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Original Article

An Improved Poisson Approximation to Binomial Distribution

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Abstract *- We determine a new improved Poisson distribution with mean $\lambda = \frac{np}{q}$ from the binomial distribution with parameters n and p. In view of three approximations to the binomial distribution, the improved Poisson approximation of this study is more accurate than both the improved Poisson approximation [26] and simple Poisson approximation.*

Keywords *- Binomial distribution, Improved Poisson approximation, Poisson distribution.*

1. Introduction

Let X be a non-negative integer-valued random variable that has a binomial distribution with parameters n and p. Its probability function can be expressed as

$$b_{n,p}(x) = \binom{n}{x} p^x q^{n-x}, x = 0,1,\dots,n, \tag{1}$$

where $q = 1 - p$and $n \in \mathbb{N}$ and $p \in (0,1)$. Some applications of binomial distribution can be found in [1], [7-12] and [27].

In the well-known Poisson limit theorem states that, if $n \to \infty, p \to 0$ and $\lambda = np$ remains a constant, then $b_{n,p}(x) \to p_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$for every $x = 0,1,\dots,n$. This theorem indicates that the Poisson distribution with mean $\lambda = np$ may be used as an simple approximation to the binomial distribution with parameters $n$and p whenever n is large and p is small. However, this approximation can be improved to be more accurate result. In 2014, Teerapabolarn and Jaioun [26] gave an improved Poisson distribution with mean $\lambda = np$by deriving from the binomial distribution in (1) as the form

$$b_{n,p}(x) \approx \frac{p_\lambda(x)}{e^{\frac{\lambda^2}{2n}} q^x \left\{1 + \frac{x(x-1)}{2n}\right\}}, \tag{2}$$

where $p_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$. He also showed, by numerical results, that the improved Poisson approximation is more accurate than the simple Poisson approximation. The similar contexts of this approximation can be found in [2-6], [13-23] and [25].

In addition, if $n \to \infty, p \to 0$ and $\lambda = \frac{np}{q}$ remains a constant, then $b_{n,p}(x) \to p_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$for every $x = 0,1,\dots,n$ as well. This is, the Poisson with mean $\lambda = \frac{np}{q}$ may also be used as an approximation to the binomial approximation, see [6] and [24]. With this mean, the result of an improved Poisson approximation to the binomial distribution in [6] is of the form

$$b_{n,p}(x) \approx \frac{p_\lambda(x) e^{\frac{\lambda^2}{2n}}}{\left\{1 + \frac{x(x-1)}{2n}\right\}}, \tag{3}$$

and the approximation is also more accurate than the simple Poisson approximation.

In this paper, we focus on determining a new improved Poisson distribution with mean $\lambda = \frac{np}{q}$ by improving the result in (3). Similarly, the accuracy of the approximation is measured in the form of $|b_{n,p}(x) - \widehat{p_\lambda}(x)|$for $x = 0,1,\dots,n$, as in [6] and [26], where $\widehat{p_\lambda}(x)$ is the improved Poisson probability function, which is in Section 2. In Section 3, some numerical examples are given to illustrate the improved approximation together with some comparisons of the approximation and the conclusion of this study is presented in the last section.

2. Result

We use the same method in [2] to determine a new improved Poisson probability function by following the result in (1). The following lemma is directly obtained from [6] and [26].

Lemma 2.1. *For* $x \in \mathbb{N}$, *we have the following:*

$$\prod_{i=0}^{x-1}\left(1-\frac{i}{n}\right)=\frac{1}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)}. \quad (4)$$

Theorem 2.1. *For* $x \in \{0,1,\dots,n\}$ *and* $\lambda = \frac{np}{q}$, *we then have the following:*

$$b_{n,p}(x) \approx \widehat{p_\lambda}(x), \quad (5)$$

where $\widehat{p_\lambda}(x) = \frac{p_\lambda(x)(e^\lambda q^n)}{1+\frac{x(x-1)}{2n}}$.

Proof. From (1), for $x = 0$, we have $b_{n,p}(0) = q^n = e^{-\lambda}e^\lambda q^n = p_\lambda(x)(e^\lambda q^n)$. For $x = 1,\dots,n$, we obtain

$$b_{n,p}(x) = \frac{n!}{x!(n-x)!}\left(\frac{np}{q}\right)^x \frac{q^n}{n^x}$$

$$= \frac{\lambda^x\left[\frac{n}{n}\cdots\left(1-\frac{x-1}{n}\right)\right]q^n}{x!}$$

$$= \frac{e^{-\lambda}\lambda^x}{x!}\prod_{i=0}^{x-1}\left(1-\frac{i}{n}\right)(e^\lambda q^n)$$

$$= \frac{p_\lambda(x)(e^\lambda q^n)}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)}. \text{ (by Lemma 2.1)}$$

For large n, we have that

$$b_{n,p}(x) \approx \frac{p_\lambda(x)(e^\lambda q^n)}{1+\frac{x(x-1)}{2n}},$$

which gives the result in (5). □

It is noted that the result in (5) is different from those appeared in (2) and (3) with both different Poisson means.

