as cube of some positive integers. If $x = 6u^3$, $y = v^3$, then

$$(6u^3 + v^3)^3 - (6u^3 - v^3)^3 = 216u^6v^3 + 2v^9 = (6u^2v)^3 + 2(v^3)^3 \Rightarrow (6u^3 + v^3)^3 = (6u^3 - v^3)^3 + (6u^2v)^3 + 2(v^3)^3$$

From the above equation, we find that there exist positive integers a, a_1, a_2, a_3, a_4 which satisfy the equation
$a^3 = a_1^3 + a_2^3 + a_3^3 + a_4^3$, where $a = 6u^3 + v^3, a_1 = 6u^3 - v^3, a_2 = 6u^2v$, $a_3 = v^3 = a_4$

Illustration: Take $u = 1, v = 1$, then $a = 6u^3 + v^3 = 6 \times 1^3 + 1^3 = 7, a_1 = 6u^3 - v^3 = 6 \times 1^3 - 1^3 = 5$,
$a_2 = 6u^2v = 6 \times 1^2 \times 1 = 6, a_3 = v^3 = 1^3 = 1 = a_4$. Therefore, $7^3 = 5^3 + 6^3 + 1^3 + 1^3 \Rightarrow a^3 = a_1^3 + a_2^3 + a_3^3 + a_4^3$.
If $x = 5u$, $y = 4u$, then $(5u + 4u)^3 - (5u - 4u)^3 = 600u^3 + 128u^3 = 728u^3 = (6u^3)^3 + (8u^3)^3$

$$\Rightarrow \quad (9u)^3 = (u)^3 + (6u)^3 + (8u)^3$$

From the above equation, we find that there exist positive integers a, a_1, a_2, a_3 which satisfy the equation
$a^3 = a_1^3 + a_2^3 + a_3^3$, where $a = 9u, a_1 = u, a_2 = 6u$, $a_3 = 8u$.

If $n = 4$, then expression $(x + y)^n - (x - y)^n$ becomes $(x + y)^4 - (x - y)^4 = 8x^3y + 8xy^3$, therefore if $n = 4$, then the expression $(x + y)^n - (x - y)^n$ has two terms $8x^3y, 8xy^3$. If $x = u^2$, $y = 2v^2$, then
$(u^2 + 2v^2)^4 - (u^2 - 2v^2)^4 = 16u^6v^2 + 64u^2v^6 = (4u^3v)^2 + (8uv^3)^2$,
i. e. $(u^2 + 2v^2)^4 - (u^2 - 2v^2)^4 = (4u^3v)^2 + (8uv^3)^2$
If $a_1 = u^2 + 2v^2$, $a_2 = u^2 - 2v^2$, $a_3 = 4u^3v$, $a_4 = 8uv^3$, then $a_1^4 - a_2^4 = a_3^2 + a_4^2$

Illustration: If $u = 3, v = 1$, then $a_1 = 3^2 + 2 \times 1^2 = 11, a_2 = 3^2 - 2 \times 1^2 = 7$, $a_3 = 4 \times 3^3 \times 1 = 108, a_4 = 8 \times 3 \times 1^3 = 24$. Therefore $11^4 - 7^4 = 108^2 + 24^2$.
If $x = u^3$ and $y = v^3$, then
$(u^3 + v^3)^4 - (u^3 - v^3)^4 = 8u^9v^3 + 8u^3v^9 = (2u^3v)^3 + (2uv^3)^3$, i. e. $(u^3 + v^3)^4 - (u^3 - v^3)^4 = (2u^3v)^3 + (2uv^3)^3$
If $a_1 = u^3 + v^3$, $a_2 = u^3 - v^3$, $a_3 = 2u^3v$, $a_4 = 2uv^3$, then $a_1^4 - a_2^4 = a_3^3 + a_4^3$.

Illustration: If $u = 3, v = 2$, then $a_1 = 3^3 + 2^3 = 35, a_2 = 3^3 - 2^3 = 19$, $a_3 = 2 \times 3^3 \times 2 = 108$,
$a_4 = 2 \times 3 \times 2^3 = 48$. Therefore $35^4 - 19^4 = 108^3 + 48^3$
If $x = u^4$ and $y = 2v^4$, then $(u^4 + 2v^4)^4 - (u^4 - 2v^4)^4 = 16u^{12}v^4 + 64u^4v^{12} = (2u^3v)^4 + 4(2uv^3)^4$
i. e. $(u^4 + 2v^4)^4 - (u^4 - 2v^4)^4 = (2u^3v)^4 + 4(2uv^3)^4 \Rightarrow (u^4 + 2v^4)^4 = (u^4 - 2v^4)^4 + (2u^3v)^4 + 4(2uv^3)^4$
If $a = u^4 + 2v^4, a_1 = u^4 - 2v^4, a_2 = 2u^3v, a_3 = 2uv^3$, then $a^4 = a_1^4 + a_2^4 + 4a_3^4$.
i. e. $a^4 = a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 + a_6^4$, where $a_3 = a_4 = a_5 = a_6$.

Illustration: If $u = 3, v = 1$, then $a = 3^4 + 2 \times 1^4 = 83, a_1 = 3^4 - 2 \times 1^4 = 79$, $a_2 = 2 \times 3^3 \times 1 = 54$,
$a_3 = 2 \times 3 \times 1^3 = 6$. Therefore $83^4 = 79^4 + 54^4 + 4 \times 6^4 = 79^4 + 54^4 + 6^4 + 6^4 + 6^4 + 6^4$

If $x = u^4$ and $y = 4v^4$, then $(u^4+4v^4)^4 - (u^4-4v^4)^4 = 32u^{12}v^4 + 512u^4v^{12} = 2(2u^3v)^4 + 2(4uv^3)^4$
i.e. $(u^4+4v^4)^4 - (u^4-4v^4)^4 = 2(2u^3v)^4 + 2(4uv^3)^4 \Rightarrow (u^4+4v^4)^4 = (u^4-4v^4)^4 + 2(2u^3v)^4 + 2(4uv^3)^4$
If $a = u^4+4v^4,\ a_1 = u^4-4v^4,\ a_2 = 2u^3v,\ a_4 = 4uv^3$, then $a^4 = a_1^4 + 2a_2^4 + 2a_4^4$
i.e. $a^4 = a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4$, where $a_2 = a_3$, $a_4 = a_5$.

Illustration: Let $u = 3, v = 1$. Then $a = 3^4 + 4\times 1^4 = 85, a_1 = 3^4 - 4\times 1^4 = 77, a_2 = 2\times 3^3\times 1 = 54$, $a_4 = 4\times 3\times 1^3 = 12$. Therefore $85^4 = 77^4 + 2\times 54^4 + 2\times 12^4 = 77^4 + 54^4 + 54^4 + 12^4 + 12^4$

Continuing like this, we can analyze $(x+y)^n - (x-y)^n$ for $n = 5, 6, \cdots$

From the above analysis, we note the following important result:

Theorem-2.1

If $u = ku_1^{\alpha_1}u_2^{\alpha_2}\cdots u_r^{\alpha_r},\quad v = lv_1^{\beta_1}v_2^{\beta_2}\cdots v_s^{\beta_s}$ or $u = k_1u_1^{\alpha_{11}}u_2^{\alpha_{12}}\cdots u_r^{\alpha_{1r}} + k_2u_1^{\alpha_{21}}u_2^{\alpha_{22}}\cdots u_r^{\alpha_{2r}} + \cdots + k_mu_1^{\alpha_{m1}}u_2^{\alpha_{m2}}\cdots u_r^{\alpha_{mr}}$, $v = l_1v_1^{\beta_{11}}v_2^{\beta_{12}}\cdots v_s^{\beta_{1s}} + l_2v_1^{\beta_{21}}v_2^{\beta_{22}}\cdots v_s^{\beta_{2s}} + \cdots + l_tv_1^{\beta_{t1}}v_2^{\beta_{t2}}\cdots v_s^{\beta_{ts}}$, where k, l, k_i, l_i are fixed integers, integers $\alpha_i,\ \beta_i, \alpha_{ij}$, $\beta_{ij} \geq 0\ \forall\ i, j$; then number of terms in $(u+v)^n - (u-v)^n$ as a function of $u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s$ cannot be less than the number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y.

