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Original Article

Numerical Ranges and Numerical Radius of Truncated Toeplitz Operators in the Case u = z^n

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Abstract *- In this paper, we first give a matrix representation of certain class of truncated Toeplitz operators in the case $u = z^n$ and then we use this matrix representation to determine the numerical ranges and the numerical radius of such operators.*

Keywords *- Model space, Truncated toeplitz operator, Numerical range, Numerical radius.*

1. Introduction

Truncated Toeplitz operators have long ago appeared in literature as a model operators for completely nonunitary contractions with defect numbers one and for their commutant. Since 2007, D. Sarason gave in his seminal paper [1], the algebraic properties of such operators and raised interesting questions. The truncated Toeplitz operators live on the coinvariant subspace of the shift operator of the form $K_u^2 = H^2 \Theta u H^2$ for some inner function u. Such spaces are also called model spaces. For more details on the truncated Toeplitz operators see [1,2,5,6,7,8,9,10,11,12,22,23,27]. This work organized as follow: in the preliminary section, we give the generalities of model spaces and theirs operators. In third section, we give the matrix representation of truncated Toeplitz operators on the finite dimensional case, namely when$u = z^n$. In the last section, we study the numerical range and the numerical radius of Truncated Toeplitz operators on the finite dimensional case.

2. Preliminaries

Let Co denote the complex plane, D denote the unit disc and let T denote the unit circle. $H^2$is the usual Hardy space, the subspace of $L^2(T)$ of normalized Lebesgue measure m on T whose harmonic extensions to D are holomorphic. A function which is analytic and bounded in D is said to be inner if $|u| = 1$ m -almost everywhere on T in the sense of nontangential boundary values. For an inner function $u \in H^2$, the model space is defined by$K_u^2 = H^2 \Theta u H^2$. In other words$K_u^2 = H^2 \cap (uH^2)^\perp = (uH^2)^\perp$, its reproducing kernel being defined by (see [1])

$$k_\lambda^u = \frac{1-\overline{u(\lambda)}u}{1-\bar{\lambda}z} \tag{1}$$

with $\lambda \in D$. This function verifies $\langle f, k_\lambda^u \rangle = f(\lambda)$ for all $f \in K_u^2$. For inner function u, K_u^2 has an conjugation operator denoted C defined by (see [1])

$$C\varphi = u\overline{z\varphi} \tag{2}$$

with$z \in T$. The kernel conjugate reproducing of k_λ^u on K_u^2denoted by Ck_λ^u and defined by (see [1])

$$Ck_\lambda^u = \frac{u-u(\lambda)}{z-\lambda} \tag{3}$$

with $\lambda \in D$. Given $\varphi \in L^2(T)$, we then define the truncated Toeplitz operator A_φ to be the operator that sends f to $P_u(\varphi f)$ for all $f \in K_u^2$, where P_u is the projection of $L^2(T)$ onto K_u^2. Truncated Toeplitz operators have many of the same properties as ordinary Toeplitz operators. The truncated Toeplitz operator is defined by density on $K_u^\infty = K_u^2 \cap H^\infty$, where $H^\infty$is the space of analytic and bounded functions on D. In other words $A_\varphi(f) = P_u(\varphi f)$ for all $f \in K_u^\infty$. For $f \in H^2$ each the shift operator is defined by $S(f) = zf$ for $z \in T$. Its adjoint is the backward shift defined by $S^*(f) = \frac{f-f(0)}{z}$. The compression on $K_u^2$of the shift operator is the operator S_u which is a truncated Toeplitz operator with symbol z that is to say $S_u = A_z$. Its adjoint is the operator S_u^* which is a truncated Toeplitz operator of symbol$\bar{z}$. The operator S_u commutes with truncated Toeplitz operators.

For A an bounded operator onL^2, the numerical range of A is the set defined and denoted by

$$W(A) = \{\langle Ax, x\rangle : x \in L^2, \|x\| = 1\} \quad (4)$$

It is a very important tool to study the properties of operators. In 1918, Toeplitz proved that the boundary $\partial W(A)$ is a convex curve. In 1919, Haussdorff gave his now classic theorem for convexity of$W(A)$. The numerical radius of A is defined by

$$w(A) = sup\{|\lambda| : \lambda \in W(A)\} \quad (5)$$

For more details on the numerical ranges and numerical radius of some operators see [3,4,14,15,16,17,18,19,20].

For $f, g \in H^2$, the usual tensor of f and g is defined by $f \otimes g(h) = \langle h, g\rangle f$for all $h \in H^2$and the product of two tensors is defined by$(f \otimes g)(h \otimes j) = \langle h, g\rangle(f \otimes j)$ for all $h, j \in H^2$.

In what follows, we denote by TTO: truncated Toeplitz operator and by TTOs: truncated Toeplitz operators. The following results are in [1] and in [2].

Proposition 2.1. [1] (1) Let $\lambda \in D$. Then $Ck_\lambda^u \otimes k_\lambda^u$ is a TTO with symbol $\frac{u}{z-\lambda}$ and $k_\lambda^u \otimes Ck_\lambda^u$ is a TTO with symbol $\overline{\frac{\overline{u}}{z-\lambda}}$
(2) Let $\lambda \in T$ such that $u$has an angular derivative in the sense of Caratheodory (ADC) at the point λ. Then $k_\lambda^u \otimes k_\lambda^u$ is a TTO with symbol $k_\lambda^u - \overline{k_\lambda^u} + 1$.
These three operators are TTOs of rank-one.