3. Numerical Examples

The following numerical examples are given to illustrate how well the new improved Poisson distribution with mean $\boldsymbol{\lambda} = \frac{\boldsymbol{np}}{\boldsymbol{q}}$ approximates the binomial distribution with parameters $\boldsymbol{n}$ and $\boldsymbol{p}$. We also compare the approximation in (5) with the approximation in (3) and the simple Poisson approximation with the same Poisson mean.

Example 3.1. Let $n = 80, p = 0.01$ and $\lambda = \frac{np}{q} = 0.80808081$, then the numerical results are as follows

x	$b_{n,p}(x)$	$p_\lambda(x)$	$\widehat{p_\lambda}(x)(3)$	$\widehat{p_\lambda}(x)(5)$	$\lvert b_{n,p}(x) - p_\lambda(x)\rvert$	$\lvert b_{n,p}(x) - \widehat{p_\lambda}(x)(3)\rvert$	$\lvert b_{n,p}(x) - \widehat{p_\lambda}(x)(5)\rvert$
0	0.44752321	0.44571265	0.44753542	0.44752321	0.00181056	0.00001221	0.00000000
1	0.36163492	0.36017184	0.36164478	0.36163492	0.00146308	0.00000986	0.00000000
2	0.14428868	0.14552398	0.14431517	0.14431123	0.00123530	0.00002648	0.00002255
3	0.03789400	0.03919838	0.03793608	0.03793504	0.00130438	0.00004208	0.00004105
4	0.00736828	0.00791886	0.00739651	0.00739631	0.00055059	0.00002823	0.00002803
5	0.00113129	0.00127982	0.00114227	0.00114224	0.00014853	0.00001098	0.00001094
6	0.00014284	0.00017237	0.00014574	0.00014574	0.00002953	0.00000290	0.00000290
7	0.00001525	0.00001990	0.00001583	0.00001582	0.00000465	0.00000057	0.00000057
8	0.00000141	0.00000201	0.00000149	0.00000149	0.00000060	0.00000009	0.00000009
9	0.00000011	0.00000018	0.00000012	0.00000012	0.00000007	0.00000001	0.00000001

Example 3.2. Let $n = 300, p = 0.01$ and $\lambda = \frac{np}{q} = 3.03030303$, then the numerical results are as follows:

x	$b_{n,p}(x)$	$p_\lambda(x)$	$\widehat{p_\lambda}(x)(3)$	$\widehat{p_\lambda}(x)(5)$	$\lvert b_{n,p}(x) - p_\lambda(x)\rvert$	$\lvert b_{n,p}(x) - \widehat{p_\lambda}(x)(3)\rvert$	$\lvert b_{n,p}(x) - \widehat{p_\lambda}(x)(5)\rvert$
0	0.04904089	0.04830100	0.04904591	0.04904089	0.00073989	0.00000502	0.00000000
1	0.14860877	0.14636666	0.14862397	0.14860877	0.00224211	0.00001520	0.00000000
2	0.22441425	0.22176767	0.22443970	0.22441675	0.00264658	0.00002545	0.00000249
3	0.22516986	0.22400775	0.22521036	0.22518732	0.00116211	0.00004050	0.00001746
4	0.16887739	0.16970284	0.16894122	0.16892394	0.00082545	0.00006383	0.00004655
5	0.10098527	0.10285021	0.10106748	0.10105714	0.00186494	0.00008221	0.00007187
6	0.05015262	0.05194455	0.05023395	0.05022882	0.00179193	0.00008134	0.00007620
7	0.02127687	0.02248682	0.02133983	0.02133765	0.00120995	0.00006296	0.00006078
8	0.00787137	0.00851773	0.00791076	0.00790995	0.00064637	0.00003939	0.00003858
9	0.00257962	0.00286792	0.00260014	0.00259987	0.00028831	0.00002052	0.00002025
10	0.00075825	0.00086907	0.00076737	0.00076729	0.00011082	0.00000911	0.00000904
11	0.00020192	0.00023941	0.00020544	0.00020542	0.00003749	0.00000352	0.00000350
12	0.00004912	0.00006046	0.00005032	0.00005031	0.00001134	0.00000120	0.00000119
13	0.00001099	0.00001409	0.00001136	0.00001136	0.00000310	0.00000037	0.00000036
14	0.00000228	0.00000305	0.00000238	0.00000238	0.00000077	0.00000010	0.00000010
15	0.00000044	0.00000062	0.00000046	0.00000046	0.00000018	0.00000003	0.00000003
16	0.00000008	0.00000012	0.00000008	0.00000008	0.00000004	0.00000001	0.00000001

