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Original Article

Introducing Safe Domination in Graphs

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Abstract - *For a nontrivial connected graph G with no isolated vertex, a nonempty subset S of the vertex set of G is a safe dominating set if and only if it is both secure and dominating. Moreover, S is called a minimum safe dominating set if S is a safe dominating set of the smallest size in a given graph. The cardinality of the minimum safe dominating set of G is the safe domination number of G. In this paper, we extend the idea of safe and dominating sets by providing characterizations of the safe dominating sets of some graph families. In particular, this paper discusses the minimum cardinality of safe dominating sets of path, cycle, complete, and complete bipartite graphs.*

Keywords - *Domination, Safe domination, Safe domination number, Safe dominating set.*

1. Introduction

Facility location problems entail choosing optimal, cost-effective and suitable sites for establishing or maintaining new facilities. It plays a crucial role in logistics management as it determines the facility's distribution pattern and various attributes, such as time, cost, and efficiency [8]. Fujita et al. addressed this problem and introduced the concept of safe set and connected safe set. On the other hand, the study of domination in graphs has experienced significant expansion in recent years, emerging as a thriving research domain within graph theory. Numerous research articles have been published on this subject, reflecting its increasing relevance in practical applications.

This paper combines safe set and domination to form a new parameter called the safe dominating set of graphs. S is a minimum safe dominating set denoted by γ_s*-set* if *S* is a safe dominating set of the smallest size in a given chart. The order of S is the safe domination number of a graph G, denoted by $\gamma_s(G)$. In this context, we determine the safe dominating number for significant graph families, including paths, cycles, complete graphs, and complete bipartite graphs. Our aim is that the formulas we derive here will be valuable for those seeking to find the safe dominating set of a graph. Additionally, we provide methods for identifying safe dominating sets in wheel and fan graphs. Lastly, we show the existence of a graph with an equal safe number and a safe dominating number. All graphs discussed in this study are nontrivial, simple, undirected, and finite.

2. Preliminary Concepts

Definition 2.1. [9] The subgraph of a graph G induced by $S \subseteq V(G)$ is denoted by $G[S]$. A component of G is a connected induced subgraph of G with an inclusion-wise maximal vertex set.

Example 2.1.1.

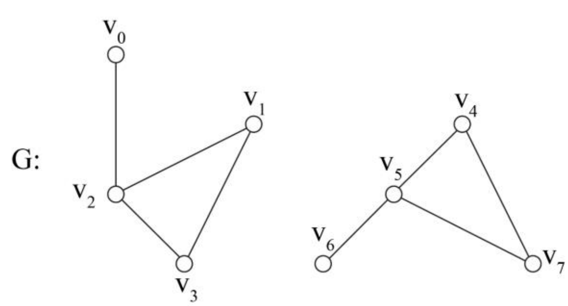

Fig. 1 A graph G with two components

Definition 2.2 [9] For vertex-disjoint subgraphs A and B of G, if there is an edge between A and B, then A and B are **adjacent**.

Example 2.2.1.

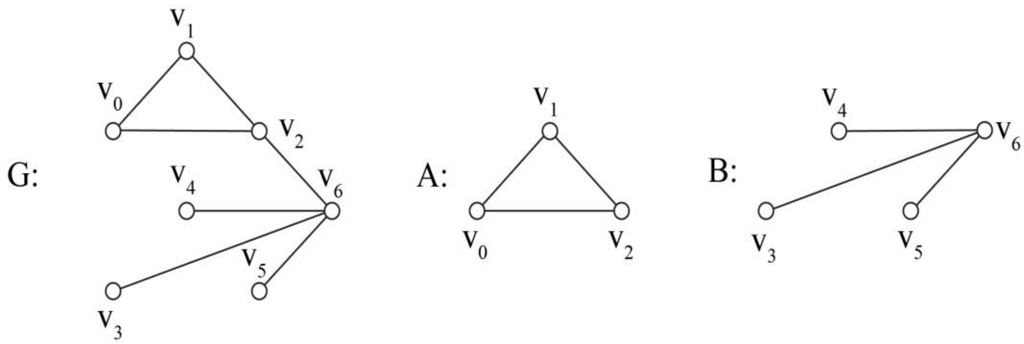

Fig. 2 A graph G has two subgraphs adjacent and connected by edge v_2v_6.

Definition 2.3 [9] A nonempty set $S \subseteq V(G)$ of vertices is a safe set if, for every component *A* of $G[S]$ and every component *B* $G[V(G)\backslash S]$ adjacent to *A*, it holds that $|A| \geq |B|$. The safe number denoted by $s(G)$ *G* is the minimum cardinality of a safe set of *G*.

Example 2.3.1.