Remark 2.2. The function u is said to have an angular derivative in the sense of Caratheodory (ADC) at the point $\zeta \in T$ if u has a nontangential limit of unit modulus at ζ and u' has a nontangential limit $u'(\zeta)$ at ζ.

Lemma 2.3. [2] Let $\varphi \in K_u^2$. Then

$$S_u C\varphi = u\overline{(\varphi - \phi(0))} \quad (6)$$

Lemma 2.4. [2] For $\lambda \in D$,

$$S_u Ck_\lambda^u = \lambda Ck_\lambda^u - u(\lambda)k_0^u \quad (7)$$

Definition 2.5. [2] For $\alpha \in Co$with $\alpha \neq 0$ $B^\alpha = \{A_{\varphi+\alpha\overline{S_u C\varphi}+c}, c \in Co\}$. An operator is of type α if it is inB^α.
For more details on truncated Toeplitz operators of type α see [2] (section 4.1).

Remark 2.6. By [2], the TTOs in proposition 1 are of type$\alpha = u(\lambda)$.
The following results give properties concerning numerical ranges and numerical radius of all operators A and B.

Proposition 2.7. [3] For all bounded operators A and B on complex Hilbert space$H$, $\alpha$and$\beta \in Co$, we have the following properties:

1. $W(\alpha I + \beta A) = \alpha + \beta W(A)$
2. $W(A + B) \subset W(A) + W(B)$
3. $W(A^*) = \{\overline{\lambda}, \lambda \in W(A)\}$

Remark 2.8. In finite dimension, we have $W(A + B) = W(A) + W(B)$.
Lemma 2.9. [3] Let A an bounded operator in complex Hilbert space H. Then

$$w(A^n) \leq (w(A))^n \quad (8)$$

Lemma 2.10. [4] The following inequality is hold:

$$sup\{\sum_{i=0}^{n-2} x_i x_{i+1} : x_i \in R, \sum_{i=0}^{n-1} x_i^2 = 1\} \leq cos\frac{\pi}{n+1} \quad (9)$$

with R: set of real numbers.

Remark 2.11. The two inequalities in equation 8 and 9 are very important for the results concerning the numerical radius.

3. Matrix representations of TTOs in the case where u = z^n

If $u = z^n$, the model space K_u^2 is finite dimensional. In this case, u is a finite order Blaschke product and $K_u^2 = K_{z^n}^2 = span\{1, z, z^2, \ldots, z^{n-1}\}$. So if $\varphi \in K_{z^n}^2$ then

$$\varphi(z) = \sum_{k=0}^{n-1} \hat{\varphi}(k) z^k \tag{10}$$

where$\hat{\varphi}(k)$: the Fourier coefficients of the function φ (see [27], example 4.5). The matrix representation of an TTO $A_\varphi$of symbol $\varphi$with respect to the orthonormal basis $\{1, z, z^2, \dots, z^{n-1}\}$ of $K^2_{z^n}$is given by (see [1], page 2)

$$Ma(A_\varphi) = \begin{pmatrix} \hat{\varphi}(0) & \hat{\varphi}(-1) & \hat{\varphi}(-2) & \cdots & \hat{\varphi}(3-n) & \hat{\varphi}(2-n) & \hat{\varphi}(1-n) \\ \hat{\varphi}(1) & \hat{\varphi}(0) & \hat{\varphi}(-1) & \cdots & \hat{\varphi}(4-n) & \hat{\varphi}(3-n) & \hat{\varphi}(2-n) \\ \hat{\varphi}(2) & \hat{\varphi}(1) & \hat{\varphi}(0) & \cdots & \hat{\varphi}(5-n) & \hat{\varphi}(4-n) & \hat{\varphi}(3-n) \\ \hat{\varphi}(3) & \hat{\varphi}(2) & \hat{\varphi}(1) & \cdots & \hat{\varphi}(6-n) & \hat{\varphi}(5-n) & \hat{\varphi}(4-n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hat{\varphi}(n-3) & \hat{\varphi}(n-4) & \hat{\varphi}(n-5) & \cdots & \hat{\varphi}(0) & \hat{\varphi}(-1) & \hat{\varphi}(-2) \\ \hat{\varphi}(n-2) & \hat{\varphi}(n-3) & \hat{\varphi}(n-4) & \cdots & \hat{\varphi}(1) & \hat{\varphi}(0) & \hat{\varphi}(-1) \\ \hat{\varphi}(n-1) & \hat{\varphi}(n-2) & \hat{\varphi}(n-3) & \cdots & \hat{\varphi}(2) & \hat{\varphi}(1) & \hat{\varphi}(0) \end{pmatrix} \tag{11}$$

Example 3.1. (1) The matrix representation of shift operator A_z is given by

$$Ma(A_z) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{12}$$

(2) The matrix representation of backward shift operator $A_{\overline{z}}$ is given by

$$Ma(A_{\overline{z}}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

Proof. Let $\varphi \in K^2_{z^n}$. Under equation 10, we have

(1) $\varphi(z) = \sum_{k=0}^{n-1} \hat{\varphi}(k) z^k = \hat{\varphi}(1) z = z$. By identification, we get $\hat{\varphi}(1) = 1$ and $\hat{\varphi}(k) = 0$ for all$k \neq 1$. We obtain the matrix in equation 12.

