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Research Article

Steiner Domination in Fuzzy Graphs

G. Priscilla Pacifica[1], J. Jenit Ajitha[2]

[1]*Department of Mathematics, St.Marys' College(Autonomous), Thoothukudi*
[2] *Department of Mathematics, St.Marys' College(Autonomous)(Affiliated under Manonmanium Sundaranar University, Tirunelveli)*

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Abstract *- Main objective of the article is to introduce Steiner and upper Steiner domination in fuzzy graphs with real life application. We define fuzzy Steiner set for fuzzy graphs with isolated nodes and verified the mandatory and adequate condition for a fuzzy Steiner set to be minimal. We determine fuzzy Steiner dominating numbers in fuzzy graphs. Also we examine some of its characteristics.. The relationship between the fuzzy Steiner domination number of a fuzzy graph and the complement graph is obtained.*

Keywords *- Fuzzy Steiner set, Minimal fuzzy Steiner set, Fuzzy Steiner domination, Fuzzy upper Steiner domination, Fuzzy Steiner domination number.*

1. Introduction

The problem we often face in many real life situations is uncertainty or fuzziness. The concept of fuzziness has been applied in each and every field. Fuzzy graph theory introduced by Rosenfeld has many real life applications. Several fuzzy graph theoretic concepts has been studied from [6] and [7]. In [1] and [2] the authors discussed the concept of domination and fuzzy graphs. Steiner domination in crisp graphs has been studied from [3], [4] and [5]. This paper introduces Steiner and upper Steiner domination in fuzzy graphs. In this article we use the expressions 'nodes' and 'arcs' for vertices and edges respectively. Also we consider the fuzzy graphs which are connected and those with isolated nodes and assume that node set is finite and non-empty. If S is a non-empty set of nodes of G, then the Steiner tree of S is defined to be a minimal connected sub graph in which the node set contains S. Steiner interval, I(S) of S is defined to be I(S) = {u ∈ V (G) / u lies in atleast one Steiner tree of S}.If the steiner interval of S equals the node set of G then S is said to be a Steiner set for G. A fuzzy graph is defined as a fuzzy relation μ on a fuzzy subset σ denoted by G (V, σ, μ) where V is a node set which satisfies μ(u,v) ≤ σ(u)∧σ(v) ∀ u,v ∈ V. An arc is called effective arc if μ(u,v) = σ(u)∧σ(v) for all u,v. For a fuzzy graph G, the fuzzy neighbourhood of a node u in V(G) is defined as $N^f(u) = \{v \epsilon V(G)/\mu(u,v) > 0\}$. If G (V, σ, μ) is a connected fuzzy graph the order and size of G denoted by p and q are defined as $p = \sum_{u\in V(G)} \sigma(u)$ and $q = \{\sum \mu(u,v) \,|\mu(u,v) > 0\}$. For a fuzzy graph G(V, σ, μ), its complement fuzzy graph denoted by $\bar{G}(V,\bar{\sigma},\bar{\mu})$ is defined as $\bar{\sigma} = \sigma$ and $\bar{\mu}(u,v) = \begin{cases} 0, & if \mu(u,v) > 0 \\ \sigma(u) \wedge \sigma(v), & otherwise \end{cases}$

2. Fuzzy Steiner Domination

We define fuzzy Steiner set for a fuzzy graph with isolated nodes.

2.1. Definition

If G is a fuzzy graph which has isolated nodes, then a fuzzy Steiner set of G is defined as $S = S' \cup I_G$ where S' is the Steiner set of the connected subgraph induced by the nodes of V(G)-I_G. I_G is the set of all isolated nodes of G. A fuzzy Steiner set S, the non-empty set of nodes of a fuzzy graph G is said to be minimal steiner set if no proper subset of S is a fuzzy Steiner set of G.

2.2. Observation

By definition, the number of minimal fuzzy Steiner sets of a fuzzy graph G is the same as the number of minimal fuzzy Steiner sets of $\langle V(G) - I_G \rangle$. Also each minimal fuzzy Steiner set of G contains all the isolated nodes of G.

