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Original Article

The Role of Boolean and Pseudo Commutativity in Near Rings

C. Dhivya[1], D. Radha[2]

[1,2]PG & Research Department of Mathematics, A.P.C.Mahalaxmi College for Women, Thoothukudi, Tamilnadu, India.
[1,2]Affiliated to Manonmaniam Sundaranar University, Abishekapatti Tirunelveli, Tamilnadu, India.

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Abstract - *In this paper we have discussed about Boolean near rings and pseudo commutative in near rings. Hansen, D.J. and Jiang Luh [6] have stated that if N is a Boolean near ring, then $xy = xyx$ for each $x, y \in N$. It has been proved in this paper by using the lemma if N is a Boolean near ring, then $ab = 0$ implies $ba = b0$ for all $a, b \in N$. We have also generalized this lemma to $m \geq 1$ and $n \geq 1$. It is obtained that every Boolean near ring with commutativity is pseudo commutative.*

Keywords - *Boolean, Commutative, Pseudo Commutative, Weak Commutative, Zero symmetric.*

1. Introduction

Near rings can be thought of as generalized rings: if in a ring we ignore the commutativity of addition and one distributive law, we get a near ring. Gunter Pilz [5] "Near Rings" is an extensive collection of the work done in the area of near rings.

Throughout this paper N stands for a right near ring $(N, +, .)$, with at least two elements and '0' denotes the identity element of the group $(N, +)$ and we write xy for $x.y$ for any two elements x, y of N. Obviously $0n = 0$ for all $n \in N$. If, in addition, $n0 = 0$ for all $n \in N$ then we say that N is zero symmetric.

2. Preliminaries

Definition 2.1 [6]

A right near ring is a non-empty set N together with two binary operations "+" and "." such that (i) $(N, +)$ is a group (ii) $(N, .)$ is a semigroup (iii)$(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1, n_2, n_3 \in N$.

Definition 2.2 [6]

A near ring N is called weak commutative if $xyz = xzy$ for every $x, y, z \in N$.

Definition 2.3 [23]

A near ring N is said to be pseudo commutative near ring if $xyz = zyx$ for all $x, y, z \in N$.

Definition 2.4 [4]

A near ring N is said to be quasi weak commutative near ring if $xyz = yxz$ for all $x, y, z \in N$.

Definition 2.5 [4]

A near ring N is said to be reduced if N has no non-zero nilpotent elements.

Definition 2.6 [6]

A near ring N is called Boolean if $x^2 = x$ for all $x \in N$.

Lemma 2.7 [7]

If N is a Boolean near ring, then $xy = xyx$ for each $x, y \in N$.

Definition 2.8 [25]

A near ring N is called reversible if for any $a, b \in N$, $ab = 0 \Longrightarrow ba = 0$.

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3. Main Results

Lemma 3.1

If N is a Boolean near ring, then $ab = 0$ implies $ba = b0$ for all $a, b \in N$.

Proof:

Let N be a Boolean near ring. Let $a, b \in N$ with $ab = 0$.
Now $ba = (ba)^2 = (ba)(ba) = b(ab)a = b0a = b(0a) = b0$. That is $ba = b0$ for all $a, b \in N$.

Corollary 3.2

If N is a Boolean near ring with zero symmetric, then N is reversible.

Lemma 3.3

Let N be a Boolean near ring. If for all $a, b \in N$ with $ab = 0$, then $(ba)^n = b0$ for all $n \geq 1$.

Proof:

Let N be a Boolean near ring. Let $a, b \in N$ with $ab = 0$. The statement is true for $n = 1$ by lemma 3.1. Assume that the statement is true for $n = k$, so $(ba)^k = b0$.

Then $(ba)^{k+1} = (ba)^k(ba) = (b0)(ba) = b(0b)a = b0a = b0$. That is $(ba)^{k+1} = b0$. Hence $(ba)^n = b0$ for all $a, b \in N$ and for all $n \geq 1$.

Lemma 3.4

Let N be a Boolean near ring. If for all $a \in N$ with $a^2 = a0$, then $a = a0$.

Proof:

Let N be a Boolean near ring. Let $a \in N$ with $a^2 = a0$. By definition of Boolean, $a = a0$ for all $a \in N$.

