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Original Article

New Ring and Vector Space Structure of Compatible Systems of First Order Partial Differential Equations

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Abstract *- Ring theory plays a vital role in mathematics, physics, chemistry, and computer science. Ring theory has applications in geometry, symmetry and transformation puzzles like Rubik's Cube. Also the vector space and partial differential equations has many applications in mathematics, engineering etc. Partial differential equations are used in problems involving functions of several variables, such as heat or sound, elasticity, electrodynamics, fluid flow, etc. In this article we have established relation between first order partial differential equations and ring theory, vector space. If $g(x,y,z,p,q)$ is the given first order partial differential equation, the set of all partial differential equations $f(x,y,z,p,q)$ which are compatible with $g(x,y,z,p,q)$ form ring structure under usual addition and multiplication of two functions. Furthermore this ring is commutative. Also if we use usual vector addition of functions and scalar multiplication then this newly formed set is a vector space.*

Keywords *- Ring, Commutative, Vector Space, Compatible, Partial Differential.*

1. Introduction

The problem of characterizing rings of commuting ordinary differential operators (ODO) was introduced and investigated by Burchnal and Chaundy. On the structure of compatible rational functions introduced by Shaoshi Chen, Ruyong Feng, and Ziming Li. (Shaoshi Chen, 2011). Phoolan Prasad defined first order partial differential equations: a simple approach for beginners. This is infinite dimension again, but relating sturm-liouville to symmetric matrices, and solving $Ax = b$ by eigenvector expansions is fun. This kind of problem comes up in Electrodynamics (Electrical Engineering), fluids (mechanical/civil/chemical engr.), and quantum mechanics (electrical/materials/chemical engineering). Etc.

In This article first we defined collection of all partial differential equations $f(x,y,z,p,q)$ which are compatible with $g(x,y,z,p,q)$. In next part, by defining trivial operation of function we proved that it form ring structure and vector space.

2. Basic Definitions

2.1. Group structure

A non-empty set G with operation $*$ is said to be *group* if it satisfies following four conditions:

a) Closure property hold with respect to $*$ i.e. $x * y$ is in G ,for every $x , y \in$ G
b) Associativity property hold with respect to $*$ i.e.$(x * y) * z = x * (y * z)$ $for\ every\ x , y , z \in$ G
c) Identity element exits in G i.e. there is e in G such that $x * e = e * x = x$ for all x ∈ G
d) Inverse element exits in G i.e. there is x' in G such that $x * x' = x' * x = e$ for all x ∈ G.

2.2. Abelian Group

Group G is Abelian group if $x * y = y * x$, for every $x, y \in$ G.

2.3. Ring Structure

A non-empty set R with operations $" + "$ and $"."$ is said to be *ring* if it satisfies following three conditions:

R_1) R is an Abelian group

R_2) Multiplication is associative i.e.$(x. y). z = x. (y. z)$, $\forall\ x , y , z \in$R

R_3) Left and right distributive laws holds.

i.e. $x.(y+z) = x.y + x.z \; and \; (x+y).z = x.z + y.z$, $\forall\, x, y, z \in$ R.

We say that $(R, +, .)$ is a ring.

2.4. Commutative Ring

We say that, ring $(R, +, .)$ is commutative ring, if multiplication is commutative.

2.5. Vector Space

A non-empty set together with two operations vector addition and scalar multiplication is a vector space if it satisfies the following properties.