In particular if $(u+v)^n - (u-v)^n = a_2^n + a_3^n + \cdots$, then number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y $\leq$ number of a_is in $a_2^n + a_3^n + \cdots$.

Moreover if there are m terms in $(x+y)^n - (x-y)^n$ as a function of x, y; then there exist at least m positive integers $a_2, a_3, \cdots, a_{m+1}$ such that $(u+v)^n - (u-v)^n = a_2^n + a_3^n + \cdots + a_{m+1}^n$.

Proof: **First Part**: we have

$$(x+y)^n - (x-y)^n = 2\binom{n}{1}x^{n-1}y + 2\binom{n}{3}x^{n-3}y^3 + \cdots + 2\alpha \qquad \ldots (2)$$

where $\alpha = \begin{cases} y^n, & \text{if } n \text{ is odd} \\ \binom{n}{n-1}xy^{n-1}, & \text{if } n \text{ is even} \end{cases}$,

Therefore number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y is the number of terms in the right hand side of Equation 2. If we put $x = ku_1^{\alpha_1}u_2^{\alpha_2}\cdots u_r^{\alpha_r} = u,\ y = lv_1^{\beta_1}v_2^{\beta_2}\cdots v_s^{\beta_s} = v$ in Equation 2, then it becomes

$$(u+v)^n - (u-v)^n = 2\binom{n}{1}u^{n-1}v + 2\binom{n}{3}u^{n-3}v^3 + \cdots + 2\alpha \qquad \ldots (3)$$

where $\alpha = \begin{cases} v^n, & \text{if } n \text{ is odd} \\ \binom{n}{n-1}uv^{n-1}, & \text{if } n \text{ is even} \end{cases}$

From the Equation 3, we find that number of terms in $(u+v)^n - (u-v)^n$ as a function of $u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s$ is equal to number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y.

If we put $x = k_1u_1^{\alpha_{11}}u_2^{\alpha_{12}}\cdots u_r^{\alpha_{1r}} + k_2u_1^{\alpha_{21}}u_2^{\alpha_{22}}\cdots u_r^{\alpha_{2r}} + \cdots + k_mu_1^{\alpha_{m1}}u_2^{\alpha_{m2}}\cdots u_r^{\alpha_{mr}} = u$,

$y = l_1v_1^{\beta_{11}}v_2^{\beta_{12}}\cdots v_s^{\beta_{1s}} + l_2v_1^{\beta_{21}}v_2^{\beta_{22}}\cdots v_s^{\beta_{2s}} + \cdots + l_tv_1^{\beta_{t1}}v_2^{\beta_{t2}}\cdots v_s^{\beta_{ts}} = v$

in Equation 2, then it again becomes Equation 3 with changed values of u and v. Then from the Equation 3, we find that number of terms in $(u+v)^n - (u-v)^n$ as a function of $u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s$ is greater than number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y.

This proves that number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y $\leq$ number of terms in $(u+v)^n - (u-v)^n$ as a function of $u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s$.

Second Part:

Let $(u+v)^n - (u-v)^n = a_2^n + a_3^n + \cdots$

Now by first part, number of terms in $(x+y)^n - (x-y)^n$ as a function of $x, y \leq$ number of terms in $(u+v)^n - (u-v)^n$ as a function of $u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s$

$\Rightarrow$ number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y $\leq$ number of terms in $a_2^n + a_3^n + \cdots$ as a function of $u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s$

$\Rightarrow$ number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y $\leq$ number of a_i^ns in $a_2^n + a_3^n + \cdots$ [$\because$ $a_2^n, a_3^n, \cdots$ are the terms of $a_2^n + a_3^n + \cdots$]

$\Rightarrow$ number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y $\leq$ number of a_is in $a_2^n + a_3^n + \cdots$

This implies if there are m terms in $(x+y)^n - (x-y)^n$ as a function of x, y; then there exist at least m positive integers $a_2, a_3, \cdots, a_{m+1}$ such that $(u+v)^n - (u-v)^n = a_2^n + a_3^n + \cdots + a_{m+1}^n$.

3. Solution of Diophantine Equations

$a^2 = a_1^2 + a_2^2 + \cdots + a_n^2$ and $a^3 = a_1^3 + a_2^3 + a_3^3 + a_4^3$

We have $(x+y)^2 = x^2 + 2xy + y^2$.Put $x = u^2, y = 2v^2$, we have

$$(u^2 + 2v^2)^2 = u^4 + 4u^2v^2 + 4v^4 = (u^2)^2 + (2uv)^2 + (2v^2)^2 = (u^2)^2 + (2v^2)^2 + (2uv)^2$$

Put $d = u^2 + 2v^2, a = u^2, b = 2v^2, c = 2uv$, we have $d^2 = a^2 + b^2 + c^2$

Illustration: Let $u = 1, v = 2$.Then $(1^2 + 2 \times 2^2)^2 = 1^4 + 4 \times 1^2 \times 2^2 + 4 \times 2^4 \Rightarrow 9^2 = 1^2 + 8^2 + 4^2$

(i) To find positive integers $a, a_1, \cdots, a_n$ satisfying the equation $a^2 = a_1^2 + a_2^2 + \cdots + a_n^2$.

Consider the identity

$$(x_1 + x_2 + \cdots + x_{n-1} + x_n)^2 - (x_1 + x_2 + \cdots + x_{n-1} - x_n)^2 = 4x_1x_n + 4x_2x_n + \cdots + 4x_{n-1}x_n$$

Now put $x_1 = y_1^2, x_2 = y_2^2, \cdots, x_n = y_n^2$, we have

$$(y_1^2 + y_2^2 + \cdots + y_{n-1}^2 + y_n^2)^2 - (y_1^2 + y_2^2 + \cdots + y_{n-1}^2 - y_n^2)^2 = 4y_1^2y_n^2 + 4y_2^2y_n^2 + \cdots + 4y_{n-1}^2y_n^2$$
$$= (2y_1y_n)^2 + (2y_2y_n)^2 + \cdots + (2y_{n-1}y_n)^2$$

Take $a = y_1^2 + y_2^2 + \cdots + y_{n-1}^2 + y_n^2,\ a_1 = y_1^2 + y_2^2 + \cdots + y_{n-1}^2 - y_n^2$

$a_2 = 2y_1y_n,\ a_3 = 2y_2y_n, \cdots,\ a_n = 2y_{n-1}y_n$, we have $a^2 - a_1^2 = a_2^2 + \cdots + a_n^2$

Therefore, for suitable choice of integers $y_1, y_2, \cdots, y_n$; there exist positive integers $a, a_1, \cdots, a_n$ satisfying the equation $a^2 = a_1^2 + a_2^2 + \cdots + a_n^2$.