Example 3.3. Let $n = 1000, p = 0.005$ and $\lambda = \frac{np}{q} = 5.02512563$, then the numerical results are as follows:

x	$b_{n,p}(x)$	$p_\lambda(x)$	$\widehat{p_\lambda}(x)(3)$	$\widehat{p_\lambda}(x)(5)$	$\lvert b_{n,p}(x) - p_\lambda(x)\rvert$	$\lvert b_{n,p}(x) - \widehat{p_\lambda}(x)(3)\rvert$	$\lvert b_{n,p}(x) - \widehat{p_\lambda}(x)(5)\rvert$
0	0.00665397	0.00657076	0.00665425	0.00665397	0.00008321	0.00000028	0.00000000
1	0.03343703	0.03301890	0.03343844	0.03343703	0.00041813	0.00000141	0.00000000
2	0.08392862	0.08296206	0.08393224	0.08392870	0.00096656	0.00000362	0.00000008
3	0.14030279	0.13896492	0.14030968	0.14030377	0.00133786	0.00000689	0.00000098
4	0.17573100	0.17457905	0.17574279	0.17573539	0.00115195	0.00001179	0.00000438
5	0.17590762	0.17545633	0.17592641	0.17591900	0.00045129	0.00001880	0.00001138
6	0.14658968	0.14694835	0.14661623	0.14661005	0.00035867	0.00002655	0.00002037
7	0.10460168	0.10549056	0.10463362	0.10462921	0.00088888	0.00003193	0.00002753
8	0.06524464	0.06626291	0.06527709	0.06527434	0.00101827	0.00003245	0.00002970
9	0.03613774	0.03699772	0.03616584	0.03616432	0.00085998	0.00002811	0.00002658
10	0.01799623	0.01859182	0.01801727	0.01801651	0.00059559	0.00002104	0.00002028
11	0.00813900	0.00849329	0.00815280	0.00815246	0.00035429	0.00001381	0.00001346
12	0.00337080	0.00355666	0.00337884	0.00337870	0.00018586	0.00000804	0.00000790
13	0.00128734	0.00137482	0.00129155	0.00129149	0.00008748	0.00000421	0.00000415
14	0.00045607	0.00049347	0.00045806	0.00045804	0.00003741	0.00000199	0.00000197
15	0.00015065	0.00016532	0.00015151	0.00015150	0.00001467	0.00000086	0.00000086
16	0.00004660	0.00005192	0.00004695	0.00004695	0.00000532	0.00000034	0.00000034
17	0.00001356	0.00001535	0.00001368	0.00001368	0.00000179	0.00000013	0.00000013
18	0.00000372	0.00000428	0.00000376	0.00000376	0.00000056	0.00000004	0.00000004
19	0.00000097	0.00000113	0.00000098	0.00000098	0.00000017	0.00000001	0.00000001

The numerical results in Examples 3.1-3.3 are indicated that the theorem gives a good approximation when p is sufficiently small. Additionally, by comparing some numerical results of three approximations, it can be seen that the new improved Poisson approximation is more accurate than both the improved Poisson approximation in (3) and simple Poisson approximation.

Because $e^{-\lambda} < q^n < e^{-q\lambda}$ [23], we have $q^n \approx \frac{e^{-\lambda}+e^{-q\lambda}}{2}$. Thus, we can also approximate $b_{n,p}(x)$ by $\frac{p_\lambda(x)(1+e^{p\lambda})}{2+\frac{x(x-1)}{n}}$. From which, the improved Poisson approximation is also more accurate than the simple Poisson approximation.

4. Conclusion

In this study, the new improved Poisson distribution with mean $\lambda = \frac{np}{q}$ is obtained by deriving the binomial distribution with parameters $n$and p. The result gives a good approximation of binomial distribution whenever p is sufficiently small. With this Poisson mean and by comparing numerical results of three approximations, the new improved Poisson approximation is more accurate than both the improved Poisson approximation in [26] and simple Poisson approximation. Additionally, the result of this study gave an alternative improved Poisson distribution for approximating the binomial distribution.

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