International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V68I9P501

Volume 68 Issue 9, 1-4, September 2022

Original Article

An Improved Bound on Poisson Approximation for the Poisson Mean λ=1 with Stein-Chen Method

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Received: 18 July 2022 Revised: 19 August 2022 Accepted: 01 September 2022 Published: 10 September 2022

Abstract *- This paper uses the Stein-Chen method to obtain an improved bound on the Poisson approximation under the restriction of Poisson mean $\lambda = 1$. In addition, it indicated that the bound in this study is better than that reported in Teerapabolarn [21].*

Keywords **-** *Non-uniform bound, Poisson approximation, Poisson mean, Stein-Chen method.*

1. Introduction

Stein's method, introduced by Stein [9], is a power full method to give a bound on the normal approximation. Later, Chen [2] applied this method to give a bound on the Poisson approximation. This method is referred to as the Stein-Chen method. In the past few decades, many authors have developed the method for applying to many fields of statistics and probability theory, which can be found in [1], [3-8], [10-27] and references therein. In this section, we first start with Stein's equation for Poisson distribution with mean $\lambda = 1$, for given h,

$$h(x) - P_1(h) = f(x+1) - xf(x), \tag{1}$$

where $P_1(h) = e^{-1}\sum_{k=0}^{\infty} h(k)\frac{1}{k!}$ and f and h are bounded real valued functions defined on $\mathbb{N} \cup \{0\}$.

For $A \subseteq \mathbb{N} \cup \{0\}$, let $h_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 \text{ , if } x \in A, \\ 0 \text{ , if } x \notin A. \end{cases} \tag{2}$$

From Barbour, Holst and Janson [1], the solution f_A of (1) can be written as

$$f_A(x) = \begin{cases} (x-1)!e[P_1(h_{A\cap C_{x-1}}) - P_1(h_A)P_1(h_{C_{x-1}})] \text{ , if } x \geq 1, \\ 0 \text{ , if } x = 0, \end{cases} \tag{3}$$

where $x \in \mathbb{N}$. Similarly, for $A = \{x_0\}$ and $A = C_{x_0} = \{0,\dots,x_0\}$ as $x_0 \in \mathbb{N} \cup \{0\}$, $f_{x_0} = f_{\{x_0\}}$ and $f_{C_{x_0}}$ can be expressed as

$$f_{x_0}(x) = \begin{cases} -\frac{(x-1)!}{x_0!}P_1(h_{C_{x-1}}) \text{ , if } x \leq x_0, \\ \frac{(x-1)!}{x_0!}P_1(1-h_{C_{x-1}}) \text{, if } x > x_0, \\ 0 \text{ , if } x = 0 \end{cases} \tag{4}$$

and

$$f_{C_{x_0}}(x) = \begin{cases} (x-1)!e[P_1(h_{C_{x-1}})P_1(1-h_{C_{x_0}})] \text{ , if } x \leq x_0, \\ (x-1)!e[P_1(h_{C_{x_0}})P_1(1-h_{C_{x-1}})] \text{ , if } x > x_0, \\ 0 \text{ , if } x = 0. \end{cases} \tag{5}$$

Let $\Delta f_{x_0}(x) = f_{x_0}(x+1) - f_{x_0}(x)$ and $\Delta f_{C_{x_0}}(x) = f_{C_{x_0}}(x+1) - f_{C_{x_0}}(x)$. Following Teerapabolarn [13], we have

$$\Delta f_{C_{x_0}}(x) = \begin{cases} (x-1)!eP_1(1-h_{C_{x_0}})[xP_1(h_{C_x})-P_1(h_{C_{x-1}})] > 0 & \text{, if } x \le x_0, \\ (x-1)!eP_1(h_{C_{x_0}})[xP_1(1-h_{C_x})-P_1(1-h_{C_{x-1}})] < 0 & \text{, if } x > x_0. \end{cases} \tag{6}$$

From (6), Teerapabolarn [19] showed that

$$\left|\Delta f_{C_{x_0}}(x)\right| \le \begin{cases} e^{-1} & \text{, if } x_0 = 0, \\ \min\left\{1-e^{-1}, \frac{2(e-2)}{x_0+1}, \frac{1}{x_0}\right\} & \text{, if } x_0 > 0. \end{cases} \tag{7}$$

Later, Teerapabolarn [21] improved the bound in (7) to be sharper bound in the form of

$$\left|\Delta f_{C_{x_0}}(x)\right| \le \begin{cases} e^{-1} & \text{, if } x_0 = 0, \\ 1-2e^{-1} & \text{, if } x_0 = 1, \\ 3(1-2.5e^{-1}), & \text{if } x_0 = 2, \\ \frac{1}{x_0+1} & \text{, if } x_0 \ge 3. \end{cases} \tag{8}$$

In this paper, we also use the Stein-Chen method that mentioned above to improve the bound in (8) to be a better result.