(2) $\overline{\varphi(z)} = \sum_{k=0}^{n-1} \overline{\hat{\varphi}(k) z^k} = \sum_{k=0}^{n-1} \hat{\varphi}(-k) z^{-k} = \hat{\varphi}(-1) z^{-1} = \hat{\varphi}(-1)\overline{z} = \overline{z}$. By identification, we get $\hat{\varphi}(-1) = 1$ and $\hat{\varphi}(k) = 0$ for all $k \neq 1$and equation 13 is obtained.

In the following, we denote by M the matrix of A_z and by $N$the matrix of $A_{\overline{z}}$. The following result generalizes the matrix representation of a TTO as a function M and$N$.

Proposition 3.2. Let$\varphi \in K^2_{z^n}$. Then

$$Ma(A_\varphi) = \hat{\varphi}(0) I + \sum_{k=1}^{n-1} (\hat{\varphi}(k) M^k + \hat{\varphi}(-k) N^k) \tag{14}$$

Proof. For $M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$, it's not difficult to verify that $M^2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}$ … and

$M^{n-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$. Similarly, for $N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$ we obtain

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \dots \text{ and } \quad N^{n-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

So under equation 11, we have
$Ma(A_\varphi) = \hat{\varphi}(0)I + \hat{\varphi}(1)M + \hat{\varphi}(2)M^2 + \ldots + \hat{\varphi}(n-1)M^{n-1} + \hat{\varphi}(-1)N + \hat{\varphi}(-2)N^2 + \ldots + \hat{\varphi}(1-n)N^{n-1}$
$= \hat{\varphi}(0)I + \sum_{k=1}^{n-1}(\hat{\varphi}(k)M^k + \hat{\varphi}(-k)N^k)$ as desired.

Remark 3.3. (1) If $M = Ma(A_z)$ then $M^2 = Ma(A_{z^2})\ldots$ and $M^{n-1} = Ma(A_{z^{n-1}})$.
(2) If $N = Ma(A_{\overline{z}})$, then $N^2 = Ma\left(A_{\overline{z^2}}\right)$, … and $N^{n-1} = Ma\left(A_{\overline{z^{n-1}}}\right)$.

Let $\alpha \in Co$ such that $\alpha \neq 0$ and $\varphi \in K_{z^n}^2$. In this case $\varphi(z) = \sum_{k=0}^{n-1}\hat{\varphi}(k)z^k$ (see equation 10) and by equation 6, we have

$S_uC\varphi = u\overline{(\varphi - \phi(0))} = z^n\overline{(\varphi(z) - \varphi(0))} = z^n\sum_{k=1}^{n-1}\overline{\hat{\varphi}(k)}z^{-k} = \sum_{k=1}^{n-1}\overline{\hat{\varphi}(k)}z^{n-k} = \sum_{k=1}^{n-1}\overline{\hat{\varphi}(n-k)}z^k$ Then

$$\varphi + \alpha\overline{S_uC\varphi} = \sum_{k=0}^{n-1}\hat{\varphi}(k)z^k + \alpha\sum_{k=1}^{n-1}\hat{\varphi}(n-k)z^{-k} \tag{15}$$

The matrix representation of a TTO $A_{\varphi+\alpha\overline{S_{z^n}C\varphi}}$ of type α is given by

$$Ma\left(A_{\varphi+\alpha\overline{S_uC\varphi}}\right) = \begin{pmatrix} \hat{\varphi}(0) & \alpha\hat{\varphi}(n-1) & \alpha\hat{\varphi}(n-2) & \cdots & \alpha\hat{\varphi}(3) & \alpha\hat{\varphi}(2) & \alpha\hat{\varphi}(1) \\ \hat{\varphi}(1) & \hat{\varphi}(0) & \alpha\hat{\varphi}(n-1) & \cdots & \alpha\hat{\varphi}(4) & \alpha\hat{\varphi}(3) & \alpha\hat{\varphi}(2) \\ \hat{\varphi}(2) & \hat{\varphi}(1) & \hat{\varphi}(0) & \cdots & \alpha\hat{\varphi}(5) & \alpha\hat{\varphi}(4) & \alpha\hat{\varphi}(3) \\ \hat{\varphi}(3) & \hat{\varphi}(2) & \hat{\varphi}(1) & \cdots & \alpha\hat{\varphi}(6) & \alpha\hat{\varphi}(5) & \alpha\hat{\varphi}(4) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hat{\varphi}(n-3) & \hat{\varphi}(n-4) & \hat{\varphi}(n-5) & \cdots & \hat{\varphi}(0) & \alpha\hat{\varphi}n(-1) & \alpha\hat{\varphi}(n-2) \\ \hat{\varphi}(n-2) & \hat{\varphi}(n-3) & \hat{\varphi}(n-4) & \cdots & \hat{\varphi}(1) & \hat{\varphi}(0) & \alpha\hat{\varphi}(n-1) \\ \hat{\varphi}(n-1) & \hat{\varphi}(n-2) & \hat{\varphi}(n-3) & \cdots & \hat{\varphi}(2) & \hat{\varphi}(1) & \hat{\varphi}(0) \end{pmatrix} \tag{16}$$

