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Research Article

A Note on Majority Coloring of Digraphs with Large Indegree

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Abstract - *A majority coloring of a directed graph is a vertex-coloring in which every vertex has the same color as at most half of its out-neighbours. It was conjectured by Kreutzer et al. that every digraph has a majority 3-coloring. This conjecture is far from being resolved. We showed that every digraph D with minimum outdegree at least $28ln|D|$ has a majority 3-coloring. We also considered the natural generalized $\frac{1}{k}$-majority coloring problem.*

Keywords - *Majority coloring, Chernoff bound, Local lemma.*

1. Introduction

In this paper, all digraphs are finite and simple (loopless and without parallel edges, but antiparallel edges are allowed). Let D be a digraph with vertex set $V(D)$ and edge set $E(D)$. For two vertices $v_i, v_j \in V(D)$, we say v_i *dominates* v_j if $v_i v_j \in E(D)$. The vertices which dominate a vertex v are its *in-neighbours*, those which are dominated by v are its *out-neighbours*. These sets are denoted by $N^-(v)$ and $N^+(v)$, respectively. The *outdegree* of a vertex v, denoted by $d^+(v)$, is $|N^+(v)|$. Then we use δ^+ (Δ^+) to denote the minimum (maximum) outdegree of D. The indegree $d^-(v)$ of v, minimum (maximum) indegree δ^- (Δ^-) of D can be defined similarly. For more problems and results concerning graph colorings, see [1,8,11,13,16,19].

A *majority coloring* of a digraph is a function that assigns each vertex v a color, such that at most half the out-neighbours of v receive the same color as v. This type of coloring has received widespread attentions since it was first introduced by van der Zypen [24] in 2016. A significant progress was made by Kreutzer et al. [18], they showed that every digraph has a majority 4-coloring and raised the following conjecture.

Conjecture 1.1 *Every digraph has a majority 3-coloring.*

The conjecture would be best possible, as evidenced by an odd directed cycle. Girão et al. [14] studied Conjecture 1.1 for tournaments, the oriented complete graphs. They showed that every tournament can be 3-coloured in such a way that all but at most 7 vertices receive the same colour as at most half of their out-neighbours. They also proved that every tournament with minimum outdegree 55 has a majority 3-coloring. Anastos et al. [3] proved Conjecture 1.1 for digraphs with maximum outdegree at most 4 or digraphs with chromatic number at most 6 or dichromatic number at most 3. In [18], the authors proved that every n-vertex digraph G with minimum outdegree $\delta^+ > 72\ln(3n)$ has a majority 3-coloring. Xia et al. [2] showed that every r-regular digraph with minimum outdegree $r > 36\ln(2n)$ has a majority 3-coloring. In this paper, we prove the followings.

Theorem 1.2 *Every digraph D with minimum outdegree $\delta^+ > 28lnn$ has a majority 3-coloring.*

For a positive real α, a digraph D is *α-almost-regular* if $\max\{\Delta^+, \Delta^-\} \leq \alpha\min\{\delta^+, \delta^-\}$.

Theorem 1.3 *Every α-almost-regular digraph D with minimum outdegree $\delta^+ \geq max\{740, 56ln12\alpha\}$* has a majority 3-coloring. Moreover, at most half the out-neighbours of each vertex receive the same color.

In [18], the authors also generalized the concept of majority coloring. For $k \geq 2$, a $\frac{1}{k}$*-majority coloring* of a digraph is a function that assigns a color to each vertex v, such that at most $\frac{d^+(v)}{k}$ out-neighbours of v receive the same color as v. A digraph D is $\frac{1}{k}$*-majority m-colorable* if there exists a $\frac{1}{k}$-majority coloring of D using m colors. Note that for a regular tournament of $2k-1$ vertices, any $\frac{1}{k}$-majority coloring must be a proper vertex coloring, thus $2k-1$ colors are necessary. The following conjecture (if true) would be best possible.

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Conjecture 1.4 *Every digraph is* $\frac{1}{k}$*-majority* $(2k-1)$*-colorable.*

Girão et al. [14] proved that every digraph is $\frac{1}{k}$-majority $2k$-colorable for all $k \geq 2$. Xia et al. [2] proved that every digraph D with minimum outdegree $\delta^+ > \frac{2k^2(2k-1)^2}{(k-1)^2}\ln[(2k-1)n]$ is $\frac{1}{k}$-majority $(2k-1)$-colorable. We improve the minimum outdegree condition and prove the following.