i. For any $u, v \in V, u + v \in V$ (Closure under addition)
ii. For $v \in V$ and scalar α, $\alpha v \in V$ (Closure under scalar Multiplication)
iii. For any $u, v \in V, u + v = v + u$ (Commutative Property)
iv. For any $u, v, w \in V, u + (v + w) = (u + v) = w$ (Associativity Property)
v. There is zero vector 0 in V such that $u + 0 = 0 + u, for\ all\ u \in V$ (Existence of Identity Element in V)
vi. For any $v \in V, \exists\, u \in V$ such that $u + v = 0$ (Existence of additive Inverse)
vii. The scalar 1 satisfies $1.v = v, \forall v \in V$ (Multiplicative Identity)
viii. For $v \in V$ and scalars α, β , $(\alpha\beta)v = \alpha(\beta v)$ (Associativity of Multiplication)
ix. For $u, v \in V$ and scalar α, $\alpha(u + v) = \alpha u + \alpha v$ (Distributivity over vector addition)
x. For $v \in V$ and scalars α, β , $(\alpha + \beta)v = \alpha v + \beta v$ (Distributive over scalar addition)

2.6. Compatible

Consider the partial differential equation $f(x, y, z, p, q) = 0$, where $z = z(x, y)$ and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$.

The partial differential equations $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ are said to be *compatible* if they have a common solution.

The necessary and sufficient condition that the two partial differential equation $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ are compatible if $[f, g] = 0$.

Where, $[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$

$$= \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} + p\begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} + q\begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$

$$= (g_p f_x - g_x f_p + \text{p}\, g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z f_q)$$

2.7. New Ring Structure of Compatible System
Result 1:

Consider the set $R = \{ f(x, y, z, p, q) = 0 : [f, g] = 0\}$ where $g \; is \; g(x, y, z, p, q) = 0$ }

i.e. the set of all P.D.E.'s f which are compatible with g. Then the set R is a ring with respect to trivial addition of functions and multiplication.

Proof:

Let $f = f(x, y, z, p, q)$, $h = h(x, y, z, p, q) \in R$.

Therefore, $[f, g] = 0$ and $[h, g] = 0$

i.e. $g_p f_x - g_x f_p + \text{p}\; g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z f_q = 0$ and $g_p h_x - g_x h_p + \text{p}\, g_p h_z - p g_z h_p + g_q h_y - g_y h_q + q g_q h_z - q g_z h_q = 0$ ------- (1.1)

Consider,

$[f + h, g] = \frac{\partial(f+h,g)}{\partial(x,p)} + p\,\frac{\partial(f+h,g)}{\partial(z,p)} + \frac{\partial(f+h,g)}{\partial(y,q)} + q\,\frac{\partial(f+h,g)}{\partial(z,q)}$

$= \begin{vmatrix} f_x + h_x & f_p + h_p \\ g_x & g_p \end{vmatrix} + p \begin{vmatrix} f_z + h_z & f_p + h_p \\ g_z & g_p \end{vmatrix} +$

$\begin{vmatrix} f_y + h_y & f_q + h_q \\ g_y & g_q \end{vmatrix} + q \begin{vmatrix} f_z + h_z & f_q + h_q \\ g_z & g_q \end{vmatrix}$

$= g_p(f_x + h_x) - g_x(f_p + h_p) + \mathrm{p}\,[g_p(f_z + h_z) - g_z(f_p + h_p)] +$

$g_q(f_y + h_y) - g_y(f_q + h_q) + q[g_q(f_z + h_z) + g_z(f_q + h_q)]$

$= g_p f_x + g_p h_x - g_x f_p - g_x h_p + \mathrm{p}\, g_p f_z + \mathrm{p}\, g_p h_z - p g_z f_p - p g_z h_p +$

$g_q f_y + g_q h_y - g_y f_q - g_y h_q + q g_q f_z + q g_q h_z - q g_z f_q - q g_z h_q$

$= (g_p f_x - g_x f_p + \mathrm{p}\, g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z f_q) +$

$(g_p h_x - g_x h_p + \mathrm{p}\, g_p h_z - p g_z h_p + g_q h_y - g_y h_q + q g_q h_z - q g_z h_q)$

$= 0 + 0$ ---------- by (1.1)

$= 0$

Therefore, $f + h \in R$ ------------ (1.2)

Now for any $f = f(x, y, z, p, q)$, $h = h(x, y, z, p, q)$ and $k = k(x, y, z, p, q) \in R$.