The following proposition gives the general form of a matrix representation of an TTO of type α.
Proposition 3.4. Let $\alpha \in Co$ such that $\alpha \neq 0$ and $\varphi \in K_{z^n}^2$. Then

$$Ma(A_{\phi+\alpha\overline{S_uC\phi}}) = \hat{\phi}(0)I + \sum_{k=1}^{n-1}\left(\hat{\phi}(k)M^k + \alpha\hat{\phi}(n-k)N^{n-k}\right) \tag{17}$$

Proof. We pose $K = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \alpha \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$. So $K^2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \alpha \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}$… and

$K^{n-1} = \begin{pmatrix} 0 & \alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \alpha \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$. In the proof of the proposition 3.4, we have $K = M + \alpha N^{n-1}$, $K^2 = M^2 + \alpha N^{n-2}$,
… and $K^{n-1} = M^{n-1} + \alpha N$. So by equation 15 and 16, we have $Ma\left(A_{\varphi+\alpha\overline{SC\varphi}}\right) = \hat{\varphi}(0)I + \sum_{k=1}^{n-1}(\hat{\varphi}(k)M^k + \alpha\hat{\varphi}(n-k)N^{n-k})$ as desired.
The following example gives some matrix representations of TTOs of rank-one (TTOs of type $\alpha = u(\lambda)$).

Example 3.5. (1) Let $\lambda \in T$ and we think that u has an ADC at the pointλ. We have

$$Ma(k_\lambda^u \otimes k_\lambda^u) = \begin{pmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-3} & \lambda^{n-2} & \lambda^{n-1} \\ \overline{\lambda} & 1 & \lambda & \cdots & \lambda^{n-4} & \lambda^{n-3} & \lambda^{n-2} \\ \overline{\lambda^2} & \overline{\lambda} & 1 & \cdots & \lambda^{n-5} & \lambda^{n-4} & \lambda^{n-3} \\ \overline{\lambda^3} & \overline{\lambda^2} & \overline{\lambda} & \cdots & \lambda^{n-6} & \lambda^{n-5} & \lambda^{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \overline{\lambda^{n-3}} & \overline{\lambda^{n-4}} & \overline{\lambda^{n-5}} & \cdots & 1 & \lambda & \lambda^2 \\ \overline{\lambda^{n-2}} & \overline{\lambda^{n-3}} & \overline{\lambda^{n-4}} & \cdots & \overline{\lambda} & 1 & \lambda \\ \overline{\lambda^{n-1}} & \overline{\lambda^{n-2}} & \overline{\lambda^{n-3}} & \cdots & 0 & \overline{\lambda} & 1 \end{pmatrix} \tag{18}$$

(3) Let$\lambda \in D$. We obtain

$$Ma(Ck_\lambda^u \otimes k_\lambda^u) = \begin{pmatrix} \lambda^{n-1} & \lambda^n & \lambda^{n+1} & \cdots & \lambda^{2n-4} & \lambda^{2n-3} & \lambda^{2n-2} \\ \lambda^{n-2} & \lambda^{n-1} & \lambda^n & \cdots & \lambda^{2n-5} & \lambda^{2n-4} & \lambda^{2n-3} \\ \lambda^{n-3} & \lambda^{n-2} & \lambda^{n-1} & \cdots & \lambda^{2n-6} & \lambda^{2n-5} & \lambda^{2n-4} \\ \lambda^{n-4} & \lambda^{n-3} & \lambda^{n-2} & \cdots & \lambda^{2n-7} & \lambda^{2n-6} & \lambda^{2n-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda^2 & \lambda^3 & \lambda^4 & \cdots & \lambda^{n-1} & \lambda^n & \lambda^{n+1} \\ \lambda & \lambda^2 & \lambda^3 & \cdots & \lambda^{n-2} & \lambda^{n-1} & \lambda^n \\ 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-3} & \lambda^{n-2} & \lambda^{n-1} \end{pmatrix} \tag{19}$$

Proof. From [2] (example 5.3, page 14), these two operators are of type $\alpha = u(\lambda)$ so of type λ^n because $u(z) = z^n$ and for u has an ADC at the point $\lambda \in T$ then $k_\lambda^{z^n} \otimes k_\lambda^{z^n} = A_{k_\lambda^{z^n} + u(\lambda)\overline{S_{z^n} C k_\lambda^{z^n}}}$. We determine the Fourier coefficients of$k_\lambda^{z^n} + u(\lambda)\overline{S_{z^n} C k_\lambda^{z^n}}$. From equation 15, we get $k_\lambda^{z^n} + u(\lambda)\overline{S_{z^n} C k_\lambda^{z^n}} = \sum_{k=0}^{n-1} \hat{\varphi}(k) z^k + u(\lambda) \sum_{k=1}^{n-1} \hat{\varphi}(n-k)\, z^{-k}$So by equations 1, 7 and 3, we obtain

$$k_\lambda^{z^n} + u(\lambda)\overline{S_{z^n} C k_\lambda^{z^n}} = \frac{1-\overline{\lambda^n} z^n}{1-\overline{\lambda} z} + u(\lambda)\overline{\left(\lambda C k_\lambda^{z^n} - u(\lambda) k_0^{z^n}\right)}$$