2.3. Definition

A fuzzy Steiner dominating set S of a fuzzy graph G(V,σ,μ) is minimal if no proper subset of S is a fuzzy Steiner dominating set. The minimum fuzzy cardinality of a minimal fuzzy Steiner dominating set is called fuzzy Steiner dominating number denoted by γ^{fs} and the corresponding minimal fuzzy Steiner dominating set is called γ^{fs}-set. The maximum fuzzy cardinality of a minimal fuzzy Steiner dominating set is called fuzzy upper dominating number denoted by Γ^{fs} and the corresponding minimal fuzzy Steiner dominating set is called the Γ^{fs}-set.

2.4. Observations

- For a fuzzy graph G, any minimal fuzzy Steiner set does not contains a fuzzy cut node. Also any minimal fuzzy Steiner set contains all the end nodes.
- If G is a fuzzy graph, clearly $\gamma^f \leq \gamma^{fs}$ where γ^f is the fuzzy dominating number which is the minimum fuzzy cardinality of a minimal fuzzy dominating set. If G is a fuzzy graph, then any super set of a fuzzy Steiner set need not be a fuzzy Steiner set.

Next we shall prove the necessary and sufficient condition for a fuzzy Steiner set to be minimal.

2.5. Theorem (Necessary and sufficient condition for minimality)

A fuzzy Steiner set S of a fuzzy graph G is minimal iff for each node u in S any one the following conditions hold a) $N^f(u) \cap S = \emptyset$ b) A node v in S exists such that $N^f(u) \cap S = \{v\}$

Proof

Let S be a non-empty minimal fuzzy Steiner set of G. Let u be any node in G.

Case (i) $u \in S$ is an isolated node in S. In this case, we have $N^f(u) \cap S = \emptyset$.
Case (ii) $u \in S$ is not an isolated node in S. We have to show that there exists a node v in S such that $N^f(u) \cap S = \{v\}$. We shall prove this by contradiction. Suppose there exists two or more nodes in $N^f(u) \cap S$. Let $N^f(u) \cap S = \{v, w\}$ then by removing the node which has more neighbors among the nodes u, v and w from S we get a fuzzy Steiner set which is a contradiction to the minimality of S. Hence no node in S can have more than one neighbor in S. Thus either $N^f(u) \cap S = \emptyset$ or $N^f(u) \cap S = \{v\}$ for all nodes u in S. The general case follows by induction on the number of nodes for $N^f(u) \cap S$.

Conversely suppose any one of the above conditions hold. Let us assume that S is not minimal. Then there exists a node v in S such that S-{v} is a fuzzy Steiner set of G. So v is in some fuzzy Steiner tree of S-{v}. Hence v has a neighbor in S-{v} By observation, all end nodes should be in a fuzzy Steiner set, v is not an end node. Thus v must have another neighbor in S-{v} say y .Therefore $N^f(v) \cap S = \{x, y\}$ which is a contradiction. Hence S must be a minimal fuzzy Steiner set of G.

