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Original Article

Topological Data Analysis and Computer Science

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Abstract *- Computational topology combines theoretical topological methods with efficient algorithms to analyse data and solve problems in some fields of computer science. In this article we look at the various application of computational or applied topology in computer science with reference to the following fields: Artificial Intelligence, Robotics, Machine learning, and Computer Graphics or Image Processing. We realized that there has been a fast boost in the application of Topological Data Analysis in the above stated areas. This paper seeks to collect and summarize the most recent works connecting the application of Topological Data Analysis to computer science and the various methods used to incorporate the tools of Topological Data Analysis into various applications in computer science.*

Keywords - *Topological Data Analysis, Machine learning, Robotics, Artificial intelligence, Persistent homology.*

1. Introduction

The concept of topology is widely known as a branch of mathematics that takes its root from geometry. Topology is the study of the global and local properties of shapes or objects under continuous deformation. During the early ages of the birth of topology, it was regarded as rubber sheet geometry, because objects or shapes were deformed without changing the underlying properties of such objects or shapes. The detailed understanding of the geometry of surfaces is well illustrated under the concept of topology. According to a topologist, a donut and a torus are the same, because they share some invariant properties. The examples of objects presented in figure 1. are the same but differ geometrically in length, angle of measure, and curvature.

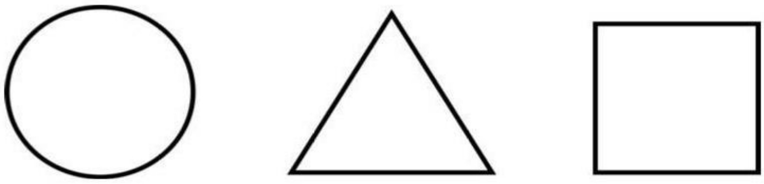

Fig. 1 These shapes are topologically equivalent but geometrically non-identical

While objects are identically through continuous deformation, on the other hand, geometric objects are identical through congruence. Congruent objects have the same lengths of corresponding sides, measures of angles, volume, perimeter, and curvature.

Definition 1 (Zomorodian, 2005) A topology in a set X is a subset τ belonging to 2^X such that, the following axioms holds.

I. The empty set and X are elements of τ
II. Finite intersection of elements of τ is an element of τ
III. Arbitrary union of any elements of τ is an element of τ

Note: The pair (X,τ) of the set X and its topology τ is a called a topological space.

Definition 2 (Kinsey, 1997) An equivalence relation $\simeq$, on a set of objects is a relation on the set such that;

I. For each a in the set , $a \simeq a$ (reflexivity property)
II. If $a \simeq b$, then $b \simeq a$ (symmetric property)
III. If $a \simeq b$, and $b \simeq c$, then $a \simeq c$ (transitivity property)

But topological notions over the past years have been applied in various fields of science, especially computer science. Topology and computer science are closely linked although the two areas are different fields. In theoretical computer science, one often constructs mathematical models for computer-related objects; usually semantic values of programs or the data that is stored in a computer (Reed et al., 1991). In the fast-growing field, Computational Topology, the measure of the shape of data have seen enormous breakthrough since its invention. Computational topology measures the shape of data and the number of n-dimensional holes is computed using the notion of homology (Obeng-Denteh and Adjei, 2022). In this paper, we begin by outlining some of the applications of computational topology in computer science. We, therefore, look at the concept of persistent homology to robotics, machine learning, and artificial intelligence. Also, we provide a survey of the use of TDA tools in classification, clustering, text recognition, and many more.

