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Original Article

Solving of Minimum Polynomial

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Abstract *- In this paper, the solution method of minimum polynomial is studied, six methods for solving the minimum polynomial are given: the undetermined coefficient method, the invariant factor method, the characteristic polynomial method, the Jordan standard form method, the vector method and Matlab. The vector method is actually a further study of the minimum polynomial. In order to better understand the solution method of the minimum polynomial, the related theorems are given in this paper. The article not only proves the solution method, but also analyzes the advantages and disadvantages of each solution method.*

Keywords *- Minimum polynomial, Characteristic polynomial, The vector method, Invariant factor, MATLAB.*

1. Introduction

Matrix is not only an important research content in algebra, but also widely used in other practical fields. The minimal polynomial of matrix is widely used, such as in determining whether matrix can be diagonalized, solving matrix functions and linear equations, etc. The 19th century German mathematician Frobenius put forward the concept of minimal polynomial in the theory of matrices. Nowadays, many scientific and technical problems need linear equations to solve, precisely the minimum polynomial is very important in solving linear equations, such as automation control. Therefore, if we want to have a deeper understanding of matrix theory, we must start with the minimum polynomial.

It is pointed out in [1-3] that the minimal polynomial is of great significance in the study of squamous cyclic factor matrix and symmetric array. In the knowledge of matrix, mathematicians first put forward the definition of characteristic polynomials, and then defined the definition of minimum polynomials for the needs of subsequent mathematical development. Therefore, [4-9] pointed out the difference and connection between characteristic polynomials and minimum polynomials. The formal definition of minimum polynomial is proposed in Hamilton-Cayley theorem. Therefore, [10-16] describes the origin and development of Hamilton-Cayley theorem, as well as the proof of rationality and other related properties of this theorem. The calculation methods of characteristic polynomials of different matrices, such as random unitary matrices, and their internal relation with Hamilton-Cayley theorem are explained in [17-21]. The calculation method of characteristic polynomials, its extension and research are given in [22-25]. Therefore, the main contents of this paper are as follows: In Section 2, the definition and related properties of the minimum polynomial are introduced; In Section 3, five methods for solving minimal polynomials are introduced and evaluated. The fourth section is conclusion.

2. Definition of Minimum Polynomial and Related Theorems

Definition 1.1: Let matrix A be a matrix of order n in the number field P and $f(x)$ is polynomial. If there $f(A) = 0$, $f(x)$ is said to be zeroized polynomial of A. In order to further study zero reducing polynomial l of A, the minimum polynomial is introduced. Zero reducing polynomial of A with the lowest degree is called the minimum polynomial, denoted by$m_A(\lambda)$.

Property 1.1: Let matrices A and B be matrices of order n in the number field P, the minimum polynomials of matrices A and B are $m_A(\lambda)$ and $m_B(\lambda)$respectively, and matrices A and B are similar, then $m_A(\lambda) = m_B(\lambda)$, otherwise.

Property 1.2: If $m_A(\lambda)$ is the minimum polynomial of matrix A, then the necessary and sufficient condition for $m_A(\lambda)$ divisible $f(\lambda)$ is $f(A) = 0$.

Property 1.3: $m_A(\lambda) = \frac{|\lambda E - A|}{D_{n-1}(\lambda)} = d_n(\lambda)$, where $D_{n-1}(\lambda)$ is the determinant factor of order $n-1$ of A, and $d_n(\lambda)$ is the last invariant factor of A.

Property 1.4: Let matrix $A \in P^{n\times n}$, A similar to diagonal matrix$\Leftrightarrow$ $m_A(\lambda)$ is the product of first factors of A of mutual prime on P.

Property 1.5: Let$\lambda_1, \ldots, \lambda_s$ are all the distinct eigenvalues of A, so $m_A(\lambda) = (\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_s)^{m_s}$, in the Jordan standard form of matrix A, the highest order of the diagonal element $\lambda_s$is m_s.

3. Solving of Minimum Polynomial

3.1. The undetermined coefficient method

Let A be a matrix of order n in the number field C, and let the minimal polynomial of matrix A be of the following form:

$$m_A(\lambda) = \lambda^m + b_{m-1}\lambda^{m-1} + \ldots + b_1\lambda + b_0 (1 \le m \le n)$$

The steps are as follows:

Step 1: When $m = 1, m_A(\lambda) = \lambda + b_0$, try to solution $A + b_0E = 0$, if there is a solution, then the minimum polynomial is $m_A(\lambda) = \lambda + b_0$; If there is no solution, perform the following calculation;

Step 2: When m=2, try to solution$A^2 + b_1A + b_0E = 0$, if there is a solution, then the minimum polynomial is $m_A(\lambda) = \lambda^2 + b_1\lambda + b_0$;

And you keep going until you find something that makes $A^m + b_{m-1}A^{m-1} + \ldots + b_1A + b_0E = 0$ work, and you replace it with matrix A, and you replace it with matrix E by 1, and you get the minimum polynomial you can find.