2. Result

The theorem of this study is our main result that obtained by using the Stein-Chen method in Section I. Before giving the result, the following lemma is also need.

Lemma 1. $\Delta f_{C_{x_0}}$ *is an increasing function for* $x > x_0$.

Proof. We shall show that $\Delta f_{C_{x_0}}(x+1) - \Delta f_{C_{x_0}}(x) > 0$ for $x > x_0$. From (6), we have

$$\begin{aligned} \Delta f_{C_{x_0}}(x+1) - \Delta f_{C_{x_0}}(x) &= (x-1)!\sum_{j=0}^{x_0}\frac{e^{-1}}{j!}\left\{x\sum_{k=x+2}^{\infty}\frac{x+1-k}{k!} - \sum_{k=x+1}^{\infty}\frac{x-k}{k!}\right\} \\ &= (x-1)!\sum_{j=0}^{x_0}\frac{e^{-1}}{j!}\left\{x\sum_{k=x+2}^{\infty}\frac{x+1-k}{k!} - \sum_{k=x+1}^{\infty}\frac{[x+1-(k+1)](k+1)}{(k+1)!}\right\} \\ &= (x-1)!\sum_{j=0}^{x_0}\frac{e^{-1}}{j!}\left\{\sum_{k=x+2}^{\infty}\frac{(x-k)(x+1-k)}{k!}\right\} \\ &> 0, \end{aligned}$$

which implies that $\Delta f_{C_{x_0}}$ *is an increasing function for* $x > x_0$. □

Theorem 1. *Let* $x_0 \in \mathbb{N} \cup \{0\}$ *and* $x \in \mathbb{N}$, *we then have the following:*

$$\left|\Delta f_{C_{x_0}}(x)\right| \le \begin{cases} e^{-1} & \text{, } if\ x_0 = 0, \\ 1-2e^{-1} & \text{, } if\ x_0 = 1, \\ 3(1-2.5e^{-1}) & \text{, } if\ x_0 = 2, \\ \frac{1}{x_0+1} - \frac{2}{x_0(x_0+1)(x_0+2)}\sum_{j=0}^{x_0-1}\frac{e^{-1}}{j!}, & if\ x_0 \ge 3. \end{cases} \tag{9}$$

Proof. For $x = 0,1,2,$ the result in (9) follows from Teerapabolarn [21]. In the next step, we have to show that $\left|\Delta f_{C_{x_0}}(x)\right| \le \frac{1}{x_0+1} - \frac{2}{x_0(x_0+1)(x_0+2)}\sum_{j=0}^{x_0-1}\frac{e^{-1}}{j!}$ for $x_0 \ge 3$.

For $x > x_0$, following (6) and Lemma 1, we have

$$\begin{aligned} 0 < -\Delta f_{C_{x_0}}(x) &\le -\Delta f_{C_{x_0}}(x_0+1) \\ &= \Delta f_{x_0+1}(x_0+1) - \Delta f_{C_{x_0+1}}(x_0+1) \end{aligned}$$

$$= \sum_{k=x_0+2}^{\infty} \frac{e^{-1}}{k!} + \frac{1}{x_0+1} \sum_{j=0}^{x_0} \frac{e^{-1}}{j!} - x_0! \sum_{k=x_0+2}^{\infty} \frac{e^{-1}}{k!} \sum_{j=0}^{x_0} \frac{x_0+1-j}{j!} \quad \text{(by (4) and (6))}$$

$$\le \sum_{k=x_0+2}^{\infty} \frac{e^{-1}}{k!} + \frac{1}{x_0+1} \sum_{j=0}^{x_0} \frac{e^{-1}}{j!} - \sum_{k=x_0+2}^{\infty} \frac{x_0!}{k!} \sum_{j=0}^{x_0} \frac{e^{-1}}{j!}$$