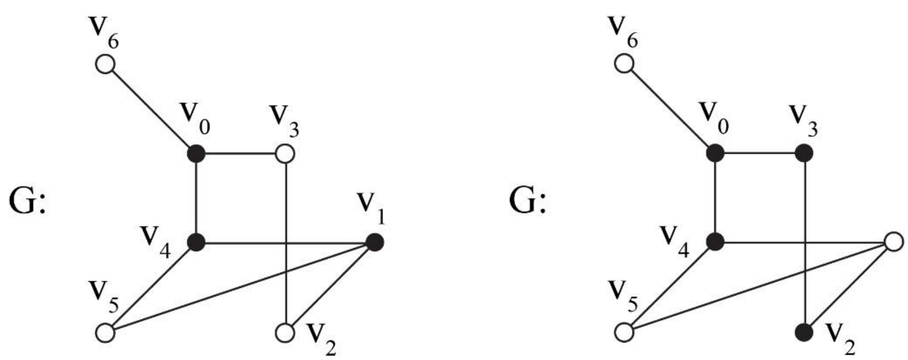

Fig. 3 A graph G and its safe sets with s(G) =3

Definition 2.4 [2] Let *G* be a simple graph. A set $S \subseteq V(G)$ is a dominating set of *G* if every vertex is adjacent to at least one vertex in S. The domination number is the minimum cardinality of the dominating group.

Example 2.4.1.

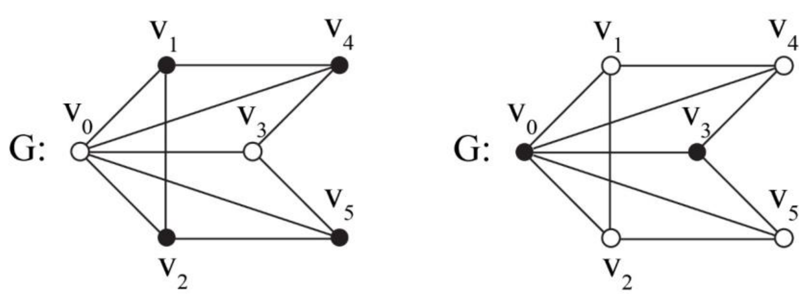

Fig. 4 A graph G and its dominating sets with γ (G) =2

Definition 2.5. For a nontrivial connected graph *G* with no isolated vertex, a nonempty subset $S \subseteq V(G)$ is a safe dominating set if and only if S is a dominating set of G and every component A of $G[S]$ and every component B of $G[V(G)\backslash S]$ adjacent to A, $|A| \geq |B|$. Moreover, *S* is called a minimum safe dominating set denoted by $\gamma_s - set$, if *S* is a safe dominating set of smallest size in a given graph. The cardinality of the minimum safe dominating set of *G* is the safe domination number of *G*, denoted by $\gamma_s(G)$.

Example 2.5.1.

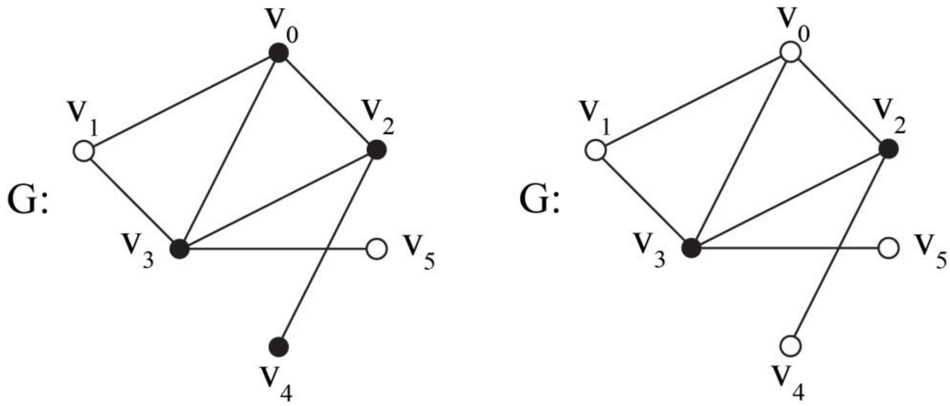

Fig. 5 A graph G and its safe dominating sets with γ_s(G) =2

3. Main Results

Theorem 3.1 If $\emptyset \neq S \subseteq V(P_n)$, then S is a safe dominating set of P_n if and only if one of the following holds:

(i) $P_n[S]$ is connected, and for every component B of $P_n[V(P_n)\backslash S]$, $|B| \leq 1$;

(ii) $P_n[S]$ is disconnected such that every component B of $P_n[V(P_n)\backslash S]$, $|B| \leq 2$ such that for $|B| = 2$, every vertex $v \in B$, $\deg_{P_n}(v) = 2$ and every trivial component of $P_n[S]$ is adjacent to a trivial component of $P_n[V(P_n)\backslash S]$.

Proof. Suppose *S* is a safe dominating set of P_n. Consider the following cases:

(i) $P_n[S]$ is connected.