$$= \frac{1-\overline{\lambda^n} z^n}{1-\overline{\lambda} z} + \lambda^n \overline{\left(\lambda C k_\lambda^{z^n} - \lambda^n k_0^{z^n}\right)}$$

$$= \frac{1-\overline{\lambda^n} z^n}{1-\overline{\lambda} z} + \lambda^n \overline{\left(\lambda \frac{z^n-\lambda^n}{z-\lambda} - \lambda^n\right)}$$

$$= \frac{1-\overline{\lambda^n} z^n}{1-\overline{\lambda} z} + \lambda^n \overline{\left(\frac{\lambda z^n - \lambda^{n+1} - \lambda^n z + \lambda^{n+1}}{z-\lambda}\right)}$$

$$= \frac{1-\overline{\lambda^n} z^n}{1-\overline{\lambda} z} + \lambda^n \overline{\left(\lambda z \frac{z^{n-1}-\lambda^{n-1}}{z-\lambda}\right)}$$

$$= 1 + \overline{\lambda} z + \overline{\lambda^2} z^2 + \ldots + \overline{\lambda^{n-1}} z^{n-1} + \lambda^n \left(\overline{\lambda z^{n-1}} + \overline{\lambda^2 z^{n-2}} + \ldots + \overline{\lambda^{n-1} z}\right)$$

$$= 1 + \overline{\lambda} z + \overline{\lambda^2} z^2 + \ldots + \overline{\lambda^{n-1}} z^{n-1} + \lambda^{n-1} \overline{z^{n-1}} + \lambda^{n-2} \overline{z^{n-2}} + \ldots + \lambda \overline{z}$$

By identification (see the matrix in equation 16), $\hat{\varphi}(0) = 1$, $\hat{\varphi}(1) = \overline{\lambda}$, $\hat{\varphi}(2) = \overline{\lambda^2}$, $\ldots\hat{\varphi}(n-1) = \overline{\lambda^{n-1}}$ and $\lambda^n \hat{\varphi}(n-1) = \lambda$, $\lambda^n \hat{\varphi}(n-2) = \lambda^2$, $\ldots\lambda^n \hat{\varphi}(1) = \lambda^{n-1}$. We get the matrix in equation 18.

(2) for all $\lambda \in D$, we get$C k_\lambda^{z^n} \otimes k_\lambda^{z^n} = A_{C k_\lambda^{z^n} + u(\lambda)\overline{S_{z^n} k_\lambda^{z^n}}}$ with $C k_\lambda^{z^n} + u(\lambda)\overline{S_{z^n} k_\lambda^{z^n}} = \sum_{k=0}^{n-1} \hat{\varphi}(k) z^k + u(\lambda) \sum_{k=1}^{n-1} \hat{\varphi}(n-k)\, z^{-k}$.

So under equations 1, 7 and 3, we have

$$C k_\lambda^{z^n} + u(\lambda)\overline{S_{z^n} k_\lambda^{z^n}} = \frac{z^n-\lambda^n}{z-\lambda} + \lambda^n \overline{\frac{1}{\overline{\lambda}}\left(k_\lambda^{z^n} - k_0^{z^n}\right)}$$

$$= \frac{z^n-\lambda^n}{z-\lambda} + \lambda^n \frac{1}{\lambda}\overline{\left(\frac{1-\lambda^n \overline{z^n}}{1-\lambda \overline{z}} - 1\right)}$$

$$= z^{n-1} + \lambda z^{n-2} + \lambda^2 z^{n-3} + \ldots + \lambda^{n-2} z + \lambda^{n-1} + \lambda^n \overline{z} + \lambda^{n+1} \overline{z^2} + \ldots + \lambda^{2n-2} \overline{z^{n-1}}$$

By identification (see the matrix in equation 16), $\hat{\varphi}(0) = \lambda^{n-1}$, $\hat{\varphi}(1) = \lambda^{n-2}$, $\hat{\varphi}(2) = \lambda^{n-3}$, $\hat{\varphi}(n-1) = 1$ and $\lambda^n \hat{\varphi}(n-1) = \lambda^n$

$\lambda^n \hat{\varphi}(n-2) = \lambda^{n+1}$, $\ldots\lambda^n \hat{\varphi}(1) = \lambda^{2n-2}$. We obtain the matrix in equation 19.

The following result generalizes the matrix of TTOs of rank-one to the power$k = 1,2, \ldots n$.

Theorem 3.6.

1. Let $\lambda \in T$. If $u = z^n$ has an ADC at the point λ, then
2.

$$Ma\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)^k = n^{k-1} Ma\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right) \tag{20}$$

with $k = 1,2,\ldots n$.

For $\lambda \in D$, then

$$Ma\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)^k = (n\lambda^{n-1})^{k-1} Ma\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right) \tag{21}$$

with $k = 1,2,\ldots n$.

