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Original Article

Domination Parameter of Book Graph B_m

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Abstract *- In this paper, we present results on various domination parameters like domination, split domination, perfect domination, connected domination, equitable domination, inverse domination, restrained domination, and strong domination number for the product of the start graph with path P_2 namely B_m*

Keywords *- Book Graph B_m, Domination parameters.*

AMS Subject Classification: *05C69*

1. Introduction

In this, paper we present results on various domination parameters like domination, split domination, perfect domination, connected domination, equitable domination, inverse domination, restrained domination, and strong domination number for the product of the start graph with path, graph namely B_m we refer Katitha B N & Indrani Pramod kelkar research papers [6], [7], [8], [9], [10]. [11].

1.1. Domination Parameters of Graph

Domination Number: A subset D of V(G) is said to be a dominating set of G if every vertex in V \ D is adjacent to a vertex in D. Cardinality of the minimum dominating set is called the domination number of G and is denoted by γ (G).

Split domination Number[17]: A dominating set D of G is called a split dominating set if the induced subgraph $< V - D >$ is disconnected otherwise it is called non-split dominating set. Cardinality of the minimum split dominating set is called the split domination number [12] of G denoted as γ_s(G).

Perfect domination number [1]: A dominating set D of a graph G is said to be a perfect dominating set if any vertex of G not in D is adjacent to exactly one vertex of D. Cardinality of the minimum perfect dominating set is called the perfect domination number [21], denoted as $\gamma_{pt}(G)$.

Equitable domination number [23]: A dominating set D of V (G) is called an equitable dominating set of a graph G if for every $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of an equitable dominating set of G is called equitable domination number [18] of G and is denoted by $\gamma^e(G)$.

Inverse domination number [18] : Consider a dominating set D of G, if the induced Subgraph < V - D > contains a dominating set D_1 of G, then D_1 is called an inverse dominating set with respect to D. Cardinality of the minimum inverse dominating set is called the inverse domination number [13] of G and is denoted as $\gamma^{-1}(G)$.

Restrained domination number [3]: Let $G = (V, E)$ be a graph. A restrained dominating set is a set $S \subseteq V$ where every vertex in $V - S$ is adjacent to vertex in S as well as another vertex in$V - S$. The restrained domination number [1] of G. denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G.

Connected domination number [20]: A dominating set D of a graph G is such that induced subgraph of D is a connected graph then D is a connected dominating set of G. Cardinality of the minimum connected dominating set is called the connected domination number [15] of G denoted as $\gamma_C(G)$.

Strong domination number [5]: Let $G = (V, E)$ a graph A set $D \subseteq V$ is strong dominating set of G's if for every vertex $v \in V - D$ there is a vertex $u \in D$ with $uv \in E$ and $d(u, G) \geq d(v, G)$. The strong domination number [5] $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set.

Weak domination number [4]: Let $G = (V, E)$ a graph A set $D \subseteq V$ is weak dominating set of G if for every vertex $v \in V - D$ there is a vertex $u \in D$ with $uv \in E$ and $d(u, G) \leq d(v, G)$. The weak domination number [4] $\gamma_{wk}(G)$ is defined as the minimum cardinality of a weak dominating set.

2. Domination Parameters of Book Graph

The cross product of star of S_{m+1} and path $P_2$is called a book graph by [6],[7],[8],[9],[10],[11], denoted as B_m. Suppose vertex set of $S_{m+1} = \{v, u_1, u_2, \dots \dots u_m\}$ and Path $P_2 = \{w_1, w_2\}$ then vertex set of B_m can be written as,

$$V(B_m) = \{(v, w_1), (v, w_2), (u_1, w_1), (u_1, w_2), (u_2, w_1), (u_2, w_2) \dots \dots (u_i, w_1)(u_i, w_2)\},$$

$|V| = 2m + 2$

The edge set of B_m contains edges of four types

(i) Central edges $f = \{((v, w_1)(v, w_2))\}$

(ii) Star edges at w_1 $\quad g_i^1 = \{((v, w_1)(u_i, w_1))$
Where $i = 1,2,3, \dots \dots \dots \dots . . m\}$

(iii) Star edges at w_2 $\quad g_i^2 = \{((v, w_2)(u_i, w_2))$ where $i = 1,2,3, \dots \dots \dots \dots . . m\}$

(iv) Path edges $h_i = \{((u_i, w_1)(u_i, w_2))$ where $\quad i = 1,2, \dots \dots \dots \dots \dots m\}$

$$|E| = |f_i| + |g_i^1| + |g_i^2| + |h_i|$$
$$= 1 + m + m + m = 3m + 1$$

Thus, the book graph $B_m$has $2m + 2$ vertices and $3m + 1$ edges.