If $|S| = n$, *S* is a safe dominating set of P_n. Now to show that $|S| \geq n-2$. Suppose that $|S| \leq n-3$. Without loss of generality, suppose further that $|S| = n-3$. Then, $|V(P_n)\backslash S| = 3$. If $P_n[V(P_n)\backslash S]$ is connected, then clearly *S* is not a dominating set. Now, if $P_n[V(P_n)\backslash S]$ is disconnected, then it has two components, say B_1 and B_2, where we can assign $|B_1| = 1$, and $|B_2| = 2$ there exists such that S is not a dominating set, a contradiction. Thus, $|S| \geq n-2$. Therefore, $n-2 \leq |S| \leq n$. Consequently, every component *B* in $P_n[V(P_n)\backslash S]$, $|B| \leq 1$.

(ii) $P_n[S]$ is disconnected. Consider the following subcases:

(a) $P_n[V(P_n)\backslash S]$ is connected and has only one component, which is itself. Suppose $|V(P_n)\backslash S| > 2$. Clearly, *S* is not a dominating set. This is a contradiction to the assumption that *S* is a dominating set.

(b) $P_n[V(P_n)\backslash S]$ is disconnected.

Suppose there exists a component *B* of $P_n[V(P_n)\backslash S]$ such that $|B| > 2$. Then, there exists a vertex $v \in V$ such that $v \notin N[S]$. Hence, *S* is not a dominating set of P_n. Another contradiction. Now, suppose that *A* is a trivial component of P_n such that *A* is adjacent to a nontrivial component of $P_n[V(P_n)\backslash S]$. Then, $|A| < |B|$. Thus, *S* is not a safe dominating set. A contradiction to the assumption that *S* is a safe dominating set. Therefore, every trivial component of P_n is adjacent to an insignificant component of $P_n[V(P_n)\backslash S]$.

For the converse, consider the following cases.

(i) $P_n[S]$is connected and for every component B of $P_n[V(P_n)\backslash S]$, $|B| \leq 1$;. Then, $P_n[S]$**it** has only one component and |S| ≥ 1. Hence, for every component B of $P_n[V(P_n)\backslash S]$, |B| ≤ 1 ≤ |S| and for every vertex v ∈ B, v ∈ N[S]. Hence, S is a safe dominating set of P_n.

(ii) $P_n[S]$ is disconnected such that every component B of $P_n[V(P_n) \backslash S]$, |B| ≤ 2 such that for |B| = 2, every vertex v∈B, $deg_{P_n}(v) = 2$ and every trivial component of P_n is adjacent to a trivial component of $P_n[V(P_n) \backslash S]$. Clearly, S is a dominating set of P_n. Now, if every trivial component A of $P_n[S]$ is adjacent to a trivial component $P_n[V(P_n) \backslash S]$, then |A| = |B|. Thus, S is a safe dominating set of P_n.

Corollary 3.2. For a path graph $\boldsymbol{P_n}$,

$$\gamma_s(P_n) = \begin{cases} \dfrac{n}{2}, & if\ n \equiv 0(mod\ 2) \\ \dfrac{n-1}{2}, & if\ n \equiv 1(mod\ 2). \end{cases}$$

Proof. Let $P_n = \{v_0,\ \ldots,\ v_{n-1}\},\ n \geq 2,\ S \subseteq V(P_n)$ be a dominating set of P_n. Consider the following cases:
(i) $n \equiv 0(mod\ 2)$
Choose $S = \{v_k,\ \ldots,\ v_{2k}\}$, where $k = \frac{n-2}{2}$. Then, $|S| = \frac{n}{2}$ and $|V(P_n)\backslash S| = \frac{n}{2}$, where $P_n[S]$ and $P_n[V(P_n)\backslash S]$ are disconnected and all of its components are trivial graphs. By Theorem 3.1(ii), S is a safe dominating set. Now we want to show that S is the minimum safe dominating set of P_n. Suppose S is not the minimum safe dominating set of P_n. Then, there exists a safe dominating set $S_0 \subseteq V(P_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n}{2}$ and $|V(P_n)\backslash S_0| > \frac{n}{2}$. Then, it follows that there exists a component B of $P_n[V(P_n)\backslash S_0]$ adjacent to a component A of $P_n[S_0]$ such that $|B| > |A|$. A contradiction to the assumption that S_0 is a safe dominating set. Thus, S is the minimum safe dominating set of P_n. Hence, $\gamma_s(P_n) = \frac{n}{2}$.
(ii) $n \equiv 1(mod\ 2)$
Choose $S = \{v_1,\ \ldots,\ v_k\}$, where $k = n$. Then, $|S| = \frac{n-1}{2}$ and $P_n[S]$ is disconnected. Now, $V(P)\backslash S = \{v_0,\ \ldots,\ v_{k-1}\}$ $|V(P)\backslash S| = \frac{n+1}{2}$. Observe that every component of $P_n[S]$ and $P_n[V(P_n)\backslash S]$ are trivial. By Theorem 3.1(ii), S is a safe dominating set. Now, we want to show that S is the minimum safe dominating set P_n. To show that S is the minimum safe dominating set, suppose S is not the minimum safe dominating set. Then, there exists a safe dominating set $S_0 \subseteq V(P_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n-1}{2}$ and $|V(P_n)\backslash S_0| > \frac{n-1}{2}$. It follows that there exists a component B of $P_n[V(P_n)\backslash S_0]$ adjacent to a component A of $P_n[S_0]$ such that $|B| > |A|$. Another contradiction. Thus, S is the minimum safe dominating set of P_n. Hence, $\gamma_s(P_n) = \frac{n-1}{2}$.