Proof. 1. For $\lambda \in T$ and $u = z^n$ has an ADC at the point λ, we have

$$\begin{aligned}\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)^k &= \left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(k_\lambda^{z^n} \otimes k_\pi^{z^n}\right)^{k-2}\\
&= \langle k_\lambda^{z^n}, k_\lambda^{z^n}\rangle\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(k_\lambda^{z^n} \otimes k_\pi^{z^n}\right)^{k-2}\\
&= k_\lambda^{z^n}(\lambda)\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(k_\lambda^{z^n} \otimes k_\pi^{z^n}\right)^{k-2}\\
&= \left(k_\lambda^{z^n}(\lambda)\right)^2\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(k_\lambda^{z^n} \otimes k_\pi^{z^n}\right)^{k-3}\\
&= \ldots\\
&= \left(k_\lambda^{z^n}(\lambda)\right)^{k-1}\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\end{aligned}$$

From equation 1, $\left(k_\lambda^{z^n}(\lambda)\right)^{k-1} = \left(\frac{1-\overline{\lambda^n}\lambda^n}{1-\bar{\lambda}\lambda}\right)^{k-1} = \left(1 + \bar{\lambda}\lambda + \overline{\lambda^2}\lambda + \ldots + \overline{\lambda^{n-1}}\lambda^{n-1}\right)^{k-1} = n^{k-1}$ because $\bar{\lambda}\lambda = 1$.

Thus, $\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)^k = n^{k-1}\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)$ which implies $Ma\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)^k = n^{k-1} Ma\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)$.

3. For $\lambda \in D$, we get

$$\begin{aligned}\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)^k &= \left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(Ck_\lambda^{z^n} \otimes k_\pi^{z^n}\right)^{k-2}\\
&= \langle Ck_\lambda^{z^n}, k_\lambda^{z^n}\rangle\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(Ck_\lambda^{z^n} \otimes k_\pi^{z^n}\right)^{k-2}\\
&= Ck_\lambda^{z^n}(\lambda)\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\left(Ck_\lambda^{z^n} \otimes k_\pi^{z^n}\right)^{k-2}\\
&= \ldots\\
&= \left(Ck_\lambda^{z^n}(\lambda)\right)^{k-1}\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)\end{aligned}$$

By equation 3, $Ck_\lambda^u\ (\lambda) = \frac{u(\lambda)-u(\lambda)}{\lambda-\lambda} = u'(\lambda) = n\lambda^{n-1}$ Thus, $\left(Ck_\lambda^{z^n}(\lambda)\right)^{k-1} = (n\lambda^{n-1})^{k-1}$ which implies $Ma\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)^k = (n\lambda^{n-1})^{k-1} Ma\left(Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}\right)$ and the result follows.

4. Numerical ranges and numerical radius of TTOs in the case where u = z^n

In this section, we denote by $W(A_\varphi)$: the numerical range of TTOs and $w(A_\varphi)$ its numerical radius. Since, $u = z^n$ we have K_u^2 is finite dimensional and by remark 2.8, we get $W(A_{\varphi_1} + A_{\varphi_2}) = W(A_{\varphi_1}) + W(A_{\varphi_2})$ for all TTOs A_{φ_1} and A_{φ_2}. Let $f \in K_{z^n}^\infty$ such that $f = (a_0, a_1, \ldots, a_{n-1})$ and $\|f\| = 1$ where $a_k \in Co$ with $k = 1,2,\ldots n-1$. So $\|f\| = |(a_0, a_1, \ldots, a_{n-1})| = \sqrt{\sum_{k=0}^{n-1}|a_k|^2} = 1$ and
$W(A_\varphi) = \left\{\langle A_\varphi f, f\rangle : \|f\| = \|(a_0, a_1, \ldots, a_{n-1})\| = \sqrt{\sum_{k=0}^{n-1}|a_k|^2} = 1\right\}$.
The following result gives the numerical range of shift operator and its adjoint.

Lemma 4.1. For the shift operator A_z and its adjoint $A_{\bar{z}}$, we have

$$W(A_z) = W(A_{\bar{z}}) \tag{22}$$

Proof. By equations 4 and 12, we obtain $W(A_z) = \left\{\langle Mf, f\rangle : \|f\| = \|(a_0, a_1, \ldots, a_{n-1})\| = \sqrt{\sum_{k=0}^{n-1}|a_k|^2} = 1\right\}$ with

$$\begin{aligned}\langle Mf, f\rangle &= \left\langle \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 0 & \cdots & 1 & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3\\ \vdots\\ a_{n-3}\\ a_{n-2}\\ a_{n-1} \end{pmatrix}, \begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3\\ \vdots\\ a_{n-3}\\ a_{n-2}\\ a_{n-1} \end{pmatrix} \right\rangle = a_0a_1 + a_1a_2 + a_2a_3 + \ldots + a_{n-3}a_{n-2} + a_{n-2}a_{n-1}\\
&= \sum_{k=0}^{n-2} a_k a_{k+1}\end{aligned}$$

Similarly, we have $W(A_{\overline{z}}) = \left\{\langle Nf, f\rangle : \|f\| = \|(a_0, a_1, \ldots, a_{n-1})\| = \sqrt{\sum_{k=0}^{n-1}|a_k|^2} = 1\right\}$ and using equation 13, we obtain

$$\langle Nf, f\rangle = \langle \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{pmatrix}, \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{pmatrix} \rangle = a_1a_0 + a_2a_1 + a_3a_2 + \ldots + a_{n-2}a_{n-3} + a_{n-1}a_{n-2}$$

$$= \sum_{k=0}^{n-2} a_k a_{k+1}$$

and result follows.