Theorem 1.5 *Every digraph* D *with minimum outdegree* $\delta^+ > \frac{2k(4k-1)(2k-1)}{3(k-1)^2} ln[(2k-1)n]$ *is* $\frac{1}{k}$*-majority* $(2k-1)$*-colorable, and at most* $\frac{d^+(v)}{k}$ *out-neighbours of each vertex receive the same color.*

The majority coloring were also generalized to list-colorings [3,4,7,17,23], countable graphs [6,15]. For more problems and results, see [1,5,6,9,10,21,22].

2. Results

In this section, we prove our main results. Our proof will depend on the Multiplicative Chernoff Bound.

Lemma 2.1 *Let* $X_1, \ldots, X_n$ *be independent random variables with*

$$\mathbb{P}(X_i = 1) = p_i, \quad \mathbb{P}(X_i = 0) = 1 - p_i.$$

We consider the sum $X = \sum_{i=1}^{n} X_i$*, with expectation* $\mu = \sum_{i=1}^{n} p_i$*. Then we have*

$$\mathbb{P}(X \geq \mu + \lambda) \leq e^{-\frac{\lambda^2}{2(\mu+\lambda/3)}},$$

$$\mathbb{P}(X \leq \mu - \lambda) \leq e^{-\frac{\lambda^2}{2\mu}}.$$

Now we give a proof of Theorem1.2.

Proof of Theorem 1.2 Independently and uniformly color each vertex of D with one of $\{1,2,3\}$ at random. Let $X(v)$ be the random variable that counts the number of out-neighbours of v, which receives the same color as v. It is obvious that the expectation of $X(v)$ is $\mu = \frac{d^+(v)}{3}$. Let $B(v)$ be the indicator random variable with $B(v) = 1$ if $X(v) > \frac{d^+(v)}{2}$ and $B(v) = 0$ otherwise. Let $B = \sum_{v \in V(D)} B(v)$.

By Lemma1.2, we use the bound of (1) with $\lambda = \frac{d^+(v)}{6}$,

$$\begin{aligned} \mathbb{P}\left(X(v) > \frac{d^+(v)}{2}\right) &= \mathbb{P}\left(X(v) > \frac{d^+(v)}{3} + \frac{d^+(v)}{6}\right) \\ &\leq \exp\left\{-\frac{d^+(v)}{28}\right\}. \end{aligned}$$

Then the expectation of B is

$$\begin{aligned} \mathbb{E}(B) &= \sum \mathbb{P}\left(X(v) > \frac{d^+(v)}{2}\right) \\ &\leq n\exp\left\{-\frac{\delta^+}{28}\right\} < 1, \end{aligned}$$

the last inequality holds as $\delta^+ > 28\ln n$. Thus with positive probability we have $B = 0$, which implies D has a majority 3-coloring.

Next, we give a proof of Theorem 1.5.

Proof of Theorem1.5 Independently and uniformly color each vertex of D with one of $\{1,2,\ldots,2k-1\}$ at random. For each $c \in \{1,2,\ldots,2k-1\}$, let $X(v,c)$ be the random variable that counts the number of out-neighbours of v colored c. Obviously, $\mathbb{E}[X(v,c)] = \frac{d^+(v)}{2k-1}$. Let $B(v,c)$ be the indicator random variable of the event that $X(v,c) > \frac{d^+(v)}{k}$.

By Lemma [lem-che] with $\lambda = \frac{(k-1)d^+(v)}{k(2k-1)}$,

$$\begin{aligned} \mathbb{P}\left(X(v,c) > \frac{d^+(v)}{k}\right) &= \mathbb{P}\left(X(v,c) > \frac{d^+(v)}{2k-1} + \frac{(k-1)d^+(v)}{k(2k-1)}\right) \\ &\leq \exp\left\{-\frac{3(k-1)^2 d^+(v)}{2k(4k-1)(2k-1)}\right\}. \end{aligned}$$

Let $B=\sum_{v\in V(D)}\sum_{c\in C}B(v,c)$, then

$$\mathbb{E}(B)=\sum_{v\in V(D)}\sum_{c\in C}\mathbb{P}\left(X(v,c)>\frac{d^+(v)}{k}\right)\le(2k-1)n\exp\left\{-\frac{3(k-1)^2\delta^+}{2k(4k-1)(2k-1)}\right\}<1,$$

where the last inequality holds as $\delta^+>\frac{2k(4k-1)(2k-1)}{3(k-1)^2}\ln[(2k-1)n]$. Thus with positive probability we have $B=0$, which implies D has a $\frac{1}{k}$-majority $(2k-1)$-coloring and at most $\frac{d^+(v)}{k}$ out-neighbours of each vertex receive the same color.