3.2. The invariant factor method

The invariant factor method can also be called the elementary transformation method. By Property 4, we know that the last invariant factor of the eigenmatrix is the minimal polynomial. The characteristic polynomials of matrix are transformed into standard form by elementary transformation. And then we use the standard form to find the invariant factor of the matrix A, which is the minimum polynomial. Another way of thinking is to first find out the (n-1)-order and n-order determinant factors of the characteristic matrix $(\lambda E - A)$, according to Property 1.3, we can find the minimum polynomial

$$m_A(\lambda) = \frac{|\lambda E - A|}{D_{n-1}(\lambda)} = d_n(\lambda).$$

The invariant factor method is universal, because no matter what kind of matrix, complex or abstract, we can find the determinant factor, the elementary factor, and the invariant factor of the characteristic polynomial of the matrix. Just because of this general practicability, this method also has the disadvantage of complicated calculation process. According to Property 2, can we try to simplify the process of solving the minimum polynomial by starting from the characteristic polynomial of the matrix? The following describes the characteristic polynomial to solve the minimum polynomial steps and advantages and disadvantages.

3.3. The Characteristic Polynomial Method

Step 1: First find the characteristic polynomial $f_A(\lambda) = |\lambda E - A|$of matrix A.

Step 2: And then, you do the standard decomposition of $f_A(\lambda) = |\lambda E - A|$.

Step 3: List the multiplicative factors with power of one that cover all the different eigenvalues. Starting with the lower power, verify that the product of the factors with the lowest degree and the first one rooted in A is the minimum polynomial.

Although the characteristic polynomial of a matrix can certainly be decomposed into a standard form in the field of complex numbers, it is undeniable that in some problems, such as matrices of very high order or matrices with relatively large numbers, it must be very difficult to decompose the characteristic polynomial into a standard form. In addition, this process of standard decomposition requires long-term computational experience and accumulated skills. In order to solve the minimum polynomial in a simpler and easier way, we have tried in another aspect, that is, to solve the minimum polynomial of the matrix by using Jordan standard form.

3.4. The Jordan Standard form Method

(1) Firstly, the Jordan standard form of matrix A is obtained.
(2) Then, use Property1.5 to write the minimum polynomial.

Solving the Jordan standard form method:

Firstly, Let's put $\lambda E - A$ in diagonal form, and then find the elementary factors, and then the Jordan block of order n_i is represented by the elementary factor $(\lambda - \lambda_i)^{n_i}$, which is of the form

$$J_i = \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{n_i \times n_i} \quad (i = 1,2,3,\ldots,m; n_1 + n_2 + n_3 + \ldots + n_m = n)$$

The Jordan standard form of the matrix A is $J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix}$

3.5. The vector method

Through the study of the above four methods of solving the minimum polynomial of matrix, it is found that the most commonly used methods can be roughly divided into two categories: one is to find the characteristic polynomials of the matrix first, and then bring the given matrix into the equation to solve the minimum polynomial; the second is to use the invariant factor, that is, the characteristic matrix of the given matrix is standardized, to find the invariant factor, so as to solve the minimum polynomial. The vector method is to obtain the minimum polynomial of the matrix by using the minimum polynomial of the vector with respect to the matrix. Before proceeding further, it is necessary to introduce the following definitions and lemmas:

Definition 3.1: Let the vector αbe an n-dimensional column vector. If the non-zero polynomial $f(\lambda)$ satisfies $f(A)\alpha = 0$, the polynomial $f(\lambda)$ with the lowest degree and the first one is the minimum polynomial of matrix A.

Theorem 3.1: $A \in P^{n\times n}$, $\alpha_1, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n$ is a basis, the minimum polynomial of vector α_i with respect to matrix A is $f_i(\lambda), i = 1,2,\ldots,n$, then the minimum polynomial of matrix A is $f(\lambda) = [f_1(\lambda), \ldots, f_n(\lambda)]$.

Theorem 3.2: Suppose $r(A_{n\times n}) = r$, the basic solution of a system of linear equations $AX = 0$is $x_{r+1}, x_{r+2}, \ldots, x_n$, and the basis of C^nwhich $x_{r+1}, x_{r+2}, \ldots, x_n$ is extended to be $x_1, x_2, \ldots, x_r, x_{r+1}, x_{r+2}, \ldots, x_n$, and let $x_1, x_2, \ldots, x_r$ respect with of matrix A be, Let the minimum polynomial in $x_{r+1}, x_{r+2}, \ldots, x_n$ with respect to the matrix A be $m_1(\lambda), \ldots, m_r(\lambda)$,the minimal polynomial of matrix A is $m_A(\lambda) = [m_1(\lambda), m_2(\lambda), \ldots, m_r(\lambda), \ \lambda]$.

