International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V69I10P504
Volume 69 Issue 10, 25-30, October 2023

Original Article

Generalized Hyers-Ulam Stability Analysis of a Cubic Functional Equation in Generalized 2-Normed Spaces

G. Yagachitradevi[1], S. Lakshminarayanan[2], P. Ravindiran[3]

[1]*Department of Mathematics, Siga College of Management and Computer Science, Villupuram, Tamil Nadu, India.*
[2,3]*Department of Mathematics, Arignar Anna Government Arts College, Villupuram, Tamil Nadu, India*

[1]*Corresponding Author : yagachitradevi@gmai.com*

Received: 24 August 2023 Revised: 29 September 2023 Accepted: 18 October 2023 Published: 31 October 2023

Abstract *- This study aims to examine the generalized Hyers-Ulam stability of a cubic functional equation, namely*

$$G = 24[g(v_2+v_1)+g(v_1-v_2)]+6[g(v_3+v_1)+g(v_1-v_3)]+48g(v_1),$$

where $G = g(3v_1+2v_2+v_3)+g(3v_1+2v_2-v_3)+g(3v_1-2v_2+v_3)+g(3v_1-2v_2-v_3)$*, within the context of generalized 2-normed spaces.*

Keywords *- Ulam stability, Generalized hyers-ulam, Generalized 2-normed spaces, Functional equation, Cubic equation.*

1. Introduction

The investigation conducted by Ulam on the stability of group homomorphisms, as described in reference [16], initiated a significant area of research focused on the stability of equations. There were significant developments in mathematical theory as a result of it. From the realm of spaces of Banach, the initial response was provided by Hyers [7], a solution now recognized as "Hyers' theorem. This ground-breaking finding paved the way for further research in the area.

Aoki [2] and Rassias [15] later contributed to the field, broadening the definition of stability to include additive and linear mappings; Rassias' work substantially impacted the evolution of generalized Ulam stability. Rassias presented a novel method in 1982 [13], expanding our understanding of the stability of equations in new directions. Gavruta [5] refined the theory by using a more malleable control function to account for unbounded Cauchy differences.

Since many academics have found stability issues in functional equations fascinating, many exciting discoveries have been made.In generalized 2-normed spaces, our current research focuses on cubic functional equations, particularly emphasizing Generalized Hyers-Ulam stability. This study builds upon prior investigations and contributes to the continuous advancement of this vibrant field.

In a recent investigation conducted by K. Balamurugan and colleagues, our attention was drawn to a cubic functional equation, as represented in Equation (1.1):

$$G = 24[g(v_1+v_2)+g(v_1-v_2)]+6[g(v_1+v_3)+g(v_1-v_3)]+16g(2v_1)-80g(v_1) \tag{1.1}$$

In the domain of Banach spaces, the authors have undertaken a comprehensive examination of the stability of this equation, with a specific focus on Generalized Hyers-Ulam stability.

A recent publication by Govindan et al. [6] introduced a fresh symmetric additive functional equation. This equation is obtained through the derivation of a third-degree characteristic polynomial and is expressed as follows:

$$g[(p^3+11p)v_1-6(p^2+1)v_2]+g[(11p-6p^2)v_2+(p^3-6)v_3]-g[(p^3-6p^2)v_3+(11p-6)v_1]$$
$$=(p^3+6)g(v_1)+(11p-12p^2-6)g(v_2)+(6p^2-6)g(v_3), \tag{1.2}$$

Where $p \neq 0$. In the domain of Banach spaces, the authors have undertaken a comprehensive examination of the stability of this equation, with a specific focus on Ulam stability.

Multiple researchers have rigorously examined the stabilities in a range of functional equations. This has resulted in many intriguing discoveries, as well-documented in references like [1, 3, 4, 5, 8, 9, 10, 12, 13, 14, 16].

In our current research, the authors explore the stability analysis in a specific cubic functional equation within the framework of generalized 2-normed spaces. Our research places significant emphasis on Generalized Hyers-Ulam stability.