2.6. Example

Consider the following connected fuzzy graph

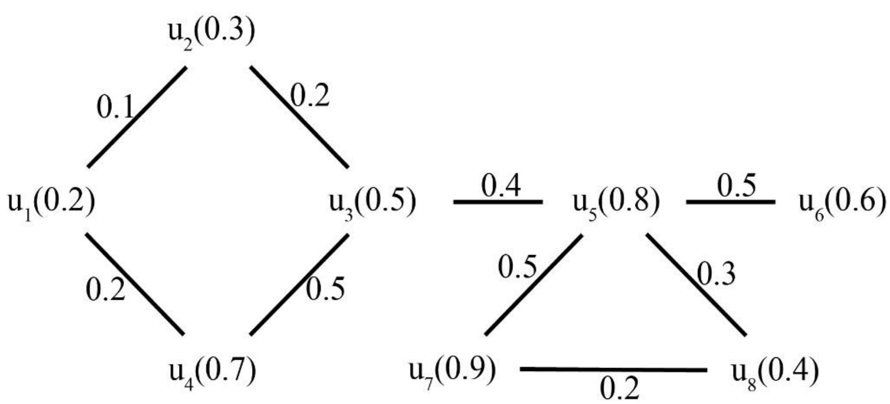

Fig. 1

In the above fuzzy graph not all the arcs are strong. Here the set $M = \{u_1, u_6, u_7, u_8\}$ is a minimal fuzzy Steiner set but not a fuzzy dominating set while super sets of M, $M_1 = \{u_1, u_2, u_6, u_7, u_8\}$ and $M_2 = \{u_1, u_4, u_6, u_7, u_8\}$ are not fuzzy Steiner sets. The fuzzy Steiner set $S_1 = \{u_1, u_3, u_6, u_7, u_8\}$ is a minimal steiner dominating set and the steiner set $S_2 = \{u_1, u_5, u_6, u_7, u_8\}$ is not a dominating set because the node u_2 has no strong neighbor in the set S_2. The fuzzy Steiner dominating number for the above graph is $\gamma^{fs} = 2.6$. Hence the set S_1 is the only minimum fuzzy Steiner dominating set athe fuzzy upper Steiner dominating number is $\Gamma^{fs} = 2.6$. The sets $U_1 = \{u_1, u_2, u_3, u_5, u_6, u_7, u_8\}$ and $U_2 = \{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}$ are minimal fuzzy total Steiner dominating sets.

2.7. Proposition

If G is a fuzzy graph of order p then $\gamma^{fs} \leq p$, where p is the order of G.

2.8. Theorem

If G is a connected non-complete strong fuzzy graph then
$p - q \leq \gamma^{fs}(G) \leq p - \delta_E$. Here p is the order, q denotes the size and δ_E denotes the minimum incident degree of G.

***Proof* :**

To prove the lower bound of the inequality, consider S the minimum steiner dominating set and γ^{fs} its fuzzy cardinality. Here G is a strong fuzzy graph. Each node of G has atmost n-1 arcs incident with it and each arc has membership value less than or equal to that of weight of any of its incident nodes.

It is clear that $p \leq \gamma^{fs} + q$. Hence $p - q \leq \gamma^{fs}$. If 'u' is the node with minimum effective incident degree $\delta_E(G)$ and since G is a connected non-complete fuzzy graph, $\gamma^{fs} < p$. The fuzzy cardinality of the minimum steiner dominating set is less or equal to the fuzzy cardinality of $V - \{u\}$.
Thus $\gamma^{fs} \leq p - \delta_E(G)$. Hence the upper bound holds.

2.8.1. Corollary

Suppose all the nodes of G have same weights, then
$p - q \leq \gamma^{fs}(G) \leq p - \Delta_E$.

2.9. Theorem

For a complete strong fuzzy graph with 'n' nodes, $\gamma^{fs} = p = \sum_{u \in V(K_n^f)} \sigma(u)$.

Proof :

Let K_n^f be a complete fuzzy graph with n nodes. Then for any two nodes u and v are strong neighbors, $\mu(u, v) > 0$. If S is any non empty set of nodes with k < n nodes, then each fuzzy Steiner tree of S contains only the nodes in S. Hence S is not a fuzzy Steiner set of K_n^f. Therefore the only fuzzy Steiner set for K_n^f is the node set $V(K_n^f)$. Thus $\gamma^{fs} = \sum_{u \in V(K_n^f)} \sigma(u)$.

2.10. Theorem

If $K_{m,n}^f$ is a complete bipartite strong fuzzy graph $K_{m,n}^f$ where m, n $\geq$ 2 with, $\gamma^{fs}(K_{m,n}^f) = \min\left(\sum_{u \in V_1} \sigma(u), \sum_{v \in V_2} \sigma(v)\right)$ where V_1 and V_2 are the partite node sets with m and n nodes respectively.

Proof :

For m = n = 1, the graph becomes K_2 for which $\gamma^{fs} = p$ where p is the order of K_2.

For m=1 and $n \geq 2$, $\gamma^{fs} = \sum_{v \in V_2} \sigma(v)$.

Now assume that m, $n \geq 2$. If S is a non-empty proper subset of $V_1(K_{m,n}^f)$, then the Steiner interval of S, I(S) contains all the nodes of $V_2(G)$. But I(S) do not contain the nodes of $V_1\left(K_{m,n}^f\right) - S$. Hence there are only two minimal fuzzy Steiner dominating sets $V_1(K_{m,n}^f)$ and $V_2(K_{m,n}^f)$ for $K_{m,n}^f$. Since γ^{fs} is the minimum fuzzy cardinality of a minimal Steiner fuzzy dominating set, $\gamma^{fs} = \min\left(\sum_{u \in V_1} \sigma(u), \sum_{v \in V_2} \sigma(v)\right)$.

2.10.1. Corollary

For a complete bipartite fuzzy graph $K_{m,n}^f$ where m, n ≥ 2, $\Gamma^{fs} = max\left(\sum_{u \in V_1} \sigma(u), \sum_{v \in V_2} \sigma(v)\right)$ where V_1 and V_2 are the partite node sets with m and n nodes respectively.