As an illustration: $(7^2 + 3^2 + 1^2)^2 - (7^2 + 3^2 - 1^2)^2 = (2 \times 7 \times 1)^2 + (2 \times 3 \times 1)^2 \Rightarrow 59^2 = 57^2 + 14^2 + 6^2$

(ii) To find positive integers a, a_1, a_2, a_3, a_4 satisfying the equation $a^3 = a_1^3 + a_2^3 + a_3^3 + a_4^3$.

Consider the identity

$$(x_1 + x_2 - x_3)^3 + (x_1 - x_2 + x_3)^3 + (-x_1 + x_2 + x_3)^3 - (x_1 + x_2 + x_3)^3 = -24x_1x_2x_3$$

Put $x_1 = 3y_1^3, x_2 = 3y_2^3, x_3 = y_3^3$, we have

$$(3y_1^3 + 3y_2^3 - y_3^3)^3 + (3y_1^3 - 3y_2^3 + y_3^3)^3 + (-3y_1^3 + 3y_2^3 + y_3^3)^3 - (3y_1^3 + 3y_2^3 + y_3^3)^3 = -216y_1^3y_2^3y_3^3$$
$$\Rightarrow (3y_1^3 + 3y_2^3 + y_3^3)^3 = (3y_1^3 + 3y_2^3 - y_3^3)^3 + (3y_1^3 - 3y_2^3 + y_3^3)^3 + (-3y_1^3 + 3y_2^3 + y_3^3)^3 + 216y_1^3y_2^3y_3^3$$
$$= (3y_1^3 + 3y_2^3 - y_3^3)^3 + (3y_1^3 - 3y_2^3 + y_3^3)^3 + (-3y_1^3 + 3y_2^3 + y_3^3)^3 + (6y_1y_2y_3)^3$$

Take $a = 3y_1^3 + 3y_2^3 + y_3^3, a_1 = 3y_1^3 + 3y_2^3 - y_3^3,\ a_2 = 3y_1^3 - 3y_2^3 + y_3^3, a_3 = -3y_1^3 + 3y_2^3 + y_3^3,\ a_4 = 6y_1y_2y_3$,

Therefore, for suitable choice of integers y_1, y_2, y_3; there exist positive integers a, a_1, a_2, a_3, a_4 satisfying the equation $a^3 = a_1^3 + a_2^3 + a_3^3 + a_4^3$.

Illustration: Take $y_1 = 1, y_2 = 2, y_3 = 3$, then $a = 3y_1^3 + 3y_2^3 + y_3^3 = 3 \times 1^3 + 3 \times 2^3 + 3^3 = 54$,
$a_1 = 3y_1^3 + 3y_2^3 - y_3^3 = 3 \times 1^3 + 3 \times 2^3 - 3^3 = 0,\ a_2 = 3y_1^3 - 3y_2^3 + y_3^3 = 3 \times 1^3 - 3 \times 2^3 + 3^3 = 6$,
$a_3 = -3y_1^3 + 3y_2^3 + y_3^3 = -3 \times 1^3 + 3 \times 2^3 + 3^3 = 48, a_4 = 6y_1y_2y_3 = 6 \times 1 \times 2 \times 3 = 36$
$\therefore\ 54^3 = 6^3 + 48^3 + 36^3 \Rightarrow 9^3 = 1^3 + 8^3 + 6^3$

Similarly, if $y_1 = 2, y_2 = 3, y_3 = 4$, then $169^3 = 41^3 + 7^3 + 121^3 + 144^3$

Also from [21], the formula of expressing cube of a positive integer as a sum of three cubes is given by $a^3 = a_1^3 + a_2^3 + a_3^3$, where $a = 9u^4$, $a_1 = 9u^4 - 3uv^3$, $a_2 = 9u^3v - v^4, a_3 = v^4$.

Illustration: If we put $u = v = 1$, then we get $9^3 = 1^3 + 8^3 + 6^3$

Again from [21], Ramanujan gave the solution of the equation $a^3 = a_1^3 + a_2^3 + a_3^3$ as follows:
$a = 6u^2 - 4uv + 4v^2$, $a_1 = 4u^2 - 4uv + 6v^2, a_2 = 5u^2 - 5uv - 3v^2$, $a_3 = 3u^2 + 5uv - 5v^2$.

Illustration: If $u = 3, v = 1$, then $a = 46, a_1 = 30, a_2 = 27, a_3 = 37$. Therefore, $46^3 = 30^3 + 27^3 + 37^3$.

4. Main Results

We observe that the expression $13^2 = 5^2 + 12^2$ can be written as $(9+4)^2 - (9-4)^2 = 12^2$
$\Rightarrow\ 26^2 = 10^2 + 24^2$ can be written as $(18+8)^2 - (18-8)^2 = 24^2$,
the expression $9^2 = 1^2 + 8^2 + 4^2$ can be written as $(5+4)^2 - (5-4)^2 = 8^2 + 4^2$
$\Rightarrow\ 18^2 = 2^2 + 16^2 + 8^2$ can be written as $(10+8)^2 - (10-8)^2 = 16^2 + 8^2$,
the expression $9^3 = 1^3 + 8^3 + 6^3$ can be written as $(5+4)^3 - (5-4)^3 = 8^3 + 6^3$
$\Rightarrow\ 18^3 = 2^3 + 16^3 + 12^3$ can be written as $(10+8)^3 - (10-8)^3 = 16^3 + 12^3$,
the expression $6^3 = 3^3 + 4^3 + 5^3$ can be written as $(5+1)^3 - (5-1)^3 = 3^3 + 5^3$

$\Rightarrow$ $12^3 = 6^3 + 8^3 + 10^3$ can be written as $(10+2)^3 - (10-2)^3 = 6^3 + 10^3$,
the expression $169^3 = 41^3 + 7^3 + 121^3 + 144^3$ can be written as $(88+81)^3 - (88-81)^3 = 41^3 + 121^3 + 144^3$
$\Rightarrow$ $338^3 = 82^3 + 14^3 + 242^3 + 288^3$ can be written as $(176+162)^3 - (176-162)^3 = 82^3 + 242^3 + 288^3$,
the expression $20615673^4 = 2682440^4 + 15365639^4 + 18796760^4$ (due to Elkies [20]) can be written as
$(17990656 + 2625017)^4 - (17990656 - 2625017)^4 = 2682440^4 + 18796760^4$
$\Rightarrow$ $41231346^4 = 5364880^4 + 30731278^4 + 37593520^4$ can be written as
$(35981312 + 5250034)^4 - (35981312 - 5250034)^4 = 5364880^4 + 37593520^4$,
the expression $422481^4 = 95800^4 + 217519^4 + 414560^4$ (due to Roger Frye) can be written as
$(320000 + 102481)^4 - (320000 - 102481)^4 = 95800^4 + 414560^4$
$\Rightarrow$ $844962^4 = 191600^4 + 435038^4 + 829120^4$ can be written as
$(640000 + 204962)^4 - (640000 - 204962)^4 = 191600^4 + 829120^4$,
the expression $353^4 = 30^4 + 120^4 + 272^4 + 315^4$ (from [21]) can be written as
$(334 + 19)^4 - (334 - 19)^4 = 30^4 + 120^4 + 272^4$
$\Rightarrow$ $706^4 = 60^4 + 240^4 + 544^4 + 630^4$ can be written as
$(688 + 38)^4 - (668 - 38)^4 = 60^4 + 240^4 + 544^4$,
the expression $144^5 = 27^5 + 84^5 + 110^5 + 133^5$ (from [21]) can be written as
$(127 + 17)^5 - (127 - 17)^5 = 27^5 + 84^5 + 133^5$
$\Rightarrow$ $288^5 = 54^5 + 168^5 + 220^5 + 266^5$ can be written as
$(254 + 34)^5 - (254 - 34)^5 = 54^5 + 168^5 + 266^5$ etc.