2. Computational Topology Application to Machine Learning

Topological Data Analysis (TDA) has more recently been utilized to gauge the overall shape of data (Adams and Moy, 2021). Through measuring shape, such as clustering(Xu and Wunsch, 2005), nonlinear dimensionality reduction (Roweis and Saul, 2000), (Tenenbaum et al., 2000), Kohonen (2012), (McInnes et al., 2018) transformation of time series(Chung et al., 2020), analysis of physical signals Chung et al. (2021), (Wang et al., 2018), coverage in sensor networks (De Silva and Ghrist, 2007), and tumor diagnosis (Dunaeva et al., 2016), persistent homology, a tool in topological data analysis, has been incorporated into machine learning. The sophisticated TDA method known as persistent homology, which was first introduced in (Edelsbrunner et al., 2000), tracks homology changes over filtration and visualizes these features using persistent diagrams and barcodes (Zomorodian and Carlsson, 2005). A filtration of space Y is an increasing sequence of spaces $\emptyset = Y_0 \subset Y_1 \subset \cdots \subset Y_n = Y$ A framework created by (Tauzin et al., 2021) makes it possible to convert a wide range of data points into formats that are appropriate for computing persistent homology, which is necessary to fully utilize persistent homology in machine learning.

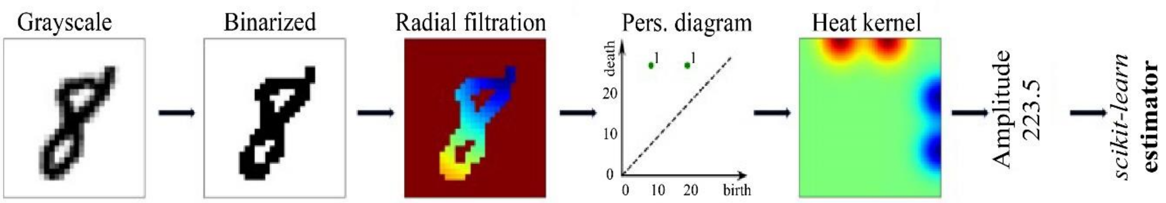

Fig. 2 Example of giotta-tda. Figure is from (Tauzin et al., 2021)

Over the past two decades, machine learning methods have been examined using persistent homology. By analyzing the data, new algorithms are developed, and persistent homology is then applied to regularization, a machine learning technique that penalizes overly complex models to prevent overfitting (Chen et al., 2019). Other recent research in machine learning include time series data analysis (Umeda et al., 2019), modeling RNA hairpin folding (Singh et al., 2007), repeated measurement (Riihim¨aki et al., 2020), chatter identification (Khasawneh et al., 2018), and morphism (Cawi et al., 2019). Some of these fields include a variety of TDA-derived techniques, such as persistent homology, which is best suited to a certain dataset's machine learning objective. Others permit direct TDA tool insertion to forecast weather patterns (Muszynski et al., 2019). TDA was used in (Kindelan et al., 2021) to solve multi-class classification issues and improve outcomes without requiring any additional machine learning stages. By breaking up large data sets into clusters or complexes, this technique allows persistent homology to be computed and used to direct complex structures. By contrast, topological data analysis has been used to assess generative adversarial networks (Goodfellow et al., 2014), population activity in the visual cortex (Singh et al., 1991), and power system contingency (Bush et al., 2021) after shifting our focus from the classification problem. So, learning from the past to enhance performance in the future may be referred to as machine learning (Das and Behera, 2017). This provides datadriven modeling and classification approaches (Carleo et al., 2019), as well as topological materials(Yun et al., 2022), (Rodriguez-Nieva and Scheurer, 2019), (Che et al., 2020), (Scheurer and Slager, 2020). Deep learning is a potent technology to be applied for pattern solving with complex data sets, as well (Goodfellow et al., 2016). Convolutional neural networks are built using a TDA tool called a mapper diagram (Carlsson and Gabrielsson, 2020). Based on the hypothesis that neural networks are comparable to networks of neurons in the brains of mammals, they created neural networks in (Carlsson and Gabrielsson, 2020). They conducted analysis on the created datasets and clustered the data using a mapper approach and a variance-normalized Euclidean metric. The mapper model replicated the given data set's structure, supporting connectivity and loops. Numerous neural networks were looked at and showed tremendous success. However, the convolutional neural network methods utilized in its computation are challenging to optimize due to the nature of optimization (Carlsson and Gabrielsson, 2020).