Theorem 2.1 Domination number of B_m is

$$\gamma(B_m) = 2 \text{ for } m \geq 3.$$

Proof. We know that the book graph is cross product of S_{m+1} and P_2 with 2m+2 vertices and $3m + 1$ edges. In book graph(v, w_1) and (v, w_2) are adjacent to m distinct vertices each with the type of edges g_i^1 and $g_i^2$for $i = 1,2,3, \dots \dots . m$ respectively. Neighbourhoods of (v, w_1) and (v, w_2)satisfy, $N[(v, w_1)] \cup N[(v, w_2)] = V$ and $N[(v, w_1)] \cap N[(v, w_2)] = \emptyset$.Therefore (v, w_1) and (v, w_2) together dominate all vertices of B_m.$D = \{(v, w_1), (v, w_2)\}$is a minimal dominating set as no single vertex of D dominate all vertices of B_m.Therefore, the domination number of book graph is

$$\gamma(B_m) = 2.$$

Lemma 2.2. Perfect domination number of book graph is

$$\gamma_P(B_m) = 2 \text{ for} m \geq 3$$

Proof. From theorem 2.1, $D = \{(v, w_1), (v, w_2)\}$ is a minimal dominating set of B_m.
D is a perfect dominating set if $\forall$ v $\in$ V − D, v is adjacent to exactly one vertex of D. Consider
$V - D = \{(u_i, w_1), (u_i, w_2) \; i = 1,2,3, \dots \dots \dots m\}$. Here all vertices (u_i, w_1) $i = 1,2, \dots . . m$ are dominated by (v, w_1) and vertices (u_i, w_2) $i = 1,2, \dots \dots \dots m$ are dominated by(v, w_2). Thus, every vertex of $V - D$ is adjacent to exactly one vertex of D. So, D satisfies the condition for perfect domination. Therefore, D is a minimal perfect dominating set, giving perfect domination number of book graph $B_m$as $|D| = 2$.

$$\therefore \gamma_P(B_m) = 2 = \gamma(B_m).$$

Lemma 2.3. The split domination number of B_m is

$$\gamma_s(B_m) = 2 \text{ for} m \geq 3,$$

Proof. From theorem 2.1, minimal dominating set of B_m is $D = \{(v, w_1), (v, w_2)\}$. For D to be a split dominating set, we need $V - D$to be a disconnected graph. We observe that removal of the vertices of D from the graph B_m gives m-disconnected components path P_2. Hence D is the minimal split dominating set of B_m giving $\gamma_s(B_m) = 2$ for $m \geq 3$.

Illustration: For a book graph B_6 we have 14 vertices dominated by two vertices of the dominating set
$D = \{(v, w_1), (v, w_2)\}$ including the centre vertex of each copy of the star graph dominating all the vertices of