2.11. Theorem(Nordhaus Gaddum result for fuzzy Steiner domination)

If G is a fuzzy graph, $\gamma^{fs} + \overline{\gamma^{fs}} \leq 2p$ where p is the order of the graph G.
This result is a direct implication of the above proposition.

2.11.1. Corollary

If G is a fuzzy graph, $\Gamma^{fs} + \overline{\Gamma^{fs}} \leq 2p$ where p is the order of the graph G.

2.12. Example

Let us consider the fuzzy graph and the complement fuzzy graph

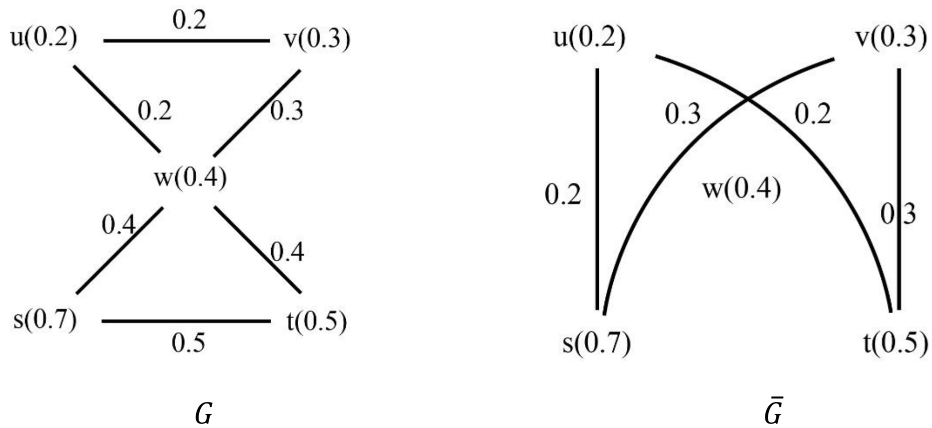

Fig. 2

In this example the graph G is a connected fuzzy graph while its complement $\bar{G}$ is a disconnected fuzzy graph with isolated node w. Hence each minimal fuzzy Steiner set of $\bar{G}$ contains the node w. The set S={u,v,s,t} is the only minimal fuzzy Steiner dominating set of G. But S is not a minimal fuzzy dominating set of G. Hence $\gamma^{fs} = \Gamma^{fs} = 1.7$. Now for the complement fuzzy graph $\bar{G}$, the minimal fuzzy Steiner sets are $S_1 = \{u, v, w\}$ and $S_2 = \{s, t, w\}$. Both S_1 and S_2 are fuzzy dominating sets and hence are minimal fuzzy Steiner dominating sets.

$\overline{\gamma^{fs}} = \min\{0.9, 1.6\} = 0.9$, $\overline{\Gamma^{fs}} = \max\{0.9, 1.6\} = 1.6$.

3. Result and Discussion

Fuzzy Steiner dominating sets have application background in various fields, especially in communication networks. For example suppose one have to find the locations to fix minimum number of closed circuit television(CCTV) cameras in a supermarket such that the whole area can be monitored effectively. In fig 3, a sample blueprint of a simple supermarket is given. We divide the whole area into smaller areas as given in fig 4. This may be designed into a fuzzy graph by representing the smaller areas as nodes and adjacent nodes are linked by an arc. The membership grade values can be given as follows. Higher membership grade is given to smaller area and lowest membership grade is given to larger area as in fig 4. The membership value increases as area decreases and vice-versa.