From the above analysis, we have the following results:

Theorem-4.1
Every Diophantine equation $a^n = a_1^n + a_2^n + \cdots + a_s^n$ can be expressed as $(u+v)^n - (u-v)^n = a_2^n + \cdots + a_s^n$ or $(u+v)^n - (u-v)^n = (2a_2)^n + \cdots + (2a_s)^n$, where $a, a_1, a_2, a_3, \cdots, a_s$ are positive integers.
Proof: $a^n = a_1^n + a_2^n + \cdots + a_s^n \Rightarrow a^n - a_1^n = a_2^n + \cdots + a_s^n$
Case-I If a is odd.
If a is odd positive integer, then one of the positive integers $a_1, a_2, a_3, \cdots, a_s$ must be odd. So suppose that a_1 is odd. Now a, a_1 are odd so $a + a_1, a - a_1$ are even.
Take $u = \dfrac{a+a_1}{2}$, $v = \dfrac{a-a_1}{2}$, then $a = u + v$, $a_1 = u - v$
$\therefore$ $a^n - a_1^n = a_2^n + \cdots + a_s^n \Rightarrow (u+v)^n - (u-v)^n = a_2^n + \cdots + a_s^n$
This implies $a^n = a_1^n + a_2^n + \cdots + a_s^n$ can be expressed as $(u+v)^n - (u-v)^n = a_2^n + \cdots + a_s^n$ in this case.

Case-II If a is even and one of the positive integers $a_1, a_2, a_3, \cdots, a_s$ is even.
If one of the positive integers $a_1, a_2, a_3, \cdots, a_s$ is even, then suppose that a_1 is even. Now a, a_1 are even so $a + a_1, a - a_1$ are even.
Take $u = \dfrac{a+a_1}{2}$, $v = \dfrac{a-a_1}{2}$, then $a = u + v$, $a_1 = u - v$
$\therefore$ $a^n - a_1^n = a_2^n + \cdots + a_s^n \Rightarrow (u+v)^n - (u-v)^n = a_2^n + \cdots + a_s^n$
This implies $a^n = a_1^n + a_2^n + \cdots + a_s^n$ can be expressed as $(u+v)^n - (u-v)^n = a_2^n + \cdots + a_s^n$ in this case.

Case-III If a is even but none of the positive integers $a_1, a_2, a_3, \cdots, a_s$ is even.
Then equation $a^n = a_1^n + a_2^n + \cdots + a_s^n$ can be expressed as $(2a)^n = (2a_1)^n + (2a_2)^n + \cdots + (2a_s)^n$
and $(2a)^n = (2a_1)^n + (2a_2)^n + \cdots + (2a_s)^n \Rightarrow (2a)^n - (2a_1)^n = (2a_2)^n + \cdots + (2a_s)^n$
Take $u = \dfrac{2a+2a_1}{2} = a + a_1$, $v = \dfrac{2a-2a_1}{2} = a - a_1$, then $2a = u + v$, $2a_1 = u - v$.
So $(2a)^n - (2a_1)^n = (2a_2)^n + \cdots + (2a_s)^n \Rightarrow (u+v)^n - (u-v)^n = (2a_2)^n + \cdots + (2a_s)^n$
This implies $(2a)^n = (2a_1)^n + (2a_2)^n + \cdots + (2a_s)^n$ can be expressed as $(u+v)^n - (u-v)^n = (2a_2)^n + \cdots + (2a_s)^n$.
This implies $a^n = a_1^n + a_2^n + \cdots + a_s^n$ can be expressed as $(u+v)^n - (u-v)^n = (2a_2)^n + \cdots + (2a_s)^n$ in this case.

Theorem-4.2 Every Diophantine equation $(2a)^n = (2a_1)^n + (2a_2)^n + \cdots + (2a_s)^n$ can be expressed as $(u+v)^n - (u-v)^n = (2a_2)^n + \cdots + (2a_s)^n$, where $a, a_1, a_2, a_3, \cdots, a_s$ are positive integers.
Proof: $(2a)^n = (2a_1)^n + (2a_2)^n + \cdots + (2a_s)^n \Rightarrow (2a)^n - (2a_1)^n = (2a_2)^n + \cdots + (2a_s)^n$
Take $u = \dfrac{2a+2a_1}{2} = a + a_1$, $v = \dfrac{2a-2a_1}{2} = a - a_1$, then $2a = u + v$, $2a_1 = u - v$.

So $(2a)^n-(2a_1)^n=(2a_2)^n+\cdots+(2a_s)^n \Rightarrow (u+v)^n-(u-v)^n=(2a_2)^n+\cdots+(2a_s)^n$
This implies $(2a)^n=(2a_1)^n+(2a_2)^n+\cdots+(2a_s)^n$ can be expressed as $(u+v)^n-(u-v)^n=(2a_2)^n+\cdots+(2a_s)^n$.

Theorem-4.3 Every Diophantine equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ can be expressed as $(u+v)^n-(u-v)^n=(2a_2)^n+\cdots+(2a_s)^n$, where $a,a_1,a_2,a_3,\cdots,a_s$ are positive integers.
Proof: Equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ can be expressed as $(2a)^n=(2a_1)^n+(2a_2)^n+\cdots+(2a_s)^n$ and by **theorem-4.2**, $(2a)^n=(2a_1)^n+(2a_2)^n+\cdots+(2a_s)^n$ can be expressed as

$$(u+v)^n-(u-v)^n=(2a_2)^n+\cdots+(2a_s)^n, \text{ where } u+v=2a, u-v=2a_1.$$

Hence $a^n=a_1^n+a_2^n+\cdots+a_s^n$ can be expressed as $(u+v)^n-(u-v)^n=(2a_2)^n+\cdots+(2a_s)^n$

Theorem-4.4 Every Diophantine equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ can be obtained by putting some positive integral values of x,y in the equation

$$(x+y)^n=(x-y)^n+2\binom{n}{1}x^{n-1}y+2\binom{n}{3}x^{n-3}y^3+\cdots+2\alpha,$$

$$\text{where} \quad \alpha=\begin{cases} y^n, & \text{if } n \text{ is odd} \\ \binom{n}{n-1}xy^{n-1}, & \text{if } n \text{ is even}\end{cases}$$

Or
Equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ is the particular case of the equation

$$(x+y)^n=(x-y)^n+2\binom{n}{1}x^{n-1}y+2\binom{n}{3}x^{n-3}y^3+\cdots+2\alpha$$

where $a,a_1,a_2,a_3,\cdots,a_s$ are positive integers.
Proof: By **theorem-4.3**, equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ can be expressed as $(u+v)^n-(u-v)^n=(2a_2)^n+\cdots+(2a_s)^n$
But $\quad (u+v)^n-(u-v)^n=2\binom{n}{1}u^{n-1}v+2\binom{n}{3}u^{n-3}v^3+\cdots+2\alpha, \quad$ ... (4)

$$\text{where} \quad \alpha=\begin{cases} v^n, & \text{if } n \text{ is odd} \\ \binom{n}{n-1}uv^{n-1}, & \text{if } n \text{ is even.}\end{cases}$$