This equation, referred to as Equation (1.3), is presented as follows:

$$G = 24[g(v_1+v_2)+g(v_1-v_2)]+6[g(v_1+v_3)+g(v_1-v_3)]+48g(v_1) \tag{1.3}$$

It's noteworthy to mention that the function $g(v_1) = av_1^3$ serves as a solution to this functional equation [a3].

2. Basic Definitions in G2NS

In this part, the authors delve comprehensively into the fundamental principles of stability, specifically in the context of the functional equation [a3], within the realm of Generalized 2-normed spaces (simply the authors notate G2NS). Additionally, the authors introduce and define crucial concepts essential for thoroughly comprehending these spaces.

Definition 1 [12] In a linear space denoted as T, a function $H(\cdot,\cdot): T \times T \to [0,\infty)$ is categorized as a generalized 2-norm when it adheres to a specific set of properties:

- (GT1) $\mathcal{H}(t,u) = 0 \Leftrightarrow t$ and u are linearly dependent.
- (GT2) $\mathcal{H}(t,u) = \mathcal{H}(u,t)$ for all $t,u \in T$.
- (GT3) $\mathcal{H}(\lambda t,u) = |\lambda|\mathcal{H}(t,u)$ for all $t,u \in T$ and $\lambda \in \varphi$, where $\varphi = R$ or C.
- (GT4) $\mathcal{H}(t+u,v) \leq \mathcal{H}(t,v)+\mathcal{H}(u,v)$ for all $t,u,v \in T$.

When these conditions are met, the pair $(T,\mathcal{H}(\cdot,\cdot))$ is referred to as a G2NS.

Definition 2 [12] In a G2NS $(T,\mathcal{H}(\cdot,\cdot))$, a sequence $\{t_n\}$ is considered convergent if $\exists\ t \in T \ni$: as n approaches infinity, $\mathcal{H}(t_n - t,u)$ tends to 0 for all $u \in$T, and further, $\mathcal{H}(t_n,u)$ converges to $\mathcal{H}(t,u)$ for all $u \in T$.

Definition 3 [12] In a G2NS $(T,\mathcal{H}(\cdot,\cdot))$, a sequence $\{t_n\}$ is designated as a Cauchy sequence when $\exists$ two linearly independent elements, $u,w \in T \ni$ the sequences $\{\mathcal{H}(t_n,\mathrm{u})\}$ and $\{\mathcal{H}(t_n,w)\}$ are Cauchy sequences in $\mathbb{R}$.

Definition 4 [12] A G2NS $(T,\mathcal{H}(\cdot,\cdot))$ earns the title of a "generalized 2-Banach space" when every Cauchy sequence within T converges in T .

3. Generalized Hyers-Ulam Stability of (1.3)

In this segment, the authors explore the Ulam stability of Equation denoted as (1.3) within the context of G2NSs. Let T be a G2NS, and H a generalized 2-Banach space. The authors define a function $g: T^3 \to H$ as follows:

$$Dg(v_1,v_2,v_3) = \mathrm{G} - 24[g(v_1+v_2)+g(v_1-v_2)] - 6[g(v_1+v_3)+g(v_1-v_3)] - 48g(v_1)$$

This definition applies to all $v_1,v_2,v_3 \in T$. Additionally, throughout this paper, the authors employ the following notation: $\psi(v_1,v_2,v_3) = \psi((v_1,s),(v_2,s),(v_3,s))$ and $\| v_1 \| = \| v_1, s \|$ for all $v_1,v_2,v_3,s \in T$.

Theorem 1. *Consider the case where j is either 1 or -1, and $\psi: T^3 \to [0,\infty)$ is a function fulfilling the condition for all $v_1,v_2,v_3 \in T$:*

$$\lim_{n\to\infty} \frac{\psi(3^{nj}v_1, 3^{nj}v_2, 3^{nj}v_3)}{3^{3nj}} = 0. \tag{3.1}$$

Let $g: T \to H$ be a mapping that obeys the condition, for all $v_1,v_2,v_3,s \in T$:

$$\mathcal{H}(Dg(v_1, v_2, v_3), s) \leq \psi(v_1, v_2, v_3). \tag{3.2}$$

Under these circumstances, ∃ a distinct cubic mapping $C: T \rightarrow H$ that fulfills (1.3), and furthermore:

$$\mathcal{H}(g(v_1) - C(v_1), s) \leq \frac{1/4 \cdot 3^3}{} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\psi(3^{kj} v_1, 0, 0)}{3^{3kj}} \tag{3.3}$$

where $C(v_1)$ is defined as:

$$C(v_1) = \lim_{n \to \infty} \frac{g(3^{nj} v_1)}{3^{3nj}} \tag{3.4}$$

for all $v_1, s \in T$.