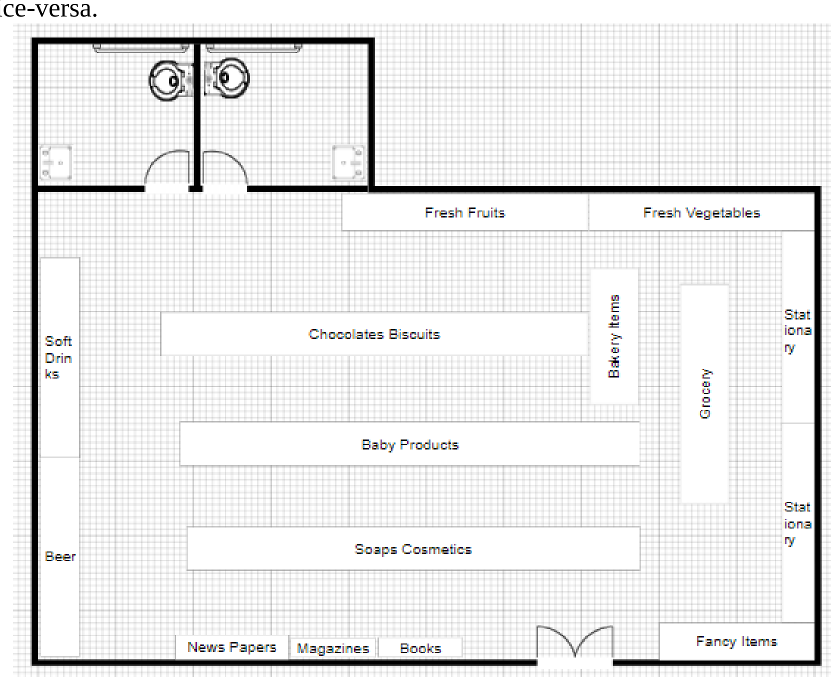

Fig. 3

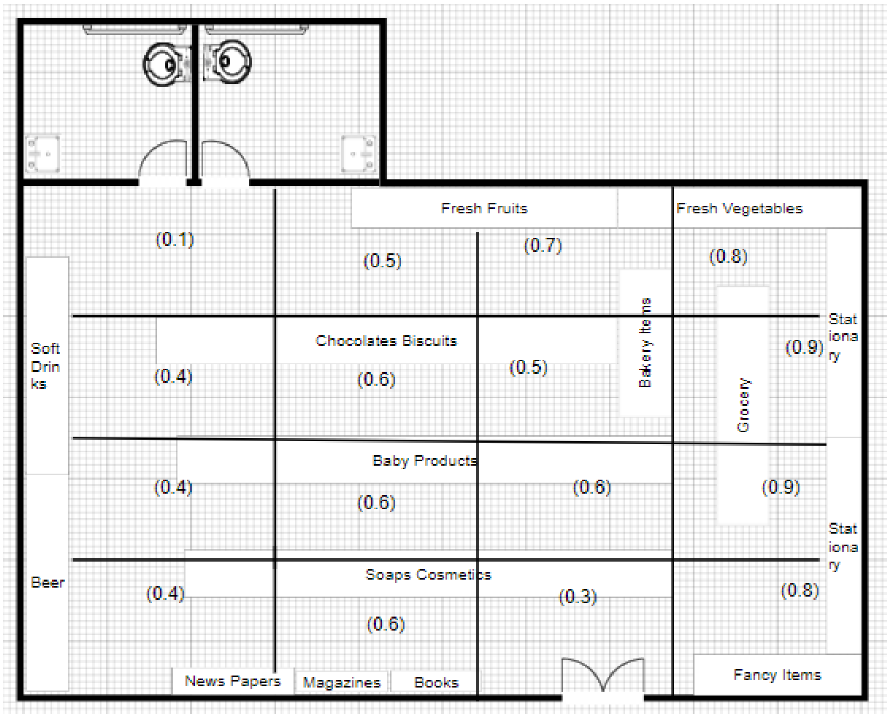

Fig. 4

The fuzzy graph modelled is depicted in fig 5. All the arcs are considered as effective arcs. Here the minimal fuzzy Steiner dominating sets are $M_1 = \{S_1, S_4, S_6, S_9, S_{11}, S_{14}, S_{16}\}$, $M_2 = \{S_2, S_4, S_5, S_7, S_{10}, S_{13}, S_{16}\}$, $M_3 = \{S_1, S_3, S_6, S_8, S_{11}, S_{13}, S_{16}\}$ and $M_4 = \{S_1, S_4, S_7, S_{10}, S_{12}, S_{13}, S_{15}\}$. The minimum fuzzy Steiner dominating set is M_4 with fuzzy Steiner dominating number 3.6 and maximum fuzzy Steiner dominating set is M_3 with fuzzy upper Steiner dominating number 4.1. The locations in the set M_4 can be selected to fix CCTV cameras as given in fig 6.