Therefore, by Equation 4,
$a^n=a_1^n+a_2^n+\cdots+a_s^n$ can be expressed as $(u+v)^n-(u-v)^n=2\binom{n}{1}u^{n-1}v+2\binom{n}{3}u^{n-3}v^3+\cdots+2\alpha$
$\Rightarrow a^n=a_1^n+a_2^n+\cdots+a_s^n$ can be expressed as $(u+v)^n=(u-v)^n+2\binom{n}{1}u^{n-1}v+2\binom{n}{3}u^{n-3}v^3+\cdots+2\alpha$
In other words, equation $a^n=a_1^n+a_2^n+\cdots+a_s^n$ is the particular case of the equation $(x+y)^n=(x-y)^n+2\binom{n}{1}x^{n-1}y+2\binom{n}{3}x^{n-3}y^3+\cdots+2\alpha$, because $a^n=a_1^n+a_2^n+\cdots+a_s^n$ is obtained from it by putting $x=u, y=v$. This proves the theorem.
Further for some positive integers u,v $(u>v)$, if $(u+v)^n-(u-v)^n=a_2^n+a_3^n+\cdots+a_{r+1}^n$ and m is the number of terms in $(x+y)^n-(x-y)^n$ as a function of x,y; then by theorem-2.1, $m\le r$ always.
Let $2\binom{n}{1}u^{n-1}v+2\binom{n}{3}u^{n-3}v^3+\cdots+2\alpha=\beta$.
Therefore, we notice that proofs of ***Fermat's last theorem*** and its extensions are given by the expression $(u+v)^n-(u-v)^n=a^n-a_1^n=\beta$ completely. Because if there exists a positive integer a_2 such that $\beta=a_2^n$, then $a^n-a_1^n=a_2^n$ and this implies $a^n=a_1^n+a_2^n$.

By using expression $(u+v)^n-(u-v)^n=a^n-a_1^n=\beta$, this theorem can be proved as follows.
Theorem-4.5 (Fermat's Last Theorem): Equation $a^n=a_1^n+a_2^n$ is possible for $n=1,2$ and it is not possible for any $n>2$ where a,a_1,a_2 are positive integers.
Proof: To prove $a^n=a_1^n+a_2^n$ holds for $n=1$, it is easy to see that every positive integer $a\ge 2$ can be expressed as $a=a_1+a_2$, where a_1,a_2 are positive integers and $a_1=a_2$ may be possible
Also $a=a_1+a_2 \Rightarrow a^1=a_1^1+a_2^1$
Therefore, $a^n=a_1^n+a_2^n$ holds for $n=1$.

To prove $a^n=a_1^n+a_2^n$ holds for $n=2$, consider the equation
$(x+y)^2-(x-y)^2=4xy$
Put $x=u^2, y=v^2$, then, $(u^2+v^2)^2-(u^2-v^2)^2=4u^2v^2=(2uv)^2$
We can choose integers u,v in such a way that $u^2+v^2, u^2-v^2, 2uv$ are all positive.
Now put $x+y=u^2+v^2=a,\ x-y=u^2-v^2=a_1,\ 2uv=a_2$
Then, $a^2-a_1^2=a_2^2 \Rightarrow a^2=a_1^2+a_2^2$

So there exist positive integers a, a_1, a_2 satisfying the equation $a^2 = a_1^2 + a_2^2$.
Therefore, $a^n = a_1^n + a_2^n$ holds for $n = 2$.

For $n > 2$, if equation $a^n = a_1^n + a_2^n$ is possible $\Rightarrow$ $(2a)^n = (2a_1)^n + (2a_2)^n$ is possible $\Rightarrow$ $b^n = b_1^n + b_2^n$ is possible, where $b = 2a, b_1 = 2a_1, b_2 = 2a_2$..

By theorem-4.2, $(2a)^n = (2a_1)^n + (2a_2)^n$ can be expressed as $(u+v)^n - (u-v)^n = (2a_2)^n$, where $u + v = 2a$, $u - v = 2a_1$, i.e. $b^n = b_1^n + b_2^n$ can be expressed as $(u+v)^n - (u-v)^n = b_2^n$, where $u + v = b, u - v = b_1$.

If it is possible to express $(u+v)^n - (u-v)^n = b_2^n + b_3^n + \cdots$, then by theorem-2.1, number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y $\leq$ number of $b_i s$ in $b_2^n + b_3^n + \cdots$.

Now for $n > 2$, there are at least 2 terms in the expression $(x+y)^n - (x-y)^n$ as a function of x, y; so the number of $b_i s$ in the expression $b_2^n + b_3^n + \cdots$ cannot be less than 2(=least number of terms in $(x+y)^n - (x-y)^n$), i.e. there exist at least 2 positive integers b_2, b_3 such that

$$(u+v)^n - (u-v)^n = b_2^n + b_3^n .$$

This implies there exists no positive integer b_2 such that $(u+v)^n - (u-v)^n = b_2^n$
$\Rightarrow$ $b^n - b_1^n = b_2^n$ is not possible $\Rightarrow$ $(2a)^n - (2a_1)^n = (2a_2)^n$ is not possible $\Rightarrow$ $a^n - a_1^n = a_2^n$ is not possible
$\Rightarrow$ $a^n = a_1^n + a_2^n$ is not possible for $n > 2$.

Therefore, $a^n = a_1^n + a_2^n$ does not hold for $n > 2$. This proves the theorem.

Theorem-4.6 Equation $a^n = a_1^n + a_2^n + a_3^n$ is possible for $n = 1, 2, 3, 4$ and it is not possible for any $n > 4$ where a, a_1, a_2, a_3 are positive integers.

Proof: To prove $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 1$, it is easy to see that every positive integer $a \geq 3$ can be expressed as $a = a_1 + a_2 + a_3$, where a_1, a_2, a_3 are positive integers and $a_1 = a_2 = a_3$ may be possible or any two of a_1, a_2, a_3 may be equal. Also $a = a_1 + a_2 + a_3$ $\Rightarrow$ $a^1 = a_1^1 + a_2^1 + a_3^1$
Therefore, $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 1$.

To prove $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 2$, consider the equation

$$(x+y+z)^2 - (x+y-z)^2 = 4xz + 4yz.$$

Put $x = u^2, y = v^2, z = w^2$, then

$$(u^2+v^2+w^2)^2 - (u^2+v^2-w^2)^2 = 4u^2w^2 + 4v^2w^2 = (2uw)^2 + (2vw)^2$$

We can choose integers u, v, w in such a way that $u^2+v^2+w^2, u^2+v^2-w^2, 2uw, 2vw$ are all positive.
Put $u^2+v^2+w^2 = a$, $u^2+v^2-w^2 = a_1$, $2uw = a_2$, $2vw = a_3$
Then, $a^2 - a_1^2 = a_2^2 + a_3^2$ $\Rightarrow$ $a^2 = a_1^2 + a_2^2 + a_3^2$
So there exist positive integers a, a_1, a_2, a_3 satisfying the equation $a^2 = a_1^2 + a_2^2 + a_3^2$.
Therefore, $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 2$.

To prove $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 3$, consider the identity given by Ramanujan in [21],

$$(6u^2 - 4uv + 4v^2)^3 = (4u^2 - 4uv + 6v^2)^3 + (5u^2 - 5uv - 3v^2)^3 + (3u^2 + 5uv - 5v^2)^3$$

We can choose integers u, v, w in such a way that $6u^2 - 4uv + 4v^2, 4u^2 - 4uv + 6v^2, 5u^2 - 5uv - 3v^2$, $3u^2 + 5uv - 5v^2$ are all positive.