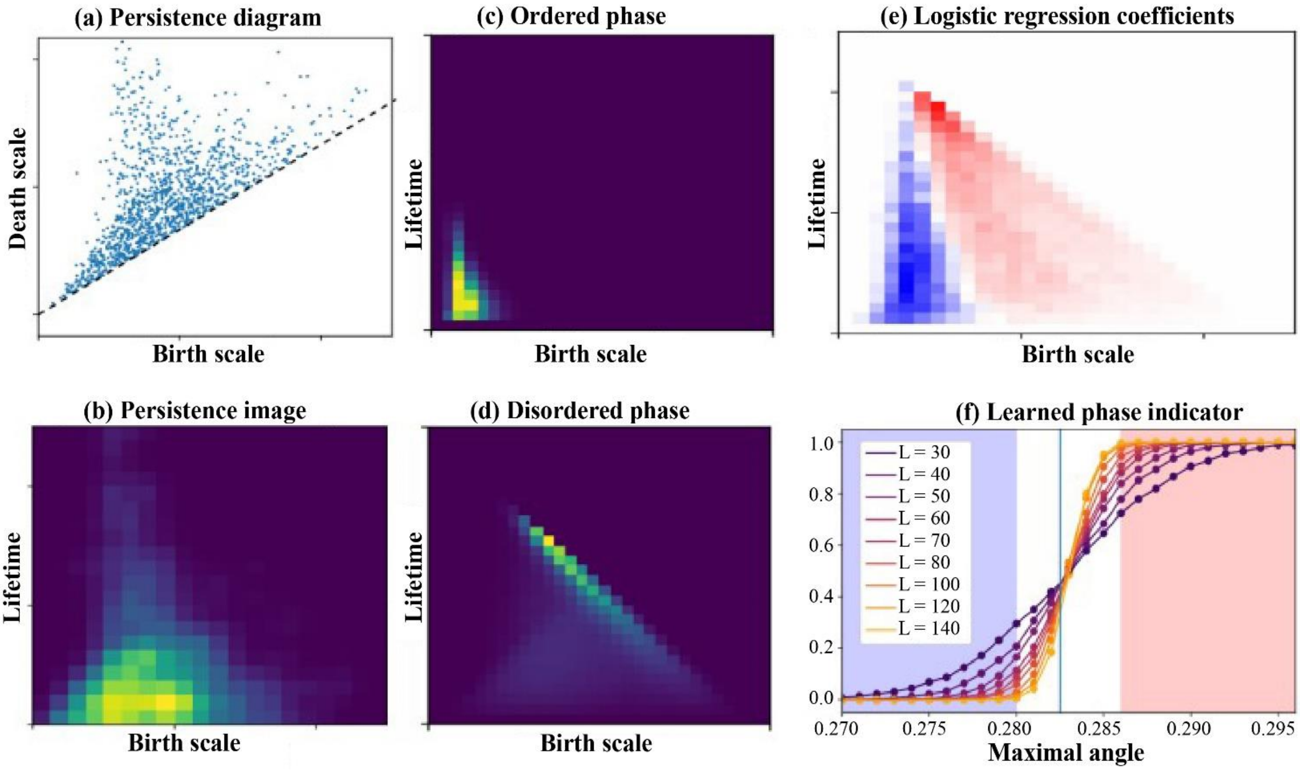

Fig. 3 Topological Data Analysis machine learning of phase transition in XY dimensional model (a)Persistence diagram computed using filtration (b)persistence diagram vectorised into landscape. (c)Persistence images obtained from spin configurations. (e) Persistent image of a coefficients of a trained logistic model. (f)Finite size scaling of the logistic regression prediction used to estimate the phase transition point. Figure is from (Sale et al., 2022)