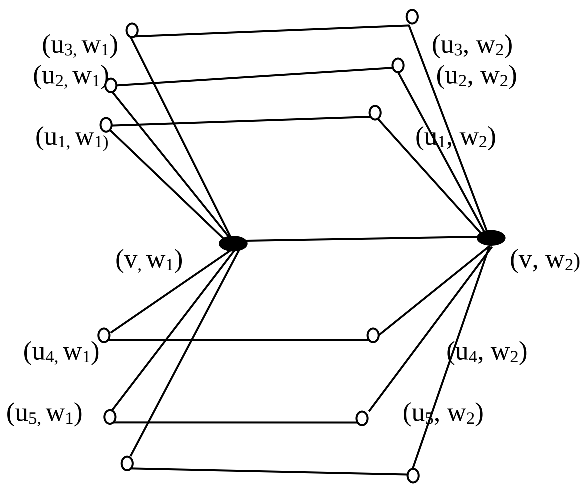

Fig. 1 Graph B_6

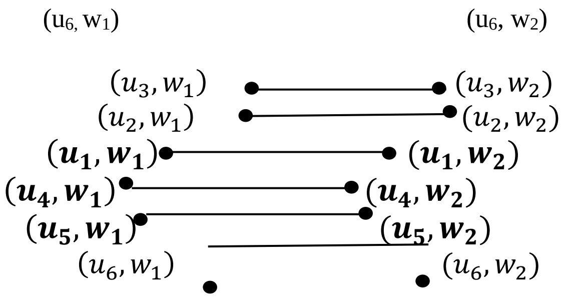

Fig. 2 Split Graph $B_6 - D$

Lemma 2.4. Equitable domination number of book graph is

$$\gamma^e(B_m) = m + 1.$$

Proof: From theorem 2.1 $D = \{(v, w_1), (v, w_2)\}$ is a minimal dominating set of B_m.
For D to be an equitable dominating set, we need if for every, $v \in V - D$ there exist a vertex $u \in D$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$.
Consider $V - D = \{(u_i, w_1), (u_i, w_2)$ where $i = 1,2,3, \dots \dots \dots \dots m\}$. Consider a vertex$(u_1, w_1)$ there exist a vertex $\{(v, w_1)\}$in D such that $(u_1\, w_1)$ is adjacent to (v, w_1) with $\deg(v, w_1) = m + 1$ and$\deg(u_1, w_1) = 2$.
$|\deg(v, w_1) - \deg(u_1, w_1)| = m + 1 - 2 \nleq 1$ for $m \geq 3$
Therefore for(u_1, w_1), a vertex chosen from $V - D$, equitable domination condition is not satisfied. This shows that D does not satisfy the equitable domination condition and so D is not an equitable dominating set.
Next consider vertex (v, w_1) which dominates $m + 1$ vertices$\{(v, w_2)(u_I, w_1)\ i = 1,2, \dots \dots \dots m\}$. Add (v, w_1)set D_1, to get equitable dominating set, we need to add m vertices from $B_m$which are not dominated by (v, w_1) so we get an extended set $D_1 = \{(v, w_1), (u_i, w_2), i = 1,2 \dots \dots \dots m\}$, with $|D_1| = m + 1$ and
$V - D_1 = \{(v, w_2)(u_i, w_1)\ i = 1,2, \dots \dots \dots m\}$As (v, w_1) dominates all vertices of $V - D_1$, it is clear that D_1 is a dominating set of B_m of cardinality $m + 1$.

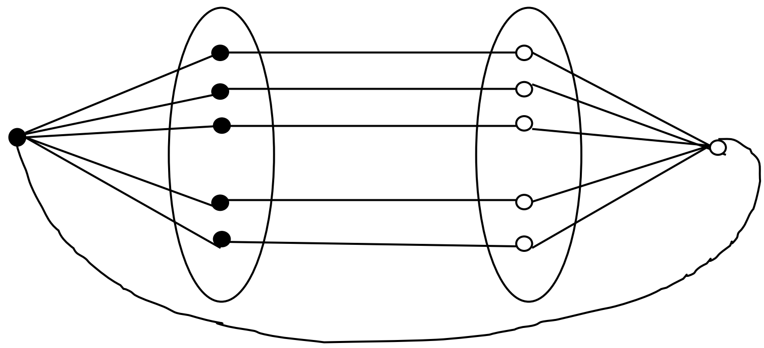

Fig. 3 Equitable domination number of B_6

To check if D_1 satisfies equitable domination condition, consider a vertex $(v, w_2) \in V - D$, with
$d(v, w_2) = m + 1$ which is adjacent to $(v, w_1) \in D$ with$d(v, w_1) = m + 1$.
$\therefore |d(v, w_2) - d(v, w_1)| = 0 \leq 1$ -----------------(1)

Next $(u_i, w_1) \in V - D$ is adjacent to $(u_i, w_2) \in D$. Hered $(u_i, w_1) = 2$ and
$d(u_i, w_2) = 2$.

$\therefore\ |d(u_i, w_2) - d(u_i, w_1)| = 0 \leq 1$ -------------- (2)

From (1) and (2), it is clear that D_1 is a dominating set of B_m which satisfies equitable condition and any subset of D_1 will not be a dominating set, as all vertices in induced graph D_1 are isolated. This implies that D_1 is the minimal equitable dominating set of cardinality $m + 1$ giving,

$$\gamma^e(B_m) = m + 1$$

Lemma 2.5: Inverse domination number of book graph B_m is

$$\gamma^{-1}(B_m) = m. \text{ Form} \geq 3.$$

Proof: From theorem 2.1, $D = \{(v, w_1), (v, w_2)\}$ is a minimal dominating set of B_m.
Consider$V - D = \{(u_i, w_1), (u_i, w_2)$ for $i = 1,2,3, \dots \dots \dots \dots m\}$. Observe that $V - D$ hasm components, each one is a paths on two vertices say, $C_1 = [(u_1, w_1), (u_1, w_2)]$, $C_2 = [(u_2, w_1), (u_2, w_2)]$, $\dots \dots \dots \dots ....$

