International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V68I8P512

Volume 68 Issue 8, 129-135, August 2022
© 2022 Seventh Sense Research Group®

Original Article

Fixed Point Theorems for z-Contractions in Intuitionistic Fuzzy Metric Space using Fuzzy Simulation Function

Deepika and Manoj Kumar

Baba Mastnath University, Asthal Bohar, Rohtak.

Received: 06 July 2022 Revised: 08 August 2022 Accepted: 20 August 2022 Published: 29 August 2022

Abstract *- In this paper, we establish fixed point theorem for five tuples using the concept of simulation function, z-contraction and fuzzy simulation function in intuitionistic fuzzy metric space.*

2010 MSC: 47H10, 54H25.

Keywords *- Intuitionistic fuzzy metric space, z-contraction, Simulation function, Fuzzy simulation function.*

1. Introduction

In 1965, the concept of fuzzy sets has been given by L.A. Zadeh [4]. A lot of researchers worked in this direction(see [7]-[11]). As a generalization of fuzzy sets, Atanassov [2] introduced the notion of intuitionistic fuzzy sets in 1983.

2. Preliminaries

Definition 2.1. [3] A binary operation $\Delta: [0,1] \times [0,1] \rightarrow [0,1]$ is called continuous t-norm if Δ satisfies the following conditions:

i) Δ is commutative and associative,
ii) Δ is continuous,
iii) $a \,\Delta\, 1 = a$, for all $a \in [0,1]$,
iv) $a \,\Delta\, b \leq c \,\Delta\, d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2. [3] A binary operation $\circ: [0,1] \times [0,1] \rightarrow [0,1]$ is called continuous t-conorm if $\circ$ satisfies the following conditions:

i) $\circ$ is commutative and associative,
ii) $\circ$ is continuous,
iii) $a \circ 0 = a$, for all $a \in [0,1]$,
iv) $a \circ b \leq c \circ d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.3. [1] **(**Intuitionistic Fuzzy Metric Space) The 5-tuple $(X, M, N, \Delta, \circ)$ is said to be an intuitionistic fuzzy metric space(Shortly, IFM-space) if X is an arbitrary set, Δ is a continuous t-norm, $\circ$ is a continuous t-conorm. M and N are fuzzy sets in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$,

I. $M(x, y, t) + N(x, y, t) \leq 1$,
II. $M(x, y, 0) = 0$,
III. $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
IV. $M(x, y, t) = M(y, x, t)$,

This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

V. $M(x,y,t) = 1 \; as \; t \to \infty$,

VI. $M(x,y,t) \, \Delta \, M(y,z,s) \le M(x,z,t+s)$,

VII. $M(x,y,.)$: $[0,\infty) \to [0,1]$ is left continuous,

VIII. $N(x,y,0) = 1$,

IX. $N(x,y,t) = 0$ for all $t > 0$ if and only if $x = y$,

X. $N(x,y,t) = N(y,x,t)$,

XI. $N(x,y,t) = 0 \; as \; t \to \infty$,

XII. $N(x,y,t) \, \Delta \, N(y,z,s) \ge N(x,z,t+s)$,

XIII. $N(x,y,.)$: $[0,\infty) \to [0,1]$ is right continuous.

Here, $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively. Moreover, [5], [6], [16] defines the area for intuitionistic fuzzy metric space.

Definition 2.4. The mapping η: $[0,\infty) \times [0,\infty) \to R$ is called a Simulation function if it satisfies the following conditions:

I. $\eta(0,0) = 0$.

II. $\eta(p,q) < q - p \; for \; all \; q,p > 0$.

III. If $\{p_n\}, \{q_n\} \subseteq (0,\infty)$ such that $\lim_{n\to\infty}\{p_n\} = \lim_{n\to\infty}\{q_n\} > 0$, then $\lim_{n\to\infty} \sup \eta(p_n, q_n) < 0$.

We denote the set of simulation functions by Z.