Theorem3.3. If ∅≠S⊆V(C_n), then S is a safe dominating set of C_n if and only if one of the following holds:
(i) $C_n[S]$ is connected and $|V(C_n)\backslash S| \leq 2$;
(ii) $C_n[S]$ is disconnected such that every component B of $C_n[V(C_n)\backslash S]$, $|B| \leq 2$ and every trivial component of $C_n[S]$ is adjacent to a trivial component of $C_n[V(C_n)\backslash S]$.

Proof. Suppose S is a safe dominating set of C_n. Consider the following cases:
(i) $C_n[S]$ is connected.
Then $C_n[V(C_n)\backslash S]$, it is also connected. If $|S| = n$, S is a safe dominating set of C_n. Now, suppose that $|V(C_n)\backslash S| > 2$. Then there exists $v \in V(C_n)\backslash S$ such that $v \notin N[S]$. A contradicts the assumption that S is a dominating set. Hence, $|V(C_n)\backslash S| \leq 2$.
(ii) $C_n[S]$ is disconnected.
Then $C_n[V(C_n)\backslash S]$, it is also disconnected. Suppose that there exists a component B of $C_n[V(C_n)\backslash S]$ such that $|B| > 2$. Then there exists a vertex $v \in B$ such that $v \notin N[S]$. Thus, S is not a dominating set—another contradiction. Now, suppose that A is a trivial component of $C_n[S]$ such that A is adjacent to a nontrivial component B of $C_n[V(C_n)\backslash S]$. Then $|A| < |B|$. Thus, S is not a dominating set of C_n. A contradiction to the assumption that S is a safe dominating set. Hence, every component B of $C_n[V(C_n)\backslash S]$, $|B| \leq 2$ and every trivial component of $C_n[S]$ is adjacent to a trivial component of $C_n[V(C_n)\backslash S]$.

For the converse, consider the following cases:
(i) If $C_n[S]$is connected and $|V(C_n)\backslash S| \leq 2$;, then for every v ∈ V (Cn) \ S, v ∈N[S]. Thus, S is a dominating set. Now, since for every component in $C_n[V(C_n)\backslash S]$, and $C_n[S]$ is connected, then $|C_n[V(C_n) \backslash S]| \leq 2 \leq |C_n[S]|$. Hence, S is a safe dominating set of Cn.
(ii)Suppose $C_n[S]$is disconnected such that for every component B in $C_n[V(C_n)\backslash S]$, $||B| \leq 2$and every trivial component of

$C_n[S]$is adjacent to a trivial component of $C_n[V(C_n)\backslash S]$. Then, for every v ∈ V (Cn) \ S, v ∈ N[S]. Thus, S is a dominating set of C_n. Now, if every trivial component A of$C_n[S]$ is adjacent to every trivial component B, then |A| = |B|. Thus, S is a safe dominating set of Cn.

Corollary 3.4. For a cycle graph C_n,

$$\gamma_s(C_n) = \begin{cases} \frac{n}{2}, & if\ n \equiv 0(mod\ 2) \\ \frac{n+1}{2}, & if\ n \equiv 1(mod\ 2). \end{cases}$$

Proof. Let $C_n = \{v_0,\ \dots,\ v_{n-1}\},\ n \geq 2,\ S \subseteq V(C_n)$ be a dominating set of C_n. Consider the following cases:
(i) $n \equiv 0(mod\ 2)$
Choose $S = \{v_k,\ \dots,\ v_{2k}\}$, where $k = \frac{n-2}{2}$. Then, the component $C_n[S]$ is S itself with $|S| = \frac{n}{2}$, and the component $C_n[V(C_n)\backslash S]$ is $V(P_n)\backslash S$ itself with $|V(C_n)\backslash S| = \frac{n}{2}$, where $C_n[S]$ and $C_n[V(C_n)\backslash S]$ are disconnected, and all of its components are trivial graphs. By Theorem 3.3(ii), S is a safe dominating set. Now we want to show that S is the minimum safe dominating set of C_n. Suppose S is not the minimum safe dominating set of C_n. Then, there exists a safe dominating set $S_0 \subseteq V(C_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n}{2}$ and $|V(C_n)\backslash S_0| > \frac{n}{2}$. Now for $|V(C_n)\backslash S_0|\ > \frac{n}{2}$ it follows that there exists a component B of $C_n[V(C_n)\backslash S_0]$ adjacent to a component A of $C_n[S_0]$ such that $|B| > |A|$. A contradiction to the assumption that S_0 is a safe dominating set. Thus, S is the minimum safe dominating set of C_n. Hence, $\gamma_s(C_n) = \frac{n}{2}$.
(ii) $n \equiv 1(mod\ 2)$
Choose $S = \{v_k,\ \dots,\ v_{2k}\}$, where $k = \frac{n-1}{2}$. Then, the component $C_n[S]$ is S itself with $|S| = \frac{n+1}{2}$, and the component $C_n[V(C_n)\backslash S]$ is $V(P_n)\backslash S$ itself with $|V(C_n)\backslash S| = \frac{n-1}{2}$, where $C_n[V(C_n)\backslash S]$ is disconnected, and all of its components are trivial graphs. By Theorem 3.3(ii), S is a safe dominating set. Now we want to show that S is the minimum safe dominating set of C_n. Suppose S is not the minimum safe dominating set of C_n. Then, there exists a safe dominating set $S_0 \subseteq V(C_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n+1}{2}$ and $|V(C_n)\backslash S_0| > \frac{n+1}{2}$. Now for $|V(C_n)\backslash S_0| > \frac{n+1}{2}$, it follows that there exists a component B of $C_n[V(C_n)\backslash S_0]$ adjacent to a component A of $C_n[S_0]$ such that $|B| > |A|$. A contradiction to the assumption that S_0 is a safe dominating set. Thus, S is the minimum safe dominating set of C_n. Hence, $\gamma_s(C_n) = \frac{n+1}{2}$.