Put $6u^2 - 4uv + 4v^2 = a$, $4u^2 - 4uv + 6v^2 = a_1$, $5u^2 - 5uv - 3v^2 = a_2$, $3u^2 + 5uv - 5v^2 = a_3$, then

$$a^3 = a_1^3 + a_2^3 + a_3^3 .$$

So there exist positive integers a, a_1, a_2, a_3 satisfying the equation $a^3 = a_1^3 + a_2^3 + a_3^3$.
Therefore, $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 3$.

To prove $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 4$, by Roger Frye, we have the equation
$(422481t)^4 = (217519t)^4 + (95800t)^4 + (414560t)^4$, where t be any positive integer.
Put $422481t = a$, $217519t = a_1$, $95800t = a_2$, $414560t = a_3$, then

$$a^4 = a_1^4 + a_2^4 + a_3^4 .$$

So there exist positive integers a, a_1, a_2, a_3 satisfying the equation $a^4 = a_1^4 + a_2^4 + a_3^4$.
Therefore, $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 4$.

For $n > 4$, if $a^n = a_1^n + a_2^n + a_3^n$ is possible $\Rightarrow$ $(2a)^n = (2a_1)^n + (2a_2)^n + (2a_3)^n$ is possible $\Rightarrow$ $b^n = b_1^n + b_2^n + b_3^n$ is possible, where $b = 2a, b_1 = 2a_1, b_2 = 2a_2, b_3 = 2a_3$.

By theorem-4.2, $(2a)^n = (2a_1)^n + (2a_2)^n + (2a_3)^n$ can be expressed as $(u+v)^n - (u-v)^n = (2a_2)^n + (2a_3)^n$, where $u+v = 2a, u-v = 2a_1$, i.e. $b^n = b_1^n + b_2^n + b_3^n$ can be expressed as $(u+v)^n - (u-v)^n = b_2^n + b_3^n$, where $u+v = b, u-v = b_1$.

If it is possible to express $(u+v)^n - (u-v)^n = b_2^n + b_3^n + \cdots$, then by theorem-2.1, number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y $\leq$ number of $b_i s$ in $b_2^n + b_3^n + \cdots$.

Now for $n > 4$, there are at least 3 terms in the expression $(x+y)^n - (x-y)^n$ as a function of x, y; so the number of $b_i s$ in the expression $b_2^n + b_3^n + \cdots$ cannot be less than 3(=least number of terms in $(x+y)^n - (x-y)^n$) , i.e. there exist at least 3 positive integers b_2, b_3, b_4 such that

$$(u+v)^n - (u-v)^n = b_2^n + b_3^n + b_4^n .$$

This implies there exist no positive integers b_2, b_3 such that $(u+v)^n - (u-v)^n = b_2^n + b_3^n$
$\Rightarrow$ $b^n - b_1^n = b_2^n + b_3^n$ is not possible $\Rightarrow$ $(2a)^n - (2a_1)^n = (2a_2)^n + (2a_3)^n$ is not possible $\Rightarrow$ $a^n - a_1^n = a_2^n + a_3^n$ is not possible
$\Rightarrow$ $a^n = a_1^n + a_2^n + a_3^n$ is not possible for $n > 4$.

Therefore, $a^n = a_1^n + a_2^n + a_3^n$ does not hold for $n > 4$. This proves the theorem.

5. Analysis of the Diophantine Equation $a^n = a_1^n + a_2^n + \cdots + a_s^n$

Every Diophantine equation $a^n = a_1^n + a_2^n + \cdots + a_s^n$ can be expressed as $(2a)^n = (2a_1)^n + (2a_2)^n + \cdots + (2a_s)^n$, and by theorem-4.2, $(2a)^n = (2a_1)^n + (2a_2)^n + \cdots + (2a_s)^n$ can be expressed as $(u+v)^n - (u-v)^n = (2a_2)^n + \cdots + (2a_s)^n$, where $u+v = 2a, u-v = 2a_1$, i.e. $b^n = b_1^n + b_2^n + \cdots + b_s^n$ can be expressed as $(u+v)^n - (u-v)^n = b_2^n + \cdots + b_s^n$, where $b = u+v = 2a, b_1 = u-v = 2a_1$, $b_2 = 2a_2, \cdots, b_s = 2a_s$.

From the above illustration, $a^n = a_1^n + a_2^n + \cdots + a_s^n$ can be expressed as $(u+v)^n - (u-v)^n = (2a_2)^n + \cdots + (2a_s)^n$, where $u+v = 2a, u-v = 2a_1$.

Let m = number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y; then $s \geq m+1$.

Also by using theorem-2.1, there exist least number of positive integers $a_2, a_3, \cdots$ such that

$$(u+v)^n - (u-v)^n = \begin{cases} a_2^n + a_3^n + \cdots + a_{\frac{n+3}{2}}^n \text{ if } n \text{ is odd} \\ a_2^n + a_3^n + \cdots + a_{\frac{n+2}{2}}^n \text{ if } n \text{ is even} \end{cases}.$$

It is easy to check that every positive integer $a > 1$ can be expressed as $a = a_1 + a_2$, where a_1, a_2 are positive integers and $a_1 = a_2$ may be possible. Also $a = a_1 + a_2 \Rightarrow a^1 = a_1^1 + a_2^1$

Or

If $n = 1$, the expression, $c^n - d^n = c^1 - d^1 = c - d$ has at least 1 term for any variable positive integral substitutions of the form $c = u_1 + u_2 + \cdots + u_r$, $d = v_1 + v_2 + \cdots + v_s$ and $c = x+y, d = x-y$, where x, y $(x > y)$ are variable positive integers for which $c^1 - d^1 = c - d$ has 1 term because

$$c^1 - d^1 = c - d = (x+y)^1 - (x-y)^1 = 2y \qquad \ldots (5)$$

From the Equation 5, we find that there is 1 term $2y$ in the expression $(x+y)^1 - (x-y)^1$. Therefore by theorem-2.1, for positive integers $x = u, y = v$ $(u > v)$, there exists at least 1 positive integer a_2 such that

$$(u+v)^1 - (u-v)^1 = a_2^1 \qquad \ldots (6)$$

$$\Rightarrow \qquad 2v = a_2^1 = a_2$$

Therefore there exists positive integer $a_2 = 2v$ which satisfies the Equation 6.
Also Equation 6 can be written as

$$(u+v)^1 = (u-v)^1 + a_2^1 \qquad \ldots (7)$$

Put $u+v = a$, $u-v = a_1$ in Equation 7, we get the equation

$$a^1 = a_1^1 + a_2^1 \qquad \ldots (8)$$

From the Equation 8, we find that every positive integer> 1, can be expressed as a sum of at least 2 positive integers.

Again let m be the number of terms in $(x+y)^1-(x-y)^1$ as a function of x,y. Since $(x+y)^1-(x-y)^1=2y$, there is 1 term $2y$ in $(x+y)^1-(x-y)^1$, therefore $m=1$. Therefore by theorem-2.1, any positive integer >1, can be expressed as a sum of at least $m+1=1+1=2$ positive integers.

From the above illustration, we find that the equation $a^1=a_1^1+a_2^1+\cdots+a_s^1$ holds only when $s\geq 2$, where $a,a_1,a_2,\cdots,a_s$ are positive integers.