The following result characterizes the numerical range of a TTO.

Lemma 4.2. Let A_φ a TTO of symbolφ. Then its numerical range is given by

$$W(A_\varphi) = \hat{\varphi}(0) + \textstyle\sum_{k=1}^{n-1}(\hat{\varphi}(k) + \hat{\varphi}(-k))W(M^k) \tag{23}$$

Proof. By proposition 2.7 and equation 14, we have

$$W(A_\varphi) = \hat{\varphi}(0) + \sum_{k=1}^{n-1}(\hat{\varphi}(k) + \hat{\varphi}(-k))W(M^k)$$

$$= \hat{\varphi}(0) + \textstyle\sum_{k=1}^{n-1}(\hat{\varphi}(k)W(M^k) + \hat{\varphi}(-k)W(N^k))$$

From equation 22, we get $W(M) = W(N)$ which implies$W(M^2) = W(N^2)$, … and $W(M^{n-1}) = W(N^{n-1})$. Thus, $W(A_\varphi) = \hat{\varphi}(0) + \sum_{k=1}^{n-1}(\hat{\varphi}(k) + \hat{\varphi}(-k))W(M^k)$ as desired.

Theorem 4.3. The numerical range of an TTOA_φ of symbol φ is a disk of radius

$$r = |\hat{\varphi}(0)| + \textstyle\sum_{k=1}^{n-1}|\hat{\varphi}(k) + \hat{\varphi}(-k)| \tag{24}$$

Proof. Under equation 23, we obtain $W(A_\varphi) = \hat{\varphi}(0) + \sum_{k=1}^{n-1}(\hat{\varphi}(k) + \hat{\varphi}(-k))W(M^k)$and in the proof of lemma 4.1, we get $W(M) = \{\sum_{k=0}^{n-2} a_k\, a_{k+1} : \sum_{k=0}^{n-1}|a_k|^2 = 1\}$. Since

$$\langle M^2 f, f\rangle = \langle \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{pmatrix}, \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{pmatrix} \rangle = a_0a_2 + a_1a_3 + \ldots + a_{n-3}a_{n-1} = \sum_{k=0}^{n-3} a_k a_{k+2}$$

… and $\langle M^{n-1} f, f\rangle = \langle \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{pmatrix}, \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{pmatrix} \rangle = a_0 a_{n-1}$ we obtain

$\langle A_\varphi f, f\rangle = \hat{\varphi}(0) + (\hat{\varphi}(1) + \hat{\varphi}(-1))\sum_{k=0}^{n-2} a_k a_{k+1} + (\hat{\varphi}(2) + \hat{\varphi}(-2))\sum_{k=0}^{n-3} a_k a_{k+2} + \ldots + (\hat{\varphi}(n-1) + \hat{\varphi}(1-n))a_0 a_{n-1}$.

Using the fact $|a_k||a_{k+1}| \leq \frac{1}{2}(|a_k|^2 + |a_{k+1}|^2)$ and $\sum_{k=0}^{n-1} |a_k|^2 = 1$, we have $\sum_{k=0}^{n-p-1}|a_k|\,|a_{k+p}| \leq 1$ with $p = 1,2,\ldots,n-1$.

Then $\left|\langle A_\varphi f,f\rangle\right| \le |\hat{\varphi}(0)| + |(\hat{\varphi}(1)+\hat{\varphi}(-1))| + |(\hat{\varphi}(2)+\hat{\varphi}(-2))| + \ldots + |(\hat{\varphi}(n-1)+\hat{\varphi}(1-n))|$ as desired.
We have the following corollary.

Corollary 4.4. Let A_φ a TTO of symbolφ. Then the numerical radius of A_φ is given by the inequality

$$w(A_\varphi) \le |\hat{\varphi}(0)| + \sum_{k=1}^{n-1}|\hat{\varphi}(k)+\hat{\varphi}(-k)|\cos^k\frac{\pi}{n+1} \qquad (25)$$

Proof. Under equation 9, we have$w(M) \le \cos\frac{\pi}{n+1}$. Thus, from equation 8, we obtain $w(M^k) \le \cos^k\frac{\pi}{n+1}$. By equation 5, we get $w(A_\varphi) = sup\{|\lambda|: \lambda \in W(A_\varphi)\}$ and by equation 23, we have $w(A_\varphi) \le |\hat{\varphi}(0)| + \sum_{k=1}^{n-1}|\hat{\varphi}(k)+\hat{\varphi}(-k)|w(M^k) \le |\hat{\varphi}(0)| + \sum_{k=1}^{n-1}|\hat{\varphi}(k)+\hat{\varphi}(-k)|\cos^k\frac{\pi}{n+1}$ as desired.

The following results characterize the numerical ranges and numerical radius of TTOs of typeα.