For a complete graph, K_n, every vertex $v \in V(K_n)$ is adjacent to every vertex in $V(K_n)\backslash\{v\}$. Thus, we have the following remark:

Remark 3.5. If $\emptyset \neq S \subseteq V(K_n)$, then S is a safe dominating set of K_n if and only if $|S| \geq |V(K_n)\backslash S|$.

Corollary 3.6. For a complete graph K_n,

$$\gamma_s(K_n) = \begin{cases} \frac{n}{2}, & if\ n \equiv 0(mod\ 2) \\ \frac{n+1}{2}, & if\ n \equiv 1(mod\ 2). \end{cases}$$

Proof. Let $K_n = \{v_0,\ \dots,\ v_{n-1}\},\ n \geq 2,\ S \subseteq K(C_n)$ be a dominating set of K_n. Consider the following cases:
(i) $n \equiv 0(mod\ 2)$
Choose $S = \{v_k,\ \dots,\ v_{2k}\}$, where $k = \frac{n-2}{2}$. Then $|S| = \frac{n}{2}$ and $|V(K_n)\backslash S| = \frac{n}{2}$. By Remark 3.5, S is a safe dominating set of K_n. Now suppose S is not the minimum safe dominating set of K_n. Then, there exists a safe dominating set $S_0 \subseteq V(K_n)$ such that $|S_0| < |S|$. Thus, $|K_n[S_0]| < \frac{n}{2} < |K_n[V(K_n)\backslash S_0]|$. A contradiction. Hence, S is the minimum safe dominating set of K_n and $\gamma_s(K_n) = \frac{n}{2}$.
(ii) $n \equiv 1(mod\ 2)$
Choose $S = \{v_k,\ \dots,\ v_{2k}\}$,
where $k = \frac{n-1}{2}$. Then $|S| = \frac{n+1}{2}$ and $|V(K_n)\backslash S| = \frac{n-1}{2}$. By Remark 3.5, S is a safe dominating set of K_n. Now suppose S is not the minimum safe dominating set of K_n. Then, there exists a safe dominating set $S_0 \subseteq V(K_n)$ such that $|S_0| < |S|$. Thus, $|K_n[S_0]| < \frac{n+1}{2} < |K_n[V(K_n)\backslash S_0]|$. A contradiction. Hence, S is the minimum safe dominating set of K_n and $\gamma_s(K_n) = \frac{n}{2}$.

Theorem 3.7. If $K_{m,n}$ is a complete bipartite graph and $S_1,\ S_2$ are the partite set of $V(\rightleftarrows \rightleftarrows K_{m,n} \rightleftarrows \rightleftarrows)$, where $|S_1| = m,\ |S_2| = n$. Then, $\emptyset \neq S \subseteq V(\rightleftarrows \rightleftarrows K_{m,n} \rightleftarrows \rightleftarrows)$ is a safe dominating set of $K_{m,n}$ if and only if one of the following holds:

(i) $S \subseteq S_1$ such that $S = S_1$;

(ii) $S \subseteq S_2$ such that $S = S_2$;

(iii) $S = A_1 \cup A_2$ such that $A_1 \subseteq S_1$ and $A_2 \subseteq S_2$, where $1 \leq |A_1| \leq |S_1|$, $1 \leq |A_2| \leq |S_2|$ such that $|A_1 \cup A_2| \geq | \quad (A_1 \cup A_2)^c \quad |$.