If $n=2$, the expression $c^n-d^n = c^2-d^2$ has at least 1 term for any variable positive integral substitutions of the form $c=u_1+u_2+\cdots+u_r$, $d=v_1+v_2+\cdots+v_s$ and $c=x+y, d=x-y$, where x,y $(x>y)$ are variable positive integers for which c^2-d^2 has 1 term because

$$c^2-d^2=(x+y)^2-(x-y)^2=4xy \qquad \dots(9)$$

From the Equation 9, we find that there is 1 term $4xy$ in the expression $(x+y)^2-(x-y)^2$. Therefore by theorem-2.1, for positive integers $x=u^2, y=v^2$ $(u>v)$, there exists at least 1 positive integer a_2 such that

$$(u^2+v^2)^2-(u^2-v^2)^2=a_2^2 \qquad \dots(10)$$

$$\Rightarrow \qquad 4u^2v^2=a_2^2 \Rightarrow (2uv)^2=a_2^2 \Rightarrow a_2=2uv$$

Therefore there exists positive integer $a_2=2uv$ which satisfies the Equation 10.
Also Equation 10 can be written as

$$(u^2+v^2)^2=(u^2-v^2)^2+a_2^2 \qquad \dots(11)$$

Put $u^2+v^2=a$, $u^2-v^2=a_1$ in Equation 11, we get the equation

$$a^2=a_1^2+a_2^2 \qquad \dots(12)$$

From the Equation 12, we find that square of every positive integer>1, can be expressed as a sum of squares of at least 2 positive integers.
Again let m is the number of terms in $(x+y)^2-(x-y)^2$ as a function of x,y. Since $(x+y)^2-(x-y)^2=4xy$, there is 1 term $4xy$ in $(x+y)^2-(x-y)^2$, then $m=1$. Therefore by **theorem-2.1**, square of every positive integer >1 can be expressed as a sum of squares of at least $m+1=1+1=2$ positive integers.
From the above illustration, we find that the equation $a^2=a_1^2+a_2^2+\cdots+a_s^2$ holds only when $s\geq 2$, where $a,a_1,a_2,\cdots,a_s$ are positive integers.

If $n=3$, the expression $c^n-d^n = c^3-d^3$ has at least 2 terms for any variable positive integral substitutions of the form $a=u_1+u_2+\cdots+u_r$, $b=v_1+v_2+\cdots+v_s$ and $c=x+y, d=x-y$, where x,y $(x>y)$ are variable positive integers for which c^3-d^3 has 2 terms because

$$c^3-d^3=(x+y)^3-(x-y)^3=6x^2y+2y^3 \qquad \dots(13)$$

From the Equation 13, we find that there are 2 terms $6x^2y, 2y^3$ in the expression $(x+y)^3-(x-y)^3$. Therefore by theorem-2.1, for positive integers $x=u, y=v$ $(u>v)$, there exist at least 2 positive integers a_2,a_3 such that

$$(u+v)^3-(u-v)^3=a_2^3+a_3^3 \qquad \dots(14)$$

Now equation $(6t)^3=(5t)^3+(4t)^3+(3t)^3$, where t is any positive integer.
$\Rightarrow \quad (5t+t)^3-(5t-t)^3=(5t)^3+(3t)^3$
Take $u=5t, v=t$. Then $(u+v)^3-(u-v)^3=(5t)^3+(3t)^3$
By Equation 14, $\quad a_2^3+a_3^3=(5t)^3+(3t)^3 \Rightarrow a_2=5t, a_3=3t$
Therefore there exist positive integers $a_2=5t, a_3=3t$ which satisfy the Equation 14 at $u=5t, v=3t$.
Also Equation 14 can be written as

$$(u+v)^3=(u-v)^3+a_2^3+a_3^3 \qquad \dots(15)$$

Put $u+v=6t=a,\ u-v=4t=a_1$ in Equation 15, we get the equation

$$a^3=a_1^3+a_2^3+a_3^3 \qquad ...(16)$$

From the Equation 16, we find that cube of every positive integer>1, can be expressed as a sum of cubes of at least 3 positive integers.

Again let m is the number of terms in $(x+y)^3-(x-y)^3$ as a function of x,y. Since $(x+y)^3-(x-y)^3=6x^2y+2y^3$, there are 2 terms $6x^2y,2y^3$ in $(x+y)^3-(x-y)^3$, then $m=2$. Therefore by theorem-2.1, cube of every positive integer >1 can be expressed as a sum of cubes of at least $m+1=2+1=3$ positive integers.

From the above illustration, we find that the equation $a^3=a_1^3+a_2^3+\cdots+a_s^3$ holds only when $s\geq 3$, where $a,a_1,a_2,\cdots,a_s$ are positive integers.

If $n=4$, the expression $c^n-d^n=c^4-d^4$ has at least 2 terms for any variable positive integral substitutions of the form $c=u_1+u_2+\cdots+u_r\,,d=v_1+v_2+\cdots+v_s$ and $c=x+y,d=x-y$, where $x,y\ (x>y)$ are variable positive integers for which c^4-d^4 has 2 terms because

$$c^4-d^4=(x+y)^4-(x-y)^4=8x^3y+8xy^3 \qquad ...(17)$$

From the Equation 17, we find that there are 2 terms $8x^3y,8xy^3$ in the expression $(x+y)^4-(x-y)^4$. Therefore by **theorem-2.1**, for positive integers $x=u,y=v\ (u>v)$, there exist at least 2 positive integers a_2,a_3 such that

$$(u+v)^4-(u-v)^4=a_2^4+a_3^4 \qquad ...(18)$$

Due to Roger Frye, equation

$(422481t)^4=(95800t)^4+(217519t)^4+(414560t)^4$, where t is any positive integer.

$\Rightarrow\ (320000t+102481t)^4-(320000t-102481t)^4=(95800t)^4+(414560t)^4$

Take $u=320000t,\ v=102481t$. Then $(u+v)^4-(u-v)^4=(95800t)^4+(414560t)^4$

By Equation 18, $a_2^4+a_3^4=(95800t)^4+(414560t)^4\ \Rightarrow\ a_2=95800t,a_3=414560t$

Therefore there exist positive integers $a_2=95800t,a_3=414560t$ which satisfy the Equation 18 at $u=320000t$, $v=102481t$.

Also Equation 18 can be written as

$$(u+v)^4=(u-v)^4+a_2^4+a_3^4 \qquad ...(19)$$

Put $u+v=422481t=a,\ u-v=217519t=a_1$ in Equation 19, we get the equation

$$a^4=a_1^4+a_2^4+a_3^4 \qquad ...(20)$$

From the Equation 20, we find that biquadrate of every positive integer>1, can be expressed as a sum of biquadrates of at least 3 positive integers.

Again let m is the number of terms in $(x+y)^4-(x-y)^4$ as a function of x,y. Since $(x+y)^4-(x-y)^4=8x^3y+8xy^3$, there are 2 terms $8x^3y,8xy^3$ in $(x+y)^4-(x-y)^4$, then $m=2$. Therefore by theorem-2.1, biquadrate of every positive integer >1 can be expressed as a sum of biquadrates of at least $m+1=2+1=3$ positive integers.

From the above illustration, we find that the equation $a^4=a_1^4+a_2^4+\cdots+a_s^4$ holds only when $s\geq 3$, where $a,a_1,a_2,\cdots,a_s$ are positive integers.