Definition 2.5. Let (X,d) be a metric space and T: $X \to X$ be a mapping and $\eta \in Z$. Then T is called a Z-contraction with respect to η if the following condition is satisfied:

$\eta\big(d(Tx,Ty), d(x,y)\big) \ge 0$, for all $x, y \in X$.

3. Main Results

Definition 3.1. A mapping $\eta : [1,\infty) \times [1,\infty) \to [0,1]$ is called a Fuzzy Simulation Function if the following conditions are satisfied:

a. $\eta(p,q) \ge \min\{\frac{1}{p}, \frac{1}{q}\}$, for all $1 \le p \le q$;

b. If $\{p_n\}, \{q_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty}\{p_n\} = \lim_{n\to\infty}\{q_n\} > 1$ in $[1,\infty)$, then $\lim_{n\to\infty} \sup \eta(p_n, q_n) \le 1$.

Definition 3.2. Let $(X, M, N, \Delta, \circ)$ be intuitionistic fuzzy metric space and T be a self-map on $X = [0,1]$ and $\eta \in Z$. Then T is called a Z −contraction with respect to η in $(X, M, N, \Delta, \circ)$ if the following condition is satisfied:

$\eta\big(M + N(Tx,Ty,t), M + N(x,y,t)\big) \le 1, \forall \; x, y \in X$.

Theorem 3.3. Let S be a Z-contraction with respect to η in an intuitionistic fuzzy metric space $(X, M, N, \Delta, \circ)$, then there exists a unique fixed point of S in X.

Proof: Case I: Let $z_0 \in X$ be any arbitrary point. By Picard's sequence, the sequence $\{z_n\}$ such that $z_n = Sz_{n-1}$, for all $n \in N$.

Let $A_n = \sup\{\frac{1}{M(z_i, y_j, s)} + \frac{1}{N(z_i, y_j, s)}; i, j \ge n\}$ which implies that $\{A_n\}$ is a monotonically decreasing sequence and

$\lim_{n\to\infty} \sup A_n \le 1$ and so $\{A_n\}$ is convergent, then there exists A ≥ 1 such that $\lim_{n\to\infty} A_n = A$.

Hence, $\{z_n\}$ is a Cauchy sequence in X.

Suppose that $A > 1$.

Then there exist h_p, l_p such that $h_p \ge l_p > p$, for every p ∈ N such that

$$A_p < \frac{1}{M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s)} < A_p + \frac{1}{p}.$$

$$\Rightarrow \lim_{p\to\infty}\left(\frac{1}{M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s)}\right) = A > 1.$$

As we know $\frac{1}{M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s)} \le \frac{1}{M\left(z_{h_{p-1}}, z_{l_{p-1}}, s\right) + N(z_{h_{p-1}}, z_{l_{p-1}}, s)}$.

$$\Rightarrow \lim_{p\to\infty}\left(\frac{1}{M\left(z_{h_p-1}, z_{l_p-1}, s\right) + N(z_{h_{p-1}}, z_{l_{p-1}}, s)}\right) = A > 1.$$

Therefore, $\lim_{p\to\infty}\left(\frac{1}{M\left(z_{h_p-1}, z_{l_p-1}, s\right) + N(z_{h_{p-1}}, z_{l_{p-1}}, s)} - \frac{1}{M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s)}\right) = A > 1.$

$$\Rightarrow \lim_{p\to\infty} \sup \eta\left(M\left(z_{h_p-1}, z_{l_p-1}, s\right) + N\left(z_{h_{p-1}}, z_{l_{p-1}}, s\right), M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s\right) > 1,$$

Which is a contradiction to the condition (ii) of fuzzy simulation function definition and so if the only possible is $A = 1$.

Since $(X, M, N, \Delta, \circ)$ is an intuitionistic fuzzy metric space, then there exists $z \in X$ such that $\lim_{p\to\infty} z_p = z$.