Lemma 3.5

Let N be a Boolean near ring. If for all $a \in N$ with $a^n = a0$ for all $n \geq 2$, then $a = a0$.

Proof:

Let N be a Boolean near ring. The statement is true for $n = 2$ by lemma 3.4. Assume that the statement is true for $n = k$, so $a^k = a0$. Then $a^{k+1} = a^k a = (a0)a = a(0a) = a0$.
That is $a^{k+1} = a0$. Hence $a^k = a0$ for all $a \in N$ and for all $n \geq 2$.

Corollary 3.6

If N is zero symmetric with Boolean, then N is reduced.

Combining the lemma 3.3 and lemma 3.5 we have Proposition 3.7

Proposition 3.7

Let N be a zero symmetric Boolean near ring. Then N is reversible and reduced.

The lemma 3.8 taken from [7] plays an important role in the forthcoming theorem, proposition and lemma. The proof is given using the lemma 3.1.

Lemma 3.8

If N is a Boolean near ring, then $xyx = xy$ for all $x, y \in N$.

Proof:

Let N be a Boolean near ring. Let $x, y \in N$.

$$\begin{aligned}\text{Now}(xyx - xy)xy &= xyx^2y - xyxy \\ &= xyxy - xyxy \\ &= (xy)^2 - (xy)^2 \\ &= 0.\end{aligned}$$

That is $(xyx - xy)xy = 0$.

By Lemma 3.1, $xy(xyx - xy) = xy0$ …… (1)

Also $(xyx - xy)xyx = xyx^2yx - xyxyx = xyxyx - xyxyx = 0$.
That is $(xyx - xy)xyx = 0$.

By lemma 3.1, $xyx(xyx - xy) = xyx0$ …… (2)

$$\begin{aligned}\text{Now } xyx - xy &= (xyx - xy)^2 = (xyx - xy)(xyx - xy) \\ &= xyx(xyx - xy) - xy(xyx - xy) \\ &= xyx0 - xy0 \text{ (By equation (1) and (2))}\end{aligned}$$

$= (xyx - xy)0.$

That is $xyx - xy = (xyx - xy)0$ …… (3)

Now $(xyx - xy)x = 0.$
$\Rightarrow (xyx - xy)0x = 0.$ (By equation (3))
$\Rightarrow (xyx - xy)0 = 0.$
$\Rightarrow xyx - xy = 0.$ (By equation (3))
Hence $xyx = xy$ for all $x, y \in N$.

Lemma 3.9

If N is a Boolean near ring, then $x^m y^n x^m = x^m y^n$ for all $x, y \in N$ where $m \geq 1, n \geq 1$ are fixed integers.

Proof:

Let N be a Boolean near ring. Let $x, y \in N$.

Now $(x^m y^n x^m - x^m y^n) x^m y^n = x^m y^n x^m x^m y^n - x^m y^n x^m y^n$
$= x^m y^n (x^m)^2 y^n - x^m y^n x^m y^n$
$= x^m y^n x^m y^n - x^m y^n x^m y^n$
$= 0.$

That is $(x^m y^n x^m - x^m y^n) x^m y^n = 0$ ……… (1)

By Lemma 3.1, $x^m y^n (x^m y^n x^m - x^m y^n) = x^m y^n 0$ ..…… (2)

Also

$(x^m y^n x^m - x^m y^n) x^m y^n x^m = x^m y^n x^m x^m y^n x^m - x^m y^n x^m y^n x^m$
$= x^m y^n (x^m)^2 y^n x^m - x^m y^n x^m y^n x^m$
$= x^m y^n x^m y^n x^m - x^m y^n x^m y^n x^m$
$= 0.$

That is $(x^m y^n x^m - x^m y^n) x^m y^n x^m = 0.$

Again, by Lemma 3.1,

$x^m y^n x^m (x^m y^n x^m - x^m y^n) = x^m y^n x^m 0$ ……… (3)

Now $x^m y^n x^m - x^m y^n = (x^m y^n x^m - x^m y^n)^2$
$= (x^m y^n x^m - x^m y^n)(x^m y^n x^m - x^m y^n)$
$= x^m y^n x^m (x^m y^n x^m - x^m y^n) - x^m y^n (x^m y^n x^m - x^m y^n)$
$= x^m y^n x^m 0 - x^m y^n 0.$