If $n=5$, the expression $c^n-d^n=c^5-d^5$ has at least 3 terms for any variable positive integral substitutions of the form $c=u_1+u_2+\cdots+u_r\,,d=v_1+v_2+\cdots+v_s$ and $c=x+y,d=x-y$, where $x,y\ (x>y)$ are variable positive integers for which c^5-d^5 has 3 terms because

$$c^5-d^5=(x+y)^5-(x-y)^5=10x^4y+20x^2y^3+2y^5 \qquad ...(21)$$

From the Equation 21, we find that there are 3 terms $10x^4y,\ 20x^2y^3,\ 2y^5$ in the expression $(x+y)^5-(x-y)^5$. Therefore by **theorem-2.1**, for positive integers $x=u,y=v\ (u>v)$, there exist at least 3 positive integers a_2,a_3,a_4 such that

$$(u+v)^5-(u-v)^5=a_2^5+a_3^5+\ a_4^5 \qquad ...(22)$$

From [21], equation
$(144t)^5 = (27t)^5 + (84t)^5 + (110t)^5 + (133t)^5$, where t is any positive integer.
$\Rightarrow (127t + 17t)^5 - (127t - 17t)^5 = (27t)^5 + (84t)^5 + (133t)^5$
Take $u = 127t, v = 17t$. Then $(u + v)^5 - (u - v)^5 = (27t)^5 + (84t)^5 + (133t)^5$
By Equation 22, $a_2^5 + a_3^5 + a_4^5 = (27t)^5 + (84t)^5 + (133t)^5 \Rightarrow a_2 = 27t, a_3 = 84t, a_4 = 133t$
Therefore there exist positive integers $a_2 = 27t, a_3 = 84t, a_4 = 133t$ which satisfy the Equation 22 at $u = 127t, v = 17t$.
Also Equation 22 can be written as

$$(u + v)^5 = (u - v)^5 + a_2^5 + a_3^5 + a_4^5 \quad \dots (23)$$

Put $u + v = 144t = a,\ u - v = 110t = a_1$ in Equation 23, we get the equation

$$a^5 = a_1^5 + a_2^5 + a_3^5 + a_4^5 \quad \dots (24)$$

From the Equation 24, we find that the fifth power of every positive integer> 1, can be expressed as a sum of the fifth powers of at least 4 positive integers.

Again let m is the number of terms in $(x + y)^5 - (x - y)^5$ as a function of x, y. Since $(x + y)^5 - (x - y)^5 = 10x^4y + 20x^2y^3 + 2y^5$, there are 3 terms $10x^4y, 20x^2y^3, 2y^5$ in $(x + y)^5 - (x - y)^5$, then $m = 3$. Therefore by theorem-2.1, the fifth power of every positive integer > 1 can be expressed as a sum of the fifth powers of at least $m + 1 = 3 + 1 = 4$ positive integers.

From the above illustration, we find that the equation $a^5 = a_1^5 + a_2^5 + \cdots + a_s^5$ holds only when $s \geq 4$, where $a, a_1, a_2, \cdots, a_s$ are positive integers.

Further on the basis of the above discussion in this section, if $n = 6$, the expression $c^n - d^n = c^6 - d^6$ has at least 3 terms for any variable positive integral substitutions of the form $c = u_1 + u_2 + \cdots + u_r$, $d = v_1 + v_2 + \cdots + v_s$ and $c = x + y, d = x - y$, where $x, y\ (x > y)$ are variable positive integers for which $c^6 - d^6$ has 3 terms because

$$c^6 - d^6 = (x + y)^6 - (x - y)^6 = 12x^5y + 40x^3y^3 + 12xy^5 \quad \dots (25)$$

From the Equation 25, we find that there are 3 terms $12x^5y,\ 40x^3y^3, 12xy^5$ in the expression $(x + y)^6 - (x - y)^6$. Therefore by theorem-2.1, for positive integers $x = u, y = v\ (u > v)$, there exist at least 3 positive integers a_2, a_3, a_4 such that

$$(u + v)^6 - (u - v)^6 = a_2^6 + a_3^6 + a_4^6 \quad \dots (26)$$

Also Equation 26 can be written as

$$(u + v)^6 = (u - v)^6 + a_2^6 + a_3^6 + a_4^6 \quad \dots (27)$$

Put $u + v = a,\ u - v = a_1$ in Equation 27, we get the equation

$$a^6 = a_1^6 + a_2^6 + a_3^6 + a_4^6 \quad \dots (28)$$

From the Equation 28, we find that the sixth power of every positive integer> 1, can be expressed as a sum of the sixth powers of at least 4 positive integers.

Again let m is the number of terms in $(x + y)^6 - (x - y)^6$ as a function of x, y. Since $(x + y)^6 - (x - y)^6 = 12x^5y + 40x^3y^3 + 12xy^5$, there are 3 terms $12x^5y$, $40x^3y^3$, $12xy^5$ in $(x + y)^6 - (x - y)^6$, then $m = 3$. Therefore by theorem-2.1, sixth power of every positive integer > 1 can be expressed as a sum of sixth powers of at least $m + 1 = 3 + 1 = 4$ positive integers.

From the above illustration, we find that the equation $a^6 = a_1^6 + a_2^6 + \cdots + a_s^6$ holds only when $s \geq 4$, where $a, a_1, a_2, \cdots, a_s$ are positive integers.

Continuing like this, in general, the equation $a^n = a_1^n + a_2^n + \cdots + a_s^n$ holds only when

$$s \geq \begin{cases} \frac{n+3}{2} & \text{if } n \text{ is odd} \\ \frac{n+2}{2} & \text{if } n \text{ is even} \end{cases}$$

If it is so, then $a^n = a_1^n + a_2^n + \cdots + a_s^n$ definitely holds for all $s \geq n$, where $a, a_1, a_2, \cdots, a_s$ are positive integers.

Also if m is the number of terms in $(x+y)^n - (x-y)^n$ as a function of x, y; then

$$m = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases} \quad \text{or } m = \left[\frac{n+1}{2}\right].$$

Therefore, by theorem-2.1, equation $a^n = a_1^n + a_2^n + \cdots + a_{m+1}^n$ holds for $n = 1, 2, \cdots, 2m$ and it does not hold if $n > 2m$, where $a_1, a_2, \cdots, a_{m+1}$ are positive integers.

6. Conclusion

From the above discussion, we draw the following conclusions:
It is possible to find positive integers $a, a_1, a_2, \cdots, a_s$ such that
the equation $a^n = a_1^n + a_2^n$ holds for $n = 1, 2$; and it does not hold for $n > 2$,
the equation $a^n = a_1^n + a_2^n + a_3^n$ holds for $n = 1, 2, 3, 4$; and it does not hold for $n > 4$,
the equation $a^n = a_1^n + a_2^n + a_3^n + a_4^n$ holds for $n = 1, 2, 3, 4, 5, 6$; and it does not hold for $n > 6$,
Further on the basis of validity of the above equations and analysis in the **section-5**, there is possibility that
the equation $a^n = a_1^n + a_2^n + a_3^n + a_4^n + a_5^n$ holds for $n = 1, 2, 3, 4, 5, 6, 7, 8$; and it does not hold for $n > 8$,
the equation $a^n = a_1^n + a_2^n + a_3^n + a_4^n + a_5^n + a_6^n$ holds for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$; and it does not hold for $n > 10$,
Continuing like this,
the equation $a^n = a_1^n + a_2^n + \cdots + a_s^n$ holds for $n = 1, 2, 3, \cdots, 2s - 2$ and it does not hold for $n > 2s - 2$.

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