As, $f + (h + k) = (f + h) + k$ for any , h , k .

Therefore, associativity property holds in R. ------------ (1.3)

Now consider, $0 = 0(x, y, z, p, q)$ and

$[0, g] = \frac{\partial(0,g)}{\partial(x,p)} + p\,\frac{\partial(0,g)}{\partial(z,p)} + \frac{\partial(0,g)}{\partial(y,q)} + q\,\frac{\partial(0,g)}{\partial(z,q)}$

$= \begin{vmatrix} 0 & f_p \\ g_x & g_p \end{vmatrix} + p \begin{vmatrix} 0 & f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} 0 & f_q \\ g_y & g_q \end{vmatrix} + q \begin{vmatrix} 0 & f_q \\ g_z & g_q \end{vmatrix}$

$= 0 + 0 + 0 + 0$

$= 0$

Therefore, $\in R$.

Also, $+ 0 = 0 + f = f$, for any $f \in G$.

Hence $e = 0$ is an identity element in G . ----------- (1.4)

Now consider,

$[-f, g] = \frac{\partial(-f,g)}{\partial(x,p)} + p\,\frac{\partial(-f,g)}{\partial(z,p)} + \frac{\partial(-f,g)}{\partial(y,q)} + q\,\frac{\partial(-f,g)}{\partial(z,q)}$

$= \begin{vmatrix} -f_x & -f_p \\ g_x & g_p \end{vmatrix} + p \begin{vmatrix} -f_z & -f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} -f_y & -f_q \\ g_y & g_q \end{vmatrix} + q \begin{vmatrix} -f_z & -f_q \\ g_z & g_q \end{vmatrix}$

$= -g_p f_x + g_x f_p - \mathrm{p}\, g_p f_z + p g_z f_p - g_q f_y + g_y f_q - q g_q f_z + q g_z f_q$

$= -(g_p f_x - g_x f_p + \mathrm{p}\, g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z f_q)$

$= 0$ --------by (1.1)

Therefore, $-f \in G$.

As , $f + (-f) = (-f) + f = 0 = e$, for any f

Hence inverse element exists for every element in R. -----------(1.5)

From (1.2), (1.3), (1.4) and (1.5) R is group w.r.t. usual addition of functions.

For any partial differential equations $f = f(x, y, z, p, q)$ and $g = g(x, y, z, p, q)$

We have,

$$f(x, y, z, p, q) + g(x, y, z, p, q) = (f + g)(x, y, z, p, q)$$
$$= (g + f)(x, y, z, p, q)$$
$$= g(x, y, z, p, q) + f(x, y, z, p, q)$$

Therefore,

$f(x, y, z, p, q) + g(x, y, z, p, q) = g(x, y, z, p, q) + f(x, y, z, p, q)$ ….. (1.6)

Hence R is Abelian group with respect to trivial addition of functions.

Therefore, R_1 holds.

For any partial differential equations

$f = f(x, y, z, p, q), g = g(x, y, z, p, q) and\ k = k(x, y, z, p, q)$

We have,

$f + (h + k) = (f + h) + k$

Therefore, multiplication is associative.

Hence, R_2 holds.

We have,

$f.(h + k) = f.h + f.k\ and\ (f + h).k = f.k + h.k$ for any functions f, h, k.

Therefore, $f.(h + k) = f.h + f.k\ and\ (f + h).k = f.k + h.k$ for any partial differential equations f, h, k.

Hence, left and right distributive laws hold in R.

Hence, R_3 holds.

Therefore, R is ring with respect to vector addition and scalar multiplication.

i.e. $(R, +, .)$ is ring.

Result 2: The ring $(R, +, .)$ is commutative ring.