$C_m = [(u_m, w_1), (u_m, w_2)]$. To get a minimal dominating set for $V - D$, from each of these components one vertex should be included to dominating set, say D'. Consider one such choice of dominating set for $V - D$ as
$D' = \{(u_1, w_1), (u_2, w_1), (u_3, w_1), \dots (u_m, w_1)\} = \{(u_i, w_1) for i = 1,2,3 \dots \dots m\}$ D' is a dominating set of $V - D$ with minimum cardinality so D' is an inverse dominating set with respect to D. Thus, inverse domination number of B_m is

$$\gamma^{-1}(B_m) = |D'| = m$$

Lemma 2.6. Restrained domination number of book graph B_m is

$$\gamma_r(B_m) = 2 \text{ For } m \geq 3$$

Proof. From theorem 2.1, we know that the domination number of graph B_m is $\gamma(B_m) = 2$ with dominating set
$D = \{(v, w_1), (v, w_2)\}$.
For D to be a restrained dominating set we require, for all $v \in V - D$ is adjacent to a vertex in D as well as another vertex in $V - D$.
Consider$V - D = \{(u_i, w_1), (u_i, w_2), i = 1,2,3 \dots \dots \dots \dots m\}$. (u_i, w_1) is adjacent one vertex (v, w_1) in D and also adjacent to one vertex (u_i, w_2) in $(V - D)$. Similarly,(u_i, w_2) vertex is adjacent to (v, w_2) in D and (u_i, w_1)in$V - D$.
Therefore, for every vertex in $V - D$ it is true that it is adjacent to one vertex in D and one vertex in
$V - D$. Therefore, the dominating set D itself forms restrained dominating set of the minimal cardinality. Hence restrained domination number of B_m is

$$\gamma_r(B_m) = 2.$$

Lemma 2.7. The connected domination number of book graph is

$$\gamma_c(B_m) = 2 \text{ For } m \geq 3$$

Proof. From theorem 2.1 the domination number of B_m is $\gamma(B_m) = 2$, with minimal dominating set
$D = \{(v, w_1), (v, w_2)\}$. As (v, w_1) is adjacent to (v, w_2), the dominating set is a connected graph. Therefore, the dominating set D itself forms a connected dominating set of minimal cardinality. Hence connected domination number of B_m is, $\gamma_c(B_m) = 2$

Lemma 2.8. Strong domination number of book graph B_m is

$$\gamma_{st}(B_m) = 2$$

Proof. From theorem 2.1, the minimal dominating set of book graph is.
$D = \{(v, w_1), (v, w_2)\}$For D to be a strong dominating set we require for $u \in D$, if every vertex $x \in V - D$ such that $ux \in E(B_m)$ and $\deg(u) \geq \deg(x)$.

Consider $V - D = \{(u_i, w_1), (u_i, w_2)\} for\ i = 1,2,3, \dots \dots . m\}$. We have $d(v, w_1) = m + 1$, $d(v, w_2) = m + 1$ and for every vertex in $V - D, d(u_i, w_1) = 2, d(u_i, w_2) = 2$ for $i = 12,3,, \dots \dots m$. In B_m we have, (u_i, w_1) is adjacent to (v, w_1) in D with $d(v, w_1) \geq d(u_i, w_1)$. Similarly,$(u_i, w_2)$ is adjacent to (v, w_2) in D with $d(v, w_2) \geq d(u_i, w_2)$.This clearly shows that the minimal dominating se D satisfies the strong domination condition. Hence strong domination number of book graph is

$$\gamma_{st}(B_m) = 2.$$

3. Results

Sl no	Domination parameters	Book Graph B_m
1	Domination number γ	2
2	Split domination number γ_s	2
3	Perfect domination number γ_p	2
4	Equitable domination number γ^e	$(m + 1)\gamma^e(P_2)$
5	Inverse domination number γ^{-1}	$m\gamma^{-1}(P_2)$
6	Restrained dominationnumber γ_r	2
7	Connected domination number γ_c	2
8	Strong domination number γ_{st}	2
9	Hinge domination number γ_h	2

4. Conclusion

Various domination parameters for Book graph are equal to the domination number. The dominating set of star graph contains only its central vertex, by the definition of the product graph of star graph with path P_2, the dominating set extends to be the subgraph attached to the centre vertex. The relationship between the domination numbers of book graph are as follows:

$$\gamma(B_m) = \gamma_s(B_m) = \gamma_{pt}(B_m) = \gamma_r(B_m) = \gamma_c(B_m) = \gamma_{st}(B_m) = \gamma_h(B_m) = |V(P_2)|$$

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