3. Computational Topology Application to Artificial Intelligence

Since the late 20th century, when topological data analysis first appeared, it has developed more quickly thanks to applications in numerous scientific fields. Although (Edelsbrunner et al., 2000), (Zomorodian and Carlsson, 2005) developed the idea of persistence. TDA has integrated Topological Data Analysis's techniques into artificial intelligence. The potential for clever data Analysis in this variable resolution topological framework is enormous (Robins et al., 2004), (Robins, 1999), (Robins et al., 1998), (Robins et al., 2000). Other approaches have been used by researchers in the field, including persistent homology for analysis of aviation data (Li et al., 2019), persistent homology for analysis of neural networks (Liu et al., 2016), (Gabrielsson and Carlsson, 2019), (Goldfarb, 2018), (Dindin et al., 2020), persistent homology for analysis of text recognition (Makarenko et al., 2016), sports (Goldfarb, 2014), and persistence diagrams for neural networks (Ballester et al., 2022). Utilizing Topological Data analysis tools in artificial intelligence system is the most pertinent research. Although these tools give a general definition of the shape of data, it is still necessary for artificial intelligence (AI) algorithms to take this information into account when analysing data. The field of precision medicine (Iniesta et al., 2022), imagery (Al-Jaberi et al., 2020), (Hu et al., 2019), (Asaad and Jassim, 2017), music (Klaren, 2018), fraud detection (Tymochko et al., 2021), anomaly detection (Davies, 2022), denoising (Al-Jaberi and Hameed, 2021), discourse structure (Savle et al., 2019), and motion recognition (Zelawski and Hachaj˙ , 2021) are a few of where these algorithm strategies are implemented.

Big Data Analytics is a current area of TDA research that captures artificial intelligence. Big Data analytics is a method for removing specific features from data collections. Survey results in (Russom et al., 2011) show that 38% of businesses under study used advanced analytics. Big Data is the new IT that makes it simple for people to analyze a lot of datasets. Over the past ten years, this Technology has received a lot of attention (Bi and Cochran, 2014). However, big data analytics mine huge amounts of data to provide commercial insights (C´ardenas et al., 2013). The following technologies have been developed: Algorithm analysis tools include Apache Flume, Apache Sqoop, Apache Piq, Apache Zookeeper, among others. To keep track of the billions of things in its inventory, large corporations like Amazon use big data analytics (Zakir et al., 2015). For clustering (Zhang et al., 1996), (ESTER et al., 1996), (Ester and Wittmann, 1998), sequential patterns (Chiu et al., 2004), (Han et al., 2001), (Yan et al., 2003), classification (Mic´o et al., 1996), (Djouadi and Bouktache, 1997), (Mehta et al., 1996), social media data mining (Almgren et al., 2017), and healthcare intelligence (Joshi and Joshi, 2019), effective computational algorithms are applied.

4. Computational Topology Application to Robotics

Asimov's writings served as a foundation for the development of robotics, and humans have been captivated by robots ever since. Without additional human intervention, robots assist in discovering solutions. Numerous scientific research subjects have been influenced by the development of robotics (Garcia et al., 2007). Industrial robots, however, didn't enter the market until the early 20th century, and robotics research took over the tech industry in the late 20th century. Robots were primarily specified and designed by the automotive industry in the early 20th century, although implicit and explicit algorithmic approaches are the two main categories in robotic research today (Khatib, 1986). While explicit algorithms rely on the oldest time, implicit algorithms depend on information about future time. Industrial and mobile robots (Garcia et al., 2007), medical robots (Kumar et al., 2000), underwater robots (Ayers, 2004), humanoid robots (Georgiades et al., 2004), and others are examples of robots.