We use the following weighted version of the Local Lemma to give a proof of Theorem 1.3.

Lemma 2.2 (see[12]). Let $\mathcal{B}=\{B_1,\dots,B_n\}$ be a set of 'bad' events, such that each B_i is mutually independent of $\mathcal{B}\setminus(D_i\cup\{B_i\})$, for some subset $D_i\subseteq\mathcal{B}$. If we have numbers $t_1,\dots,t_n\ge1$ and a real number $p\in\left[0,\frac{1}{4}\right]$ such that for $1\le i\le n$,

$$(a)\ \mathbb{P}(B_i)\le p^{t_i}\text{ and }\ (b)\ \sum_{B_j\in D_i}(2p)^{t_j}\le\frac{t_i}{2},$$

then with positive probability, none of the events in $\mathcal{B}$ occur.

Proof of Theorem1.3 Let $p=\exp\{-\frac{\delta^+}{28}\}$. Since $\delta^+\ge\max\{740,56\ln12\alpha\}$, we have $p\in\left[0,\frac{1}{4}\right]$. Independently and uniformly color each vertex of D with one of $\{1,2,3\}$ at random. For each $c\in\{1,2,3\}$, let $X(v,c)$ be the random variable that counts the number of out-neighbours of v colored c. Clearly, $\mathbb{E}(X(v,c))=\frac{d^+(v)}{3}$. Let $B(v,c)$ be the event that $X(v,c)>\frac{d^+(v)}{2}$. Let $\mathcal{B}=\{B(v,c):v\in V(D),c\in\{1,2,3\}\}$ be our set of events. Let $t(v,c)=t_v=\frac{d^+(v)}{\delta^+}$ be the associated weight. Then $t_v\ge1$. It suffices to prove that conditions (a) and (b) of Lemma 2.2 hold.

Note that $X(v,c)$ is determined by $d^+(v)$ independent trials, by Lemma 2.1 with $\lambda=\frac{d^+(v)}{6}$,

$$\mathbb{P}(B(v,c))\le\exp\left\{-\frac{d^+(v)}{28}\right\}=\exp\left\{-\frac{t_v\delta^+}{28}\right\}=p^{t_v}.$$

Thus condition (a) is satisfied.

For each event $B(v,c)$, let $D(v,c)$ be the set of events $B(w,c')\in\mathcal{B}$ such that v and w have a common out-neighbour. Then $B(v,c)$ is mutually independent of $\mathcal{B}\setminus(D(v,c)\cup\{B(v,c)\})$. Since $t_w\ge1$,

$$\sum_{B(w,c')\in D(v,c)}(2p)^{t_w}\le\sum_{B(w,c')\in D(v,c)}(2p)^1=2p|D(v,c)|\le6pd^+(v)\Delta^-,$$

where the last inequality follows from the fact that there are three colors and each out-neighbour of v has indegree at most Δ^-. Since D is α-almost regular, we have $\Delta^-\le\alpha\delta^+$. Therefore,

$$\sum_{B(w,c')\in D(v,c)}(2p)^{t_w}\le6pd^+(v)\Delta^-\le6\alpha(\delta^+)^2t_v\exp\left\{-\frac{\delta^+}{28}\right\}.$$

Note that $\delta^+\ge\max\{740,56\ln12\alpha\}$, which implies

$$\delta^+\ge28\ln(12\alpha)+56\ln\delta^+,$$

this further implies $6\alpha(\delta^+)^2t_v\exp\{-\frac{\delta^+}{28}\}\le\frac{t_v}{2}$. Hence, (b) is satisfied.

We conclude that every α-almost-regular digraph$D$ with minimum outdegree $\delta^+\ge\max\{740,56\ln12\alpha\}$ has a majority 3-coloring. Moreover, at most half the out-neighbours of each vertex receive the same color.