Proof: For any functions $f = f(x, y, z, p, q)$ and $g = g(x, y, z, p, q)$

We have , $(f.g)(x, y, z, p, q) = f(x, y, z, p, q).g(x, y, z, p, q)$

$$= g(x, y, z, p, q).f(x, y, z, p, q)$$
$$= (g.f)(x, y, z, p, q)$$

Therefore, multiplication is commutative.

Hence, $(R, +, .)$is commutative ring.

New Vector Space Structure of Compatible System:

Result 3:

The set $V = \{ f(x,y,z,p,q) = 0: [f,g] = 0, where\ g\ is\ g(x,y,z,p,q) = 0$

i.e. the set of all P.D.E.'s f which are compatible with g, is a vector space with respect to usual vector addition and scalar multiplication.

Proof:

From (1.2),(1.3),(1.4),(1.5) and (1.6),closure of vector addition, associativity and commutativity of addition, existence of additive identity and additive inverse properties holds in V.

Therefore we now show the remaining five properties of vector space for the set V.

For $f \in V$ and scalar α, consider

$$[\alpha f\ , g] = \frac{\partial(\alpha f,g)}{\partial(x,p)} + p\,\frac{\partial(\alpha f,g)}{\partial(z,p)} + \frac{\partial(\alpha f,g)}{\partial(y,q)} + q\,\frac{\partial(\alpha f,g)}{\partial(z,q)}$$

$$= \begin{vmatrix} \alpha f_x & \alpha f_p \\ g_x & g_p \end{vmatrix} + p \begin{vmatrix} \alpha f_z & \alpha f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} \alpha f_y & \alpha f_q \\ g_y & g_q \end{vmatrix} + q \begin{vmatrix} \alpha f_z & \alpha f_q \\ g_z & g_q \end{vmatrix}$$

$$= g_p(\alpha f_x) - g_x(\alpha f_p) + p g_p(\alpha f_z) - p g_z(\alpha f_p) + g_q(\alpha f_y) - g_y(\alpha f_q)$$

$$+ q g_q(\alpha f_z) - q g_z(\alpha f_q)$$

$$= \alpha\ (g_p f_x - g_x f_p + \mathrm{p}\ g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z\ f_q)$$

$$= \alpha([f,g])$$

$$= 0 \qquad \ldots\ldots since, f \in V => [f,g] = 0$$

$=> \alpha f \in V$

Therefore, V is closed under scalar multiplication.

For any function f and scalar 1, it is true that $(1.f) = f$

Therefore for $f \in V$ and scalar 1, we have

$$1. f(x,y,z,p,q) = f(x,y,z,p,q)$$

$$i.e.\ \ 1. f = f\ , \forall\ f \in V$$

Also for any function f and scalars $\alpha, \beta, (\alpha\beta) f = \alpha(\beta f)$

Hence, for $f \in V$ and any scalars α, β, we have

$$(\alpha\beta) f(x,y,z,p,q) = \alpha(\beta f(x,y,z,p,q)$$

$$i.e.\ \ (\alpha\beta) f = \alpha(\beta f), \forall\ f \in V$$

For any functions f and h and scalars α, β, left and right distributive laws

$\alpha(f+h) = \alpha f + \alpha h\ and\ (\alpha+\beta) f = \alpha f + \beta f$ holds.

Therefore, left and right distributive laws hold in V

$i.e.\ \alpha(f+h) = \alpha f + \alpha h\ and\ (\alpha+\beta) f = \alpha f + \beta f$ holds for all f, g in V and any scalars α, β. Therefore, V satisfies all the 10 conditions of vector space and hence V is vector space with respect to usual vector addition and scalar multiplication.

3. Conclusion and Future Work

In this article new set is defined which contain of all partial differential equations $f(x,y,z,p,q)$ which are compatible with fixed function $g(x,y,z,p,q)$. Using trivial addition of functions and multiplication, the given set form a commutative ring. Furthermore, if we use usual vector addition and scalar multiplication then this newly formed set is a vector space. In Future, we want to extend our work for the properties of group, ring and vector space etc.

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