In line with the development of robotic research, a brand-new field of study centered on topology has emerged. The relationship between topology and robotics develops via a number of interconnected paths, each of which challenges and informs the others in turn (Farber et al., 2007a). The concept of robotics can be found in configuration spaces in the study of topology, a space where complex systems can be represented by topological objects. Numerous studies, including (Hausmann et al., 2007), (Fadell and Neuwirth, 1962), discuss the basic forms of configuration spaces. Recent research on topological robots has focused on the topological complexity related to the motion planning problem. Examples of topological complexity research include hyperplane (Yuzvinsky, 2007), formal spaces (Lechuga and Murillo, 2007), and collisionfree motion (Farber et al., 2007b). The investigation of configuration space homology and cohomology provides the groundwork for investigating persistent homology and its use in robotics. The topology of configuration spaces at n-dimensional points is described in (Arnold, 1969), (Cohen et al., 2007), while persistent homology and the investigation of such spaces' homology are covered in (Alpert and Manin, 2021). Since the last 10 years, persistent homology applications in robotics have advanced quickly. trajectory (Pokorny et al., 2014), (Pokorny et al., 2016a), Internet of Things (Novak and Hoffman, 2016), mapping (Dirafzoon and Lobaton, 2013), manipulation (Vieira et al., 2022), and motion planning (Pokorny et al., 2016b) are some robotics applications where persistent homology is used. However in (Vasudevan et al., 2013), they illustrated the usefulness of persistent homology in robotic bipedal walking. One creates a simplicial complex by thresholding a picture over its intensity and adding each pixel to the complex as a node in order to calculate persistent homology. With this method, it was simple to track the betti numbers as a function of the threshold parameter and attempt to fill in the gaps in the data to provide a consistent result over time.

5. Computational Topology Application to Computer Graphics or Image Processing

The basic idea behind computer graphics is to interpret images at a high level, including identifying objects and figuring out how they relate to one another (Bernstein et al., 2020), (Kovalevsky, 1989), however the majority of data from contemporary research is in the form of high resolution digital images (Makarenko et al., 2016), (Garside et al., 2019). This section's goal is to investigate how well Topological Data Analysis tools work for image processing or computer graphics (Mumford and Desolneux, 2010), (Chan and Shen, 2005), (Kimmel et al., 2005), (Weickert and Hagen, 2005). On the basis of mathematical work in algebraic and computational topology, a new technology called topological data analysis has been developed for computer graphics and image processing (Edelsbrunner and Harer, 2022), (Kaczynski et al., 2004).

A solution to object data (Patrangenaru et al., 2018), shape analysis (Carri`ere et al., 2015), (Feng et al., 2018), space planning (Medjdoub and Yannou, 2000), scientific visualization (Tierny, 2016), (Koseki et al., 2020), (Vandaele et al., 2020), medical imaging (Shen, 2021), (Saggar et al., 2018), (Jazayeri et al., 2022), and image classification (Dey et al., 2017), (Yang et al., 2019), (Kachan and Onuchin, 2021) is the persistent homology technique. Topological Data Analysis algorithms are sturdily constructed to handle with images due to expanding steganography research as well. In (Rashid et al., 2018), 1000 randomly chosen natural photos were used to create stego images with included biometric data. 3D printing is another area of image processing that is gaining popularity. Persistent homology is employed by Topological Data Analysis algorithms in order to detect anomalies in 3D printing. Persistent homology tool is used in (Rosen et al., 2018) to assess the quality of 3D printed objects.

The application of Topological Data Analysis to photographs of black holes is one of the most recent developments in image processing using topological data analysis. A black hole is a region of space time that cannot be observed from infinity (Bieri, 2018). However, the Event Horizon Telescope has created photos of an astrophysical black hole in the heart of the M87 galaxy that are on the scale of the horizon (Collaboration et al., 2019), (Akiyama et al., 2019). There is a topological signature of the black hole image that is seen in the EHT observation (Collaboration et al., 2019), (Akiyama et al., 2019): a bright ring encircling a dull region. In (Christian et al., 2022), they decomposed synthetic black hole images using two steps: metronization and persistent homology.

6. Conclusion

Some of the contributions on the use of topological data analysis in computer science have been reviewed in this work. We began with a basic overview of topology (e.g., Topology, Topological spaces, persistent homology). In the application part, we looked at the strategy that applies the techniques of topological data analysis to practical situations. We discussed robotics, artificial intelligence, computer graphics or image processing, and machine learning applications. It is possible that TDA may become a well-liked area of research at the nexus of computer science and topological data analysis due to the high quality of TDA in all of these fields and the technological development of these fields.

Conflict of Interest

The authors described that the research was conducted in the abcence of any commercial or financial relationships that could be construed as a potential conflict of interest

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