That is $x^m y^n x^m - x^m y^n = (x^m y^n x^m - x^m y^n)0$ ……… (4)

Now $(x^m y^n x^m - x^m y^n) x^m = x^m y^n x^m x^m - x^m y^n x^m$
$= x^m y^n (x^m)^2 - x^m y^n x^m$
$= x^m y^n x^m - x^m y^n x^m$
$= 0.$

That is $(x^m y^n x^m - x^m y^n) x^m = 0$
$\Rightarrow (x^m y^n x^m - x^m y^n) 0 x^m = 0$ (By equation (4))
$\Rightarrow (x^m y^n x^m - x^m y^n) 0 = 0$
$\Rightarrow x^m y^n x^m - x^m y^n = 0$ (By equation (4))
$\Rightarrow x^m y^n x^m = x^m y^n$ for all $x, y \in N$.

Theorem 3.10

Let N be a Boolean near ring. If N is commutative then N is pseudo commutative.

Proof:

Let N be a Boolean near ring. Let $x, y, z \in N$.

Now, $y(z - xz)x = y(zx - xzx)$
$= y(zx - xz)$ (By lemma 3.8)
$= y(zx - zx)$
$= y0.$

That is $y(z - xz)x = y0.$
$\Rightarrow (z - xz)y(z - xz)x = (z - xz)y0$
$\Rightarrow (z - xz)yx = zy0 - xzy0$ (By lemma 3.8)
$\Rightarrow zyx - xzyx = zy0 - xzy0$

$\Rightarrow zyx - xz(yx) = zy0 - xzy0$
$\Rightarrow zyx - xz(xy) = zy0 - xzy0$ (Since N is commutative)
$\Rightarrow zyx - (xzx)y = zy0 - xzy0$
$\Rightarrow zyx - (xz)y = zy0 - xzy0$ (By lemma 3.8)
$\Rightarrow zyx - x(zy) = zy0 - xzy0$

$\Rightarrow zyx - xyz = zy0 - xzy0$ ………….. (1)

Also, $x(z - xz)x = x(zx - xzx)$
$= x(zx - xz)$ (By lemma 3.8)
$= x(zx - zx)$
$= x0$
That is $x(z - xz)x = x0$.
$\Rightarrow x(z - xz) = x0$ (By lemma 3.8)
$\Rightarrow yx(z - xz) = yx0$ …..…….. (2)
Now$yx(z - xz)yx = yx(z - xz)$ (By Lemma 3.8)= $yx0$ (By equation (2))
That is $yx(z - xz)yx = yx0$
Now $(z - xz)yx(z - xz)yx = (z - xz)yx0$
$\Rightarrow [(z - xz)yx]^2 = (z - xz)yx0$
$\Rightarrow (z - xz)yx = (z - xz)yx0$ (Since N is Boolean)
$\Rightarrow zyx - xzyx = zyx0 - xzyx0$
$\Rightarrow zyx - xz(yx) = zyx0 - xz(yx)0$
$\Rightarrow zyx - xz(xy) = zyx0 - xz(xy)0$ (Since N is commutative)
$\Rightarrow zyx - (xzx)y = zyx0 - (xzx)y0$
$\Rightarrow zyx - xzy = zyx0 - xzy0$ (By Lemma 3.8)
$\Rightarrow zyx - x(zy) = zyx0 - xzy0$
$\Rightarrow zyx - xyz = zyx0 - xzy0$ (Since N is commutative) ……… (3)
By equation (1) and (3) we get
$\Rightarrow zy0 = zyx0$ for all $y, z \in N$ ………... (4)
Replacing x by y and y by z in equation (4), we get
$\Rightarrow zz0 = zzy0$
$\Rightarrow z^2 0 = z^2 y0$
$\Rightarrow z0 = zy0$ (Since N is Boolean) for all $z, y \in N$ …………. (5)
From equation (1) we get
$zyx - xyz = zy0 - xzy0$
$= z0 - xz0$ (By equation (5))
$= z0 - (xz)0$
$= z0 - (zx)0$
$= z0 - z0$ (By equation (5))
$= 0$
That is $zyx - xyz = 0$
$\Rightarrow zyx = xyz$
Hence N is pseudo commutative.

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