Proof. Suppose S is a safe dominating set of $K_{m,n}$. Consider the following cases:

(i) $S \subseteq S_1$ such that $S = S_1$
Then $K_{m,n}[S]$ is disconnected. Suppose further that $S \neq S_1$. Then there exists $v \in S$ such that $v \notin N[S]$. A contradiction to the assumption that S is a dominating set of $K_{m,n}$. Hence, $S = S_1$.

(ii) $S \subseteq S_2$ such that $S = S_2$
Then $K_{m,n}[S]$ is disconnected. Suppose further that $S \neq S_2$. Then there exists $v \in S$ such that $v \notin N[S]$. A contradiction to the assumption that S is a dominating set of $K_{m,n}$. Hence, $S = S_2$.

(iii) $S = A_1 \cup A_2$ such that $A_1 \subseteq S_1$ and $A_2 \subseteq S_2$, where $1 \leq |A_1| \leq |S_1|$, $1 \leq |A_2| \leq |S_2|$ such that $|A_1 \cup A_2| \geq | \quad (A_1 \cup A_2)^c \quad |$.
If $A_1 = S_1$ and $A_2 = S_2$, clearly S is a safe dominating set of $K_{m,n}$. Suppose $A_1 \neq S_1$ and $A_2 \neq S_2$ and suppose further that $|A_1 \cup A_2| < \quad (A_1 \cup A_2)^c \quad |$. Then, $K_{m,n}[\rightleftarrows \rightleftarrows (A_1 \cup A_2)^c \rightleftarrows \rightleftarrows]$ and $K_{m,n}[A_1 \cup A_2]$ has only one component respectively. Hence, $\quad K_{m,n}[A_1 \cup A_2] \quad | < | \quad K_{m,n}[\rightleftarrows \rightleftarrows (A_1 \cup A_2)^c \rightleftarrows \rightleftarrows] \quad |$. Another contradiction. Consequently, $|A_1 \cup A_2| \geq | \quad (A_1 \cup A_2)^c \quad |$.

Conversely, for *(i)* and *(ii)*, clearly, S is a safe dominating set since one of the partite sets is a dominating set $K_{m,n}$ and will induce an empty graph. Now for *(iii)*, clearly, S is a dominating set. Observe that $A_1 \cup A_2$and $(A_1 \cup A_2)^c$ induce only one component each. Hence, if $|A_1 \cup A_2| \geq | \quad (A_1 \cup A_2)^c \quad |$, then it follows that, $\quad K_{m,n} < A_1 \cup A_2 > \quad | \geq | \quad K_{m,n} < (A_1 \cup A_2)^c > \quad |$. Therefore, S is a safe dominating set of $K_{m,n}$.

Corollary 3.8. For a complete bipartite graph $K_{m,n}$,

$$\gamma_s(\rightleftarrows \rightleftarrows K_{m,n} \rightleftarrows \rightleftarrows) = min\{m,\ n\}.$$

Proof. Let S_1 and S_2 be a partite set of $V(\rightleftarrows \rightleftarrows K_{m,n} \rightleftarrows \rightleftarrows)$, where $|S_1| = m$ and $|S_2| = n$. Suppose S is a dominating set of $K_{m,n}$. Then $S \subseteq S_1$ or $S \subseteq S_2$. Note that by Theorem 3.7(i), if $S \subseteq S_1$, then $|S| = |S_1| = m$ and S is a safe dominating set of $K_{m,n}$. Now by Theorem 3.7(ii), if $S \subseteq S_2$, then $|S| = |S_2| = m$ and S is a safe dominating set of $K_{m,n}$. Therefore, $\gamma_s(\rightleftarrows \rightleftarrows K_{m,n} \rightleftarrows \rightleftarrows) = min\{m,\ n\}$.

For a complete bipartite graph, observe that it is a union of two independent sets of vertices, each of which dominates the other. Choosing a partite set as a dominating set results in an empty graph. Hence, we have the following remark:

Remark 3.9. $\gamma(G) = \gamma_s(G)$ if and only if $G = K_{m,n}$.

Theorem 3.10. If $\emptyset \neq S \subseteq V(F_n)$. Then S is a safe dominating set of F_n if and only if one of the following holds:
(i) $S \subseteq V(P_n),\ P_n[S]$ is connected such that for every component A of $P_n[V(P_n)\backslash S],\ |A| = 1$ or $P_n[S]$ is disconnected with no trivial components and that $|S| > |V(P_n)\backslash S|$;

(ii) $S = S_1 \cup S_2,\ S_1 \subseteq V(P_n),\ S_2$ is a trivial graph such that $P_n[S_1]$ is connected or $P_n[S_1]$ is disconnected and the cardinality of every component of $P_n[V(P_n)\backslash S_1]$ is less than or equal to $|S|$.