Theorem 4.5 The numerical range of an of TTO of type $\alpha \in Co$ such that $\alpha \ne 0$ is given by
$W(A_{\varphi+\alpha\overline{S_{z^n}C\varphi}}) = \hat{\varphi}(0) + (1+\alpha)\sum_{k=1}^{n-1}\hat{\varphi}(k)W(M^k)$ (26)

Proof. Using equation 17, we have$W(A_{\varphi+\alpha\overline{S_{z^n}C\varphi}}) = \hat{\varphi}(0) + \sum_{k=1}^{n-1}(\hat{\varphi}(k)W(M^k) + \alpha\hat{\varphi}(n-k)N^{n-k})$. Under equation 22, we obtain $W(A_{\varphi+\alpha\overline{S_{z^n}C\varphi}}) = \hat{\varphi}(0) + \sum_{k=1}^{n-1}(\hat{\varphi}(k)W(M^k) + \alpha\hat{\varphi}(n-k)M^{n-k}) = \hat{\varphi}(0) + (1+\alpha)\sum_{k=1}^{n-1}\hat{\varphi}(k)W(M^k)$ as desired.
The numerical radius of an TTO of type α is given by the following inequality.

Corollary 4.6. The numerical radius of a TTO of type $\alpha \in Co$ such that $\alpha \ne 0$ is given by

$$w(A_{\varphi+\alpha\overline{S_{z^n}C\varphi}}) \le |\hat{\varphi}(0)| + (1+|\alpha|)\sum_{k=1}^{n-1}|\hat{\varphi}(k)|\cos^k\frac{\pi}{n+1} \qquad (27)$$

Proof. The proof is immediate using equations 5, 9, 8 and 26.

The following result gives us generalizations of the numerical ranges of rank-one TTOs.

Theorem 4.7. The numerical ranges of rank-one TTOs are discs of radius$2n-1$.
Proof. (1) Let $\lambda \in T$. We assume that u has an ADC at the pointλ. We know that $k_\lambda^{z^n} \otimes k_\lambda^{z^n}$ is of type $u(\lambda)$ (see remark 2.6). So by equation 26, we have$W\left(k_\lambda^{z^n} \otimes k_\lambda^{z^n}\right) = \hat{\varphi}(0) + (1+u(\lambda))\sum_{k=1}^{n-1}\hat{\varphi}(k)W(M^k) = \hat{\varphi}(0) + (1+\lambda^n)\sum_{k=1}^{n-1}\hat{\varphi}(k)W(M^k)$. Thus,

$\langle k_\lambda^{z^n} \otimes k_\lambda^{z^n} f,f\rangle = \hat{\varphi}(0) + (1+\lambda^n)(\hat{\varphi}(1)\sum_{k=0}^{n-2}a_k a_{k+1} + \hat{\varphi}(2)\sum_{k=0}^{n-3}a_k a_{k+2} + \ldots + \hat{\varphi}(n-1)a_0 a_{n-1})$ (See the proof of theorem 4.3). Since$\sum_{k=0}^{n-p-1}|a_k|\left|a_{k+p}\right| \le 1$, we obtain $\left|\langle k_\lambda^{z^n} \otimes k_\lambda^{z^n} f,f\rangle\right| \le |\hat{\varphi}(0)| + 2(|\hat{\varphi}(1)| + |\hat{\varphi}(2)| + \ldots + |\hat{\varphi}(n-1)|)$ because$|\lambda| = 1$. By equation 18, we get $\hat{\varphi}(0) = 1$, $\hat{\varphi}(1) = \overline{\lambda}$,$\hat{\varphi}(2) = \overline{\lambda^2}$, …and $\hat{\varphi}(n-1) = \overline{\lambda^{n-1}}$. Therefore, $\left|\langle k_\lambda^{z^n} \otimes k_\lambda^{z^n} f,f\rangle\right| \le 1 + 2\left(|\lambda| + \left|\overline{\lambda^2}\right| + \ldots + \left|\overline{\lambda^{n-1}}\right|\right) \le 1 + 2(n-1) = 2n-1$.

(2)) Let $\lambda \in D$. Under remark 2.6, we have $Ck_\lambda^{z^n} \otimes k_\lambda^{z^n}$ is of type $u(\lambda)$, according to the above , we obtain $\left|\langle Ck_\lambda^{z^n} \otimes k_\lambda^{z^n} f,f\rangle\right| \le |\hat{\varphi}(0)| + 2(|\hat{\varphi}(1)| + |\hat{\varphi}(2)| + \ldots + |\hat{\varphi}(n-1)|)$. By equation 19, we get$\hat{\varphi}(0) = \lambda^{n-1}$, $\hat{\varphi}(1) = \lambda^{n-2}$,$\hat{\varphi}(2) = \lambda^{n-3}$, …and $\hat{\varphi}(n-1) = 1$. Then, $\left|\langle k_\lambda^{z^n} \otimes k_\lambda^{z^n} f,f\rangle\right| \le |\lambda^{n-1}| + 2(|\lambda^{n-2}| + |\lambda^{n-3}| + \ldots + 1) \le 1 + 2(n-1) = 2n-1$.

5. Conclusion

From the matrices of the compression on model space K_u^2 of shift operator denoted A_z and its adjoint denoted $A_{\overline{z}}$, we obtain general formulas concerning numerical ranges and numerical radius of the truncated Toeplitz operator A_φ with symbol φ and the truncated Toeplitz operator of type αcalled $A_{\varphi+\alpha\overline{S_{z^n}C\varphi}}$ with symbol $\varphi + \alpha\overline{S_{z^n}C\varphi}$ in the case the inner function $u = z^n$. These results are important in engineering science and quantum physics.

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