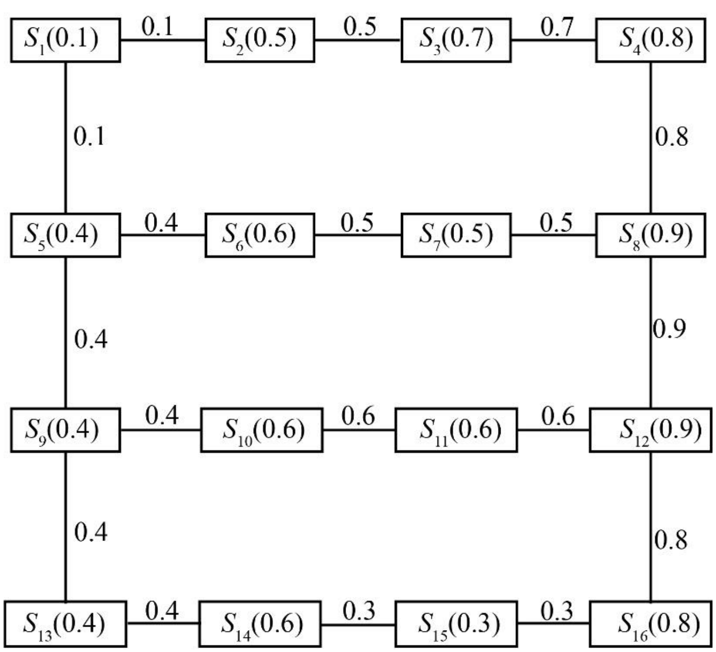

Fig. 5

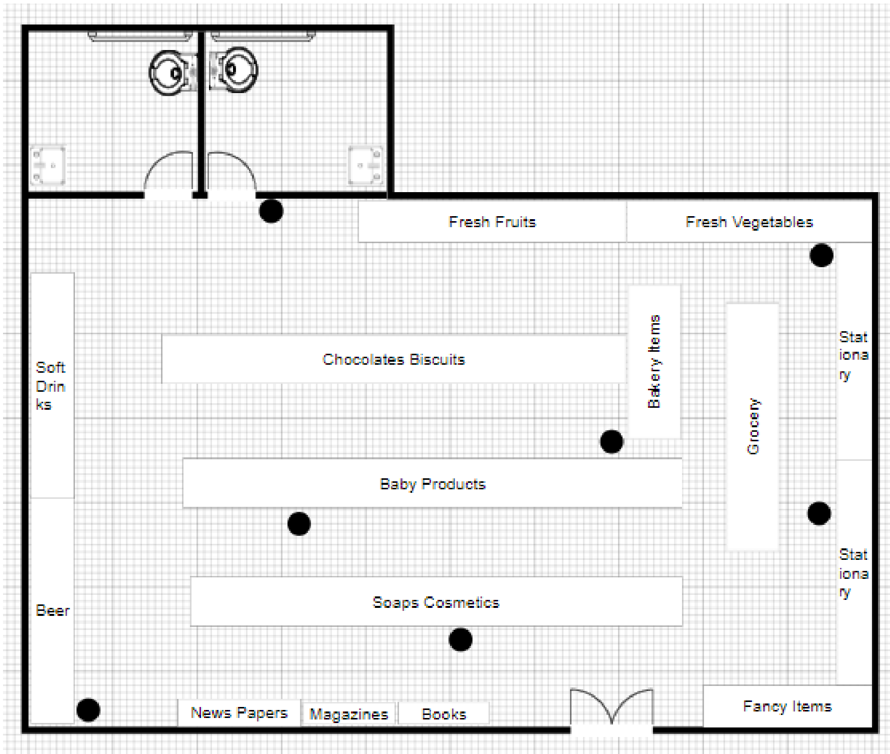

Fig. 6

4. Conclusion

This article introduced and established some new results on fuzzy Steiner domination. Also we discussed a real life application of fuzzy Steiner dominating set. The fuzzy Steiner domination also have various applications particularly in communication networks.

References

[1] Anjali Narayanan, and Sunil Mathew, "Energy of a Fuzzy Graph," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 3, pp. 455-465, 2013.

[2] R. Balakrishnan, and K. Ranganathan, *A Textbook of Graph Theory*, Springer, New York, 2000.

[3] Benjier H.Arriola, "A Note on Some Domination of Fuzzy Graphs," *Far East Journal of Applied Mathematics*, vol. 97, no. 4, pp. 113-123, 2017. *Crossref*, http://dx.doi.org/10.17654/AM097040113

[4] John Adrian Bondy, and U.S.R Murty, *Graph Theory with Applications*, The MacMillan Press Ltd, 1976.