Proof. Suppose S is a safe dominating set of F_n. Consider the following cases:

(i) If $P_n[S]$ is connected, $F_n[V(F_n)\setminus S]$ is also connected. Then, it follows that $|S| > |V(P_n)\setminus S|$. Suppose that there exists a component A of $P_n[V(P_n)\setminus S]$ such that $|A| > 1$. Then, there exists a vertex $v \in A$ such that $v \notin N[S]$. Hence, S is not a dominating set. A contradiction to the assumption that S is a dominating set. Now, suppose $P_n[S]$ is disconnected and that $|S| \leq |V(P_n)\setminus S|$. Without loss of generality, suppose further that $|S| = |V(P_n)\setminus S|$. Observe that, $F_n[V(F_n)\setminus S]$ is connected. Hence for all component A, of $F_n[S]$, $|A| < |F_n[V(F_n)\setminus S]|$. Therefore, S is not a safe dominating set. A contradiction to the assumption that S is a safe dominating set. Moreover, if A is a trivial component of $P_n[S]$, then $|A| < |F_n[V(F_n)\setminus S]|$. Another contradiction.

(ii) If $P_n[S_1]$ is connected or disconnected, observe that $F_n[S]$ is connected. Hence, it follows that $F_n[S]$ it has one component, which is itself. Since S is a safe dominating set, it follows that for every component B of $P_n[V(P_n)\setminus S_1]$, $|B| \leq |F_n[S]|$.

For the converse, consider the following cases.
(i) Suppose$S \subseteq V(P_n)$. If $P_n[S]$is connected such that for every component A of $P_n[V(P_n)\setminus S]$, |A| = 1. Clearly, S is a dominating set. Now, since $P_n[S]$ is connected, it follows that $F_n[V(F_n)\setminus S]$ is also connected. Hence, since $|S| > |V(P_n)\setminus S|$, S is a safe dominating set. Now, if $P_n[S]$ it is disconnected with no trivial component and that$|S| > |V(P_n)\setminus S|$. Clearly, S is a dominating set. Observe that $F_n[V(F_n)\setminus S]$ is connected. Since $|S| > |V(P_n)\setminus S|$ and $P_n[S]$ has no trivial component, therefore, S is a safe dominating set.

(ii) Suppose $S = S_1 \cup S_2$ where $S_1 \subseteq V(P_n)$and $S_2$is a trivial graph. Clearly, S is a dominating set. Now, if $P_n[S_1]$ is connected or disconnected, observe that $F_n[S_1]$ is connected or disconnected, observe that $F_n[S]$ is connected. Since the cardinality of every component is less than or equal to |S|, S is clearly a safe dominating set.

Theorem 3.11. If $\emptyset \neq S \subseteq V(W_n)$, Then S is a safe dominating set of W_n if and only if one of the following holds:
(i) $S \subseteq V(C_n)$, $C_n[S]$ is connected such that $|V(C_n)\setminus S| \leq 2$ and $|S| > |V(C_n)\setminus S|$ or $C_n[S]$ disconnected such that $|V(C_n)\setminus S| < |A|$, where A has the smallest order of all components of $C_n[S]$ and for all component B of $C_n[V(C_n)\setminus S]$, $|B| \leq 2$.
(ii) $S = S_1 \cup S_2$, $S_1 \subseteq V(C_n)$, S_2 is a trivial graph such that $C_n[S_1]$ is connected and $|V(C_n)\setminus S_1| \leq |S|$ or $C_n[S_1]$ is disconnected and for every component B of $C_n[V(C_n)\setminus S]$, $|B| \leq |S|$.

Proof. Suppose S is a safe dominating set of W_n. Consider the following cases:
(i) $S \subseteq V(C_n)$.
If $C_n[S]$ is connected, then $W_n[V(W_n)\setminus S]$ is also connected. Suppose further that $|V(C_n)\setminus S| > 2$, then there exists a vertex $v \in V(C_n)\setminus S$ such that $v \notin N[S]$. Therefore, S is not a dominating set. A contradiction to the assumption that S is a dominating set. Consequently, it follows that $|S| > |V(C_n)\setminus S|$. If $C_n[S]$ is disconnected observe that $W_n[V(W_n)\setminus S]$ is connected. Suppose $|V(C_n)\setminus S| \geq |A|$, where A has the smallest order of all components of $C_n[S]$. Without loss of generality, suppose further that $|V(C_n)\setminus S| = |A|$, then $|W_n[V(W_n)\setminus S]| > |A|$. Thus, S is not a safe dominating. Another contradiction. Now, if B is a component of $C_n[V(C_n)\setminus S]$, where $|B| > 2$, then there exists a vertex $v \in B$ such that $v \notin N[S]$. Thus, S is not a dominating set. Another contradiction.
(ii) $S = S_1 \cup S_2$, $S_1 \subseteq V(C_n)$, S_2 is a trivial graph.
If $C_n[S_1]$ is connected, then $C_n[V(C_n)\setminus S_1]$ and $W_n[S]$ are also connected. Consequently, $|V(C_n)\setminus S_1| \leq |S|$. If $C_n[S_1]$ is disconnected, observe that $W_n[S]$ is connected. Suppose there exists a component B of $C_n[V(C_n)\setminus S]$ such that $|B| > |S|$. Therefore, S is not a safe dominating set. A contradiction to the assumption that S is a safe dominating set. Hence, for every component B of $C_n[V(C_n)\setminus S]|B| \leq |S|$.