That is, $\lim_{p\to\infty} \frac{1}{M(z_p,z,s)+N(z_p,z,s)} = 1,$ since $z_p \to z$ and $M(z, z, s) + N(z, z, s) = 1$.

Consider, $Sz \neq z$, then $\frac{1}{M(Sz,z,s)+N(Sz,z,s)} > 1,$

$$\Rightarrow \lim_{p\to\infty} \sup \eta(M\left(Sz_p, Sz, s\right) + N(Sz_p, Sz, s), M\left(z_p, z, s\right) + N(z_p, z, s)) > 1.$$

Which is a contradiction to the condition (ii) of fuzzy simulation function definition.

Hence, $Sz = z$.

Therefore, z is a fixed point of S in X.

Let $r \in X$ be another fixed point of S and $r \neq z, Sr = r$.

Then $\frac{1}{M(Sz,Sr,s)+N(Sz,Sr,s)} > 1$.

$$\Rightarrow \eta(M(Sz,Sr,s)+N(Sz,Sr,s),M(z,r,s)+N(z,r,s)) > 1,$$

Which is a contradiction to the z- contraction of fuzzy simulation function.

Hence, $r = z$.

Therefore, z is the unique fixed point of S in X.

Case II:

Let $A_n = sup\left\{\frac{1}{1-(M(z_i,q_j,s)+N(z_i,q_j,s))} \,; i,j \geq n \; and \; 0 \leq \left(M(z_i,q_j,s)+N(z_i,q_j,s)\right) < 1\right\}$.

$\Rightarrow \{A_n\}$ is a monotonically increasing sequence and $\lim_{n\to\infty} \sup A_n \leq 1$ and so $\{A_n\}$ is convergent then there exists $A \geq 1$ such that $\lim_{n\to\infty} \sup A_n = A$.

Thus $\{z_n\}$ is a Cauchy sequence in X.

Consider that $A > 1$.

Then there exists h_p, l_p such that $h_p \geq l_p > p$, for every $p \in N$ such that

$$A_p - \frac{1}{p} < \frac{1}{M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s)} < A_p.$$

$$\Rightarrow \lim_{p\to\infty} \frac{1}{1-\left(M\left(z_{h_p}, z_{l_p}, s\right) + N\left(z_{h_p}, z_{l_p}, s\right)\right)} = A > 1.$$

As we know

$$\frac{1}{1-\left(M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s)\right)} \geq \frac{1}{M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s)} \geq \frac{1}{M\left(z_{h_p-1}, z_{l_p-1}, s\right) + N(z_{h_p-1}, z_{l_p-1}, s)}.$$

Hence, $\lim_{p\to\infty} \sup\left\{\frac{1}{M\left(z_{h_p-1}, z_{l_p-1}, s\right)+N(z_{h_p-1}, z_{l_p-1}, s)}, \frac{1}{M\left(z_{h_p}, z_{l_p}, s\right)+N(z_{h_p}, z_{l_p}, s)}\right\} = A > 1$.

$$\Rightarrow \lim_{p\to\infty} \sup \eta\left(M\left(z_{h_p-1}, z_{l_p-1}, s\right) + N\left(z_{h_p-1}, z_{l_p-1}, s\right), M\left(z_{h_p}, z_{l_p}, s\right) + N(z_{h_p}, z_{l_p}, s)\right) > 1,$$

Which is a contradiction to the condition (ii) of fuzzy simulation function definition and if the only possible is $A = 1$.

Since $(X, M, N, \Delta, \circ)$ is an intuitionistic fuzzy metric space, then there exists $z \in X$ such that $\lim_{n\to\infty} z_n = z$.

$\Rightarrow \lim_{n\to\infty} \frac{1}{M(z_n,z,s)+N(z_n,z,s)} = 1.$ since $z_n \to z$ and $M(z,z,s) + N(z,z,s) = 1.$

Let $Sz \neq z$, then $\frac{1}{M(Sz,z,s)} > 1.$

$$\lim_{p\to\infty} \sup \eta \left(M\left(Sz_p, Sz, s\right) + N\left(Sz_p, Sz, s\right), M\left(Sz_p, z, p\right) + N(Sz_p, z, p)\right) > 1.$$

Which is a contradiction to the condition (ii) of fuzzy simulation function definition.