[5] F. Buckley, and F. Harary, *Distance in Graphs*, Addison-Wesley Publishing Company, Inc., 1990.

[6] Kiran R. Bhutani, and Azriel Rosenfeld, "Strong Arcs in Fuzzy Graphs," *Information Sciences*, vol. 152, pp. 319-322, 2003. *Crossref*, https://doi.org/10.1016/S0020-0255(02)00411-5

[7] Gary Chartrand, and Ping Zhang, "The Steiner Number of a Graph," *Discrete Mathematics*, vol. 242, no. 1-3, pp. 41-54, 2002. *Crossref*, https://doi.org/10.1016/S0012-365X(00)00456-8

[8] Gary Chartrand, and Ping Zhang, *Introduction to Graph Theory*, Tata McGrawHill Edition, 2006.

[9] Clemens Barand, and Norbert Seifter, "Eigen Values and Domination in Graphs," *Mathematica Slovoca*, vol. 46, no. 1, pp. 33-39, 1996.

[10] Ivan Gutman, "The Energy of a Graph," *Approx Math Extra Research Z Graz*, vol. 103, pp. 1–22, 1978.

[11] F.Harary, and R.A.Melter, "On the Metric Dimension of a Graph," *Ars Combinatoria*, vol. 2, pp. 191-195, 1976.

[12] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, New York, 1969.

[13] Teresa W. Haynes, Stephen T. Hedetneimi, and Peter J. Slater, *Fundamentals of Domination in Graphs*, CRC Press, New York, 1998. *Crossref*, https://doi.org/10.1201/9781482246582

[14] Teresa W.Haynes, Stephen T. Hedetneimi, and Peter J.Slater, *Domination in Graphs: Advanced Topics*, Marcel Deker, New York, 1998.

[15] Michael Raines, and Ping Zhang, "The Steiner Distance Dimension of Graphs," *Australasian Journal of Combinatorics*, vol. 20, pp. 133-143, 1999.

[16] K.Ramalakshmi, and K.Palani, "On Steiner Domination Number of Graphs," *International Journal of Mathematics Trends and Technology*, vol. 41, no. 2, pp. 186-190, 2017. *Crossref*, https://doi.org/10.14445/22315373/IJMTT-V41P517

[17] Azriel Rosenfeld, "Fuzzy Graphs," *Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press*, pp. 77-95, 1975.

[18] Sadegh Rahimi Sharbaf, and Fatmem Fayazi, "Laplacian Energy of a Fuzzy Graph," *Iranian Journal of Mathematical Chemistry*, vol. 5, no. 1, pp 1-10, 2014. *Crossref*, https://doi.org/10.22052/ijmc.2014.5214

[19] Saied Alikhani, and Yee-hock Peng, "Introduction to Domination Polynomial of a Graph," *Ars Combinatoria-Waterloo then Winnipeg*, 2009. *Crossref*, https://doi.org/10.48550/arXiv.0905.2251

[20] P.Santhakuramaran, and J.John, "The Upper Steiner Number of a Graph," *Graph Theory Notes of New York LIX, 2010*.

[21] M.S Sunitha, and Sunil Mathew, "Fuzzy Graph Theory: A Survey," *Annals of Pure and Applied Mathematics*, vol. 4, no. 1, pp. 92-110, 2013.

[22] G.D.Suresh, and A.Vijayan, "On Steiner Sets and Steiner Polynomials of Squares of Paths," *International Journal of Scientific Research and Review*, vol.7, no. 12, pp. 487- 495, 2018.

[23] S.K. Vaidya, and S.H. Harkar, "Steiner Domination Number of Some Graphs," *International Journal of Mathematics and Scientific Computing*, vol. 5, no. 1, pp. 1-3, 2015.

[24] Samir K.Vaidya, and Riya Nitinbhai Mehta, "Steiner Domination Number of Some Wheel Related Graphs," *International Journal of Mathematics and Soft Computing*, vol. 5, no. 2, pp. 15-19, 2015. *Crossref*, http://dx.doi.org/10.26708/IJMSC.2015.2.5.03

[25] Yashwant Vaishnaw, and Sanjay Sharma, "Some New Approaches of Fuzzy Graph," *Journal of Mathematics*, vol. 12, no. 6, pp. 109-112, 2016. *Crossref*, http://dx.doi.org/10.9790/5728-120601109112