For the converse, consider the following cases:
(i)S $\subseteq$V (Cn).
Suppose Cn[S] is connected such that $|V(C_n)\setminus S| \leq 2$. Clearly, S is a dominating set and $W_n[V(W_n)\setminus S]$ is connected. Since |S| > $|V(C_n)\setminus S|$, it follows that $|W_n[S]| \geq |W_n[V(W_n)\setminus S]|$. Hence, S is a safe, dominating set. Now, if $C_n[S]$ is disconnected such that $|V(C_n) \setminus S| < |A|$ where A has the smallest order of all components of $C_n[S]$ and for all component B of $C_n[V(C_n) \setminus S]$, $|B| \leq 2$. Then, S is a dominating set of W_n. Observe that, $W_n[V(W_n) \setminus S]$ has one component, which is itself. Given that $|V(C_n) \setminus S| < |A|$ where A has the smallest order of all components of $C_n[S]$ consequently, $|W_n[V(W_n) \setminus S]| \leq |A|$. Therefore, S is a safe dominating set of W_n.
(ii) $S = S_1 \cup S_2$, $S_1 \subseteq V(C_n)$, S_2 is a trivial graph.
Clearly, S is a dominating set. Suppose $C_n[S_1]$ is connected and $|V(C_n) \setminus S_1| \leq |S|$. Then $W_n[S]$ and $W_n[V(W_n) \setminus S]$ are connected and $|W_n[S]| \geq |W_n[V(W_n) \setminus S]|$. Hence, S is a safe dominating set. Now, if $C_n[S_1]$ is disconnected and for every component B of

$C_n[V(Cn) \setminus S]$, $|B| \leq |S|$. Observe that $W_n[S]$ has only one component, which is itself. Clearly, S is a safe dominating set.

4. Conclusion

In this paper, we introduce the new concept called safe domination, which focuses on finding the subset of safe and dominating vertices. We characterize the safe domination in different graph classes and generate its domination number. Moreover, we have determined that a complete bipartite graph's safe and domination numbers are equal. With these findings, the paper will be helpful in real-life situations like FLP. Facility location problem (FLP) revolves around choosing suitable sites, a cost-efficient subset of places, to establish new facilities or to keep existing ones. Strategically placing facilities and optimally arranging customers improves the efficiency of delivering materials and services to customers and maximizes the utilization of facilities.

Consequently, it reduces the necessity for duplicating or having redundant facilities. With the application of this recently introduced study, the facilities would encompass the most significant possible population or area using domination. They will guarantee swift and sufficient responses from these establishments using a safe set.

References

[1] Cabrera Martínez, Abel, et al. "On the Secure Total Domination Number of Graphs." *Symmetry*, vol. 11, no. 9, 2019, p. 1165, [CrossRef] [Publisher Link]

[2] Gary Chartrand, Linda Lesniak, and Ping Zhang, *Graphs and Digraphs, (Discrete Mathematics and Its Applications)*, 6th ed., Chapman and Hall/CRC, 2015. [Publisher Link]

[3] Grobler, P.J.P., and C.M. Mynhardt. "Secure Domination Critical Graphs." *Discrete Mathematics*, vol. 309, no. 19, 2009, pp. 5820-5827, [CrossRef] [Publisher Link]

[4] Hernández-Ortiz, R., Montejano, L.P. & Rodríguez-Velázquez, J.A. Secure domination in rooted product graphs. *J Comb Optim* **41**, 401–413 (2021). [CrossRef] [Publisher Link]

[5] Jha, Anupriya, et al. "The Secure Domination Problem in Cographs." *Information Processing Letters*, vol. 145, 2019, pp. 30-38, [CrossRef] [Publisher Link]

[6] Nathann Cohen et al., "Safe Sets in Graphs: Graph Classes and Structural Parameters," *COCOA 2016 - 10th International Conference Combinatorial Optimization and Applications, Hong Kong, China*, pp. 241-253, 2016. [Google Scholar] [Publisher Link]

[7] Saeid Alikhani, and Yee-Hock Peng, "Dominating Sets and Domination Polynomials of Paths," *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[8] Shinya Fujita, Gary MacGillivray, and Tadashi Sakuma, "Safe Set Problem on Graphs," *Discrete Applied Mathematics*, vol. 215, pp. 106-111, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[9] V. R. Kulli, B. Chaluvaraju & M. Kumara (2018) Graphs with equal secure total domination and inverse secure total domination numbers, Journal of Information and Optimization Sciences, vol. 39, no. 2, pp. 467-473. [CrossRef] [Publisher Link]

[10] Yun-Hao Zou, Jia-Jie Liu, Shun-Chieh Chang, and Chiun-Chieh Hsu. 2022. The co-secure domination in proper interval graphs. Discrete Appl. Math. 311, C (Apr 2022), 68–71. [CrossRef] [Publisher Link]