The following result, also proved by the weighted Local Lemma, shows the existence of a $\frac{1}{k}$-majority $(2k-1)$-coloring if we further limiting the maximum indegree of a digraph D.

Theorem2.3 *Every digraph D with minimum outdegree $\delta^+ \geq \frac{2k(4k-1)(2k-1)ln4}{3(k-1)^2}$ and maximum indegree at most $\frac{exp\left\{\frac{3(k-1)^2\delta^+}{2k(4k-1)(2k-1)}\right\}}{4(2k-1)\delta^+}$ has a $\frac{1}{k}$-majority $(2k-1)$-coloring. Moreover, at most $\frac{d^+(v)}{k}$ out-neighbours of each vertex receive the same color.*

Define $p = \exp\left\{-\frac{3(k-1)^2\delta^+}{2k(4k-1)(2k-1)}\right\}$. Since

$$\delta^+ \geq \frac{2k(4k-1)(2k-1)\ln 4}{3(k-1)^2},$$

we have $p \in \left[0, \frac{1}{4}\right]$.

Independently and uniformly color each vertex of D with one of $\{1,2,\dots,2k-1\}$ at random. For each $c \in \{1,2,\dots,2k-1\}$, let $X(v,c)$ be the random variable that counts the number of out-neighbours of v colored c. Clearly, $\mathbb{E}(X(v,c)) = \frac{d^+(v)}{2k-1}$. Let $B(v,c)$ be the event that $X(v,c) > \frac{d^+(v)}{k}$. Let $\mathcal{B} := \{B(v,c) : v \in V(D), c \in \{1,2,\dots,2k-1\}\}$ be our set of events. Let $t_v = \frac{d^+(v)}{\delta^+}$ be the associated weight. Then $t_v \geq 1$. It suffices to prove that conditions (a) and (b) hold.

Note that $X(v,c)$ is determined by $d^+(v)$ independent trials. By Lemma 2.1 with $\lambda = \frac{(k-1)d^+(v)}{k(2k-1)}$,

$$\mathbb{P}(B(v,c)) \leq \exp\left\{-\frac{3(k-1)^2 d^+(v)}{2k(4k-1)(2k-1)}\right\} = \exp\left\{-\frac{3(k-1)^2 t_v \delta^+}{2k(4k-1)(2k-1)}\right\} = p^{t_v}.$$

Thus condition (a) is satisfied.

For each event $B(v,c)$, let $D(v,c)$ be the set of all events $B(w,c') \in \mathcal{B}$ such that v and w have a common out-neighbour. Then $B(v,c)$ is mutually independent of $\mathcal{B} \setminus (D(v,c) \cup \{B(v,c)\})$.

Since each out-neighbour of v has indegree at most $\Delta^- \leq \frac{\exp\left\{\frac{3(k-1)^2\delta^+}{2k(2k-1)(4k-1)}\right\}}{4(2k-1)\delta^+}$, we have $|D(v,c)| \leq (2k-1)d^+(v)\Delta^- = (2k-1)\delta^+ t_v \Delta^-$. Thus,

$$\sum_{B(w,c') \in D(v,c)} (2p)^{t_w} \leq \sum_{B(w,c') \in D(v,c)} (2p)^1 = 2p|D(v,c)| \leq 2p(2k-1)\delta^+ t_v \Delta^- \leq \frac{t_v}{2}.$$

We conclude that every digraph D with minimum outdegree $\delta^+ \geq \frac{2k(2k-1)(4k-1)\ln 4}{3(k-1)^2}$ and maximum indegree at most $\frac{\exp\left\{\frac{3(k-1)^2\delta^+}{2k(2k-1)(4k-1)}\right\}}{4(2k-1)\delta^+}$ has a $\frac{1}{k}$-majority $(2k-1)$-coloring. Moreover, at most $\frac{d^+(v)}{k}$ out-neighbours of each vertex receive the same color.

Corollary 2.4. *Every r-regular digraph D with $r \geq g(k)$ has a $\frac{1}{k}$-majority $(2k-1)$-coloring, where $g(k)$ can be determined by the function equation $r = \frac{exp\left\{\frac{3r(k-1)^2}{2k(2k-1)(4k-1)}\right\}}{(8k-4)r}$. Moreover, at most $\frac{r}{k}$ out-neighbours of each vertex receive the same color.*

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