$$= \sum_{k=x_0+2}^{\infty} \frac{e^{-1}}{k!} + \frac{1}{x_0+1} \sum_{j=0}^{x_0} \frac{e^{-1}}{j!} - \left\{ \frac{1}{(x_0+1)(x_0+2)} + \frac{1}{(x_0+1)(x_0+2)(x_0+3)} + \cdots \right\} \sum_{j=0}^{x_0} \frac{e^{-1}}{j!}$$

$$\le \sum_{k=x_0+2}^{\infty} \frac{e^{-1}}{k!} + \frac{1}{x_0+1} \sum_{j=0}^{x_0} \frac{e^{-1}}{j!} - \frac{1}{(x_0+1)(x_0+2)} \sum_{j=0}^{x_0} \frac{e^{-1}}{j!}$$

$$\le \frac{1}{x_0+2} \sum_{k=x_0+1}^{\infty} \frac{e^{-1}}{k!} + \left\{ \frac{1}{x_0+1} - \frac{1}{(x_0+1)(x_0+2)} \right\} \sum_{j=0}^{x_0} \frac{e^{-1}}{j!}$$

$$= \frac{1}{x_0+2},$$

which yields

$$\left| \Delta f_{C_{x_0}}(x) \right| \le \frac{1}{x_0+2}. \tag{10}$$

For $x \le x_0$, following (6) and Teerapabolarn [21], we also have

$$0 < \Delta f_{C_{x_0}}(x) \le \Delta f_{C_{x_0}}(x_0)$$

$$= \Delta f_{x_0}(x_0) + \Delta f_{C_{x_0-1}}(x_0)$$

$$= \sum_{k=x_0+1}^{\infty} \frac{e^{-1}}{k!} + \frac{1}{x_0} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!} + (x_0-1)! \sum_{k=x_0+1}^{\infty} \frac{x_0-k}{k!} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!} \quad \text{(by (4) and (6))}$$

$$= \sum_{k=x_0+1}^{\infty} \frac{e^{-1}}{k!} + \frac{1}{x_0} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!} - \sum_{k=x_0+1}^{\infty} \frac{(x_0-1)!(k-x_0)}{k!} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!}$$

$$= \sum_{k=x_0+1}^{\infty} \frac{e^{-1}}{k!} + \frac{1}{x_0} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!} - \left\{ \frac{1}{x_0(x_0+1)} + \frac{2}{x_0(x_0+1)(x_0+2)} + \cdots \right\} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!}$$

$$\le \frac{1}{x_0+1} \sum_{k=x_0}^{\infty} \frac{e^{-1}}{k!} + \frac{1}{x_0} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!} - \left\{ \frac{1}{x_0(x_0+1)} + \frac{2}{x_0(x_0+1)(x_0+2)} + \cdots \right\} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!}$$

$$= \frac{1}{x_0+1} + \left\{ \frac{1}{x_0} - \frac{1}{x_0+1} \right\} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!} - \left\{ \frac{1}{x_0(x_0+1)} + \frac{2}{x_0(x_0+1)(x_0+2)} + \cdots \right\} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!}$$

$$= \frac{1}{x_0+1} - \frac{2}{x_0(x_0+1)(x_0+2)} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!},$$

which gives

$$\left| \Delta f_{C_{x_0}}(x) \right| \le \frac{1}{x_0+1} - \frac{2}{x_0(x_0+1)(x_0+2)} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!}. \tag{11}$$

Hence, from (10) and (11) and $\max\left\{ \frac{1}{x_0+2}, \frac{1}{x_0+1} - \frac{2}{x_0(x_0+1)(x_0+2)} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!} \right\} = \frac{1}{x_0+1} - \frac{2}{x_0(x_0+1)(x_0+2)} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!}$, the result in (9) is obtained.

□

Because $\frac{1}{x_0+1} - \frac{2}{x_0(x_0+1)(x_0+2)} \sum_{j=0}^{x_0-1} \frac{e^{-1}}{j!} < \frac{1}{x_0+1}$, the result in this paper is better than that reported in (8).

3. Conclusion

In this study, an improvement of the bound on Poisson approximation is obtained by using the Stein-Chen method. Additionally, by comparing, the bound of this study is better than that presented in Teerapabolarn [21].

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