Where $h(\lambda) = [m_1(\lambda), m_2(\lambda), m_3(\lambda), \ldots, m_r(\lambda), \ \lambda]$.

3.6. The method of MATLAB

Using mathematical software to write a program, so as to achieve the solution of the minimum polynomial, the example is as follows:

Example 1: Finding the minimum polynomial of the matrix. $A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.

Solution: Firstly, finding the fundamental system of solutions of $AX = 0$: $\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$.

We can get the solution is $x_1 = 0, x_2 = -x_3$, let $x_{2=1}$, then $x_3 = -1$, so a fundamental systems of solutions of homogeneous linear equations $AX = 0$ is $X_1 = (0,1,-1)^T$. Next, we can extend this basic solution to a basis on: add vectors and vectors. Next, the method in Theorem 2 above is used to find the minimum polynomial of vector with respect to matrix A, and the matrix is constructed as follows:

$$A=(X_3,AX_3,A^2X_3,A^3X_3)=\begin{pmatrix}1&2&5&14\\0&0&-1&-4\\0&-1&-3&-9\end{pmatrix}\to\begin{pmatrix}1&2&5&14\\0&-1&-3&-9\\0&0&1&4\end{pmatrix}$$
$$\to\begin{pmatrix}1&0&-1&-4\\0&-1&-3&-9\\0&0&1&4\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&1&0&-3\\0&0&1&4\end{pmatrix}.$$

The matrix obtained after elementary transformation shows that, so the minimum polynomial of vector with respect to matrix A, similarly, we can calculate the minimum polynomial of matrix A is, and the minimum polynomial of matrix A can be obtained by Theorem 3.2:

$$m_A(\lambda)=[\lambda,f_2(\lambda),f_3(\lambda)]=[\lambda,3\lambda-4\lambda^2+\lambda^3,3\lambda-4\lambda^2+\lambda^3]=\lambda^3-4\lambda^2+3\lambda$$

Then use the mathematical software to solve, See Appendix 1 for detailed code.

Table 1. Use MATLAB to verify Example 1.

Code	Ans
A=[2 -1 -1;0 1 1;-1 1 1]; Min_ploy=minimal_polynomial(A); syms x; Min_ploy;collect(Min_ploy,x)	ans =x^3 - 4*x^2 + 3*x

Mathematical software is very convenient for calculating the smallest polynomial of complex matrix.

4. Conclusion

In this paper, the properties and solution of the minimal polynomial of matrix are studied. The methods of solving the minimum polynomial mainly include the undetermined coefficient method, the characteristic polynomial method, the invariant factor method, the Jordan standard form method, the vector method and using mathematical software to solve. These methods have advantages and disadvantages, undetermined coefficient method is the simplest knowledge level, but at the same time, it is also a method with the largest amount of calculation. The characteristic polynomial method should be easy to master and commonly used, because the characteristic polynomial is not only used to solve the minimum polynomial, but also very important in linear transformation; Invariant factor method is the further study of characteristic polynomial. This method is universal, but the amount of calculation is relatively large. The vector method is actually a unique way to solve the minimum polynomial, using the vector with respect to the matrix of the minimum polynomial to solve the matrix of the minimum polynomial. This paper also introduces the use of mathematical software to solve the smallest polynomial of the matrix, which makes the process mor concise and fast. It can be seen that the minimum polynomial plays an important role in the study of higher algebra, we should spend more energy to study the minimum polynomial.

Appendix 1

```
function Min_ploy=minimal_polynomial(A)
[m,n]=size(A);
if m~=n
   error(' Make sure the matrix is square ')
end
syms x
eqn=det(x*eye(m,n)-A);
lambda = eval(solve(eqn==0));
lambdak=unique(lambda);
for j=1:1:length(lambdak)
   nk(j,1)=length(find(lambda==lambdak(j)));
end
for j=1:1:length(lambdak)
   rk=m-rank(lambdak(j)*eye(m,n)-A);
   zsk(j)=nk(j)-rk+1;
end
Min_ploy=(x-lambdak(1))^zsk(1);
for j=1:1:length(lambdak)
   if j==1
```

```
      Min_ploy=Min_ploy*1;
    else
       Min_ploy=Min_ploy*(x-lambdak(j))^zsk(j);
    end
 end
 end
```

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