Hence, $Sz = z$.

Therefore, z is a fixed point of S in X.

Consider $x \in X$ be another fixed point of S and $x \neq z, Sx = x$, then

$$\left(\frac{1}{M(Sz,Sx,s) + N(Sz,Sx,s)}\right) > 1.$$

$$\Rightarrow \eta\left(M(Sz,Sx,s) + N(Sz,Sx,s), M(z,x,s) + N(z,x,s)\right) > 1,$$

Which is a contradiction to the z-contraction of fuzzy simulation function.

Hence, $x = z$.

Therefore, there exists a unique fixed point z of T in X.

Example 3.4

(I) For Monotonically decreasing sequence,

(1) Let $X = [0, \infty)$, $\{z_n\} \to z$, where $z_n = \frac{1}{n^3}$ and $z = 0$, for all $n \in N$.

Consider $M(z_n, z, s) + N(z_n, z, s) = \frac{s}{s+z_n}$, where s = $\frac{1}{n} \in [0, 1]$.

Then $\lim_{n\to\infty} \left(\frac{1}{M(z_n,z,s)+N(z_n,z,s)}\right) = \lim_{n\to\infty} \left(\frac{1}{\left(\frac{s}{s+z_n}\right)}\right) = 1,$

Where $\lim_{n\to\infty} M(\frac{1}{n^3}, 0, \frac{1}{n}) + N(\frac{1}{n^3}, 0, \frac{1}{n}) = \lim_{n\to\infty} \left(\frac{\frac{1}{n}}{\frac{1}{n}+\frac{1}{n^3}}\right) = 1.$

(2) Let X= [0, ∞), $\{z_n\} \to z$, where $z_n = \frac{1}{n^4}$ and $z = 0$, for all $n \in N$.

Let $M(z_n, z, s) + N(z_n, z, s) = \frac{s}{s+z_n}$, where s = $\frac{1}{n} \in [0, 1]$.

Then $\lim_{n\to\infty}\left(\frac{1}{M(z_n,z,s)+N(z_n,z,s)}\right) = \lim_{n\to\infty}\left(\frac{1}{\left(\frac{s}{s+z_n}\right)}\right)$

i.e, $\lim_{n\to\infty}\left(\frac{1}{M\left(\frac{1}{n^4},0,\frac{1}{n}\right)+N\left(\frac{1}{n^4},0,\frac{1}{n}\right)}\right) = \lim_{n\to\infty}\left(\frac{1}{\frac{\frac{1}{n}}{\frac{1}{n}+\frac{1}{n^4}}}\right) = 1.$

(3) Let X = Z, $\{z_n\} \to z$, where $z_n = \frac{1}{n+z^3}$ and z = 0, for all n $\in N$ and z in X.

Let $M(z_n,z,s) + N(z_n,z,s) = \frac{s}{s+z_n}$, where $s = \frac{1}{n} \in [0,1]$

Which gives $\lim_{n\to\infty}\left(\frac{1}{M(z_n,z,s)+N(z_n,z,s)}\right) = \lim_{n\to\infty}\left(\frac{1}{\frac{s}{s+z_n}}\right) = 1.$

(II) For Monotonically increasing sequence:

(1) Let $X = [0,\infty)$, $\{z_n\} \to z$, where $z_n = 1 - \frac{1}{n^3}$ and $z = 1$, for all n in N.

Let $M(z_n,z,s) + N(z_n,z,s) = \frac{s}{s+z_n}$, where $s = \frac{1}{n} \in [0,1]$.

Then $\lim_{n\to\infty}\left(\frac{1}{1-(M(z_n,z,s)+N(z_n,z,s))}\right) = \lim_{n\to\infty}\left(\frac{1}{1-\left(\frac{s}{s+z_n}\right)}\right) = 1.$

(2) Let X = [0, ∞), $\{z_n\} \to z$, where $z_n = 1 - \frac{1}{n^3}$ and $z = 1$, for all n in N.

Let $M(z_n,z,s) + N(z_n,z,s) = \frac{s}{s+z_n}$, where $s = \frac{1}{n} \in [0,1]$.

Then $\lim_{n\to\infty}\left(\frac{1}{1-(M(z_n,z,s)+N(z_n,z,s))}\right) = \lim_{n\to\infty}\left(\frac{1}{1-\left(\frac{s}{s+z_n}\right)}\right) = 1.$

References

[1] C. Alaca, D. Turkoglu, and C.Yildiz, "Fixed Points in Intuionistic Fuzzy Metric Spaces", *Chaos Solitons and Fractals*, vol. 29, pp. 1073-1078, 2006.

[2] K. Atanassov, "Intuitionistic Fuzzy sets, Fuzzy Sets and System", vol. 20, pp. 87-96, 1986.

[3] B. Schweizer, and A. Sklar, "Statistical Metric Spaces", *Pacific. J. Math*., vol. 10, pp. 313-334, 1960.

[4] L. A. Zadeh, "Fuzzy sets", *Inform. Control*, vol. 8, pp. 338-345, 1965.

[5] C. Alaca, I. Altun, and D. Turkoglu, "On Compatible Mappings of Type (I) and (II) in Intuionistic Fuzzy Metric Spaces", *Communications of the Korean Mathematical Society*, vol. 23, pp. 427-446, 2008.

[6] A. Branciari, "A Fixed Point Theorem for Mappings Satisfying a General Contractive Condition of Integral Type", *International Journal of Mathematics and Mathematical Sciences*, vol. 29, pp. 531-536, 2002.

[7] B. Singh, S. Jain, "Weak Compatibility and Fixed Point Theorems in Fuzzy Metric Spaces", *Ganita, vol.* 56, no. 2, pp. 167-176, 2005.

[8] W. Sintunavarat, P. Kuman, "Common Fixed Point Theorems for a Pair of Weakly Compatible Mappings in Fuzzy Metric Spaces", *Journal of Applied Mathematics,* 2011, Article ID 637958, 14 pages, 2011.

[9] A. George, and P. Veeramani, "On Some Results in Fuzzy Metric Spaces", *Fuzzy Sets and Systems,* vol. 64, pp. 395-399, 1994.

[10] M. Grabiec, "Fixed Point in Fuzzy Metric Spaces, Fuzzy Sets and Systems", vol. 27, pp. 385-389, 1988.

[11] S. N. Mishra, N. Sharma, and S. L. Singh, "Common Fixed Points of Maps in Fuzzy Metric Spaces", *Internat. J. Math. Math. Sci*., vol. 17, pp. 253-258.

[12] B. Schweizer, and A. Sklar, "Statistical Metric Spaces", *Pacific. J. Math*., vol. 10, pp. 313-334, 1960.

[13] B. Singh, S. Jain, "Weak Compatibility and Fixed Point Theorems in Fuzzy Metric Spaces", *Ganita,* vol. 56, no. 2, pp. 167-176, 2005.

[14] W. Sintunavarat, P. Kuman, “Common Fixed Point Theorems for a Pair of Weakly Compatible Mappings in Fuzzy Metric Spaces”, *Journal of Applied Mathematics*, 2011, Article ID 637958, 14 pages, 2011.
[15] L. A. Zadeh**, “**Fuzzy Sets”, *Inform. Control*, vol. 8, pp. 338-345, 1965.
[16] M. S. Khan, “Fixed Point Theorem by Altering Distance Between the Points”, *Bulletin of the Australian Mathematical Society*, vol. 30, pp. 1-9, 1984.