Proof. Substituting (v_1, v_2, v_3) with $(3^n v_1, 3^n v_2, 3^n v_3)$ and dividing by 3^{3n} in (3.2), the authors have

$$\mathcal{H}(g(3^n v_1, 3^n v_2, 3^n v_3) - 27g(3^n v_1, 3^n v_2, 3^n v_3), s) \leq \frac{\psi(3^n v_1, 3^n v_2, 3^n v_3)}{4} \tag{3.5}$$

for all $v_1, v_2, v_3, s \in T$. Dividing the above inequality by 3^{3n}, the authors derive

$$\mathcal{H}\left(\frac{g(3^n v_1, 3^n v_2, 3^n v_3)}{3^{3n}} - \frac{g(v_1, v_2, v_3)}{3^{3n}}, s\right) \leq \frac{\psi(3^n v_1, 3^n v_2, 3^n v_3)}{4 \cdot 3^{3n}} \tag{3.6}$$

for all $v_1, v_2, v_3, s \in T$. Substituting v_1 by $3^n v_1$ and dividing by 3^{3n} in (3.6), the authors get

$$\mathcal{H}\left(\frac{g(3^{2n} v_1, 3^{2n} v_2, 3^{2n} v_3)}{3^{6n}} - \frac{g(3^n v_1, 3^n v_2, 3^n v_3)}{3^{3n}}, s\right) \leq \frac{\psi(3^n v_1, 3^n v_2, 3^n v_3)}{4 \cdot 3^{6n}} \tag{3.7}$$

for all $v_1, v_2, v_3, s \in T$. When the authors add together (3.6) and (3.7), the authors get

$$\mathcal{H}\left(\frac{g(3^{2n} v_1, 3^{2n} v_2, 3^{2n} v_3)}{3^{6n}} - \frac{g(v_1, v_2, v_3)}{3^{3n}}, s\right)$$

$$\leq \mathcal{H}\left(\frac{g(3^n v_1, 3^n v_2, 3^n v_3)}{3^{3n}} - \frac{g(v_1, v_2, v_3)}{3^{3n}}, s\right) + \mathcal{H}\left(\frac{g(3^{2n} v_1, 3^{2n} v_2, 3^{2n} v_3)}{3^{6n}} - \frac{g(3^n v_1, 3^n v_2, 3^n v_3)}{3^{3n}}, s\right) \tag{3.8}$$

$$\leq \frac{1}{4 \cdot 3^{3n}}\left[\psi(3^n v_1, 3^n v_2, 3^n v_3) + \frac{\psi(3^{2n} v_1, 3^{2n} v_2, 3^{2n} v_3)}{3^{3n}}\right] \tag{3.9}$$

for all $v_1, v_2, v_3, s \in T$. Continuing with induction on a natural number n, the authors arrive at

$$\mathcal{H}\left(\frac{g(3^{2n} v_1, 3^{2n} v_2, 3^{2n} v_3)}{3^{6n}} - \frac{g(v_1, v_2, v_3)}{3^{3n}}, s\right) \leq \frac{1}{4 \cdot 3^{3n}} \sum_{k=0}^{n-1} \frac{\psi(3^{k+n} v_1, 3^{k+n} v_2, 3^{k+n} v_3)}{3^{3(k+n)}} \tag{3.10}$$

$$\leq \frac{1}{4 \cdot 3^{3n}} \sum_{k=0}^{\infty} \frac{\psi(3^{k+n} v_1, 3^{k+n} v_2, 3^{k+n} v_3)}{3^{3(k+n)}} \tag{3.11}$$

for all $v_1, v_2, v_3, s \in T$. To demonstrate the convergence of $\left\{\frac{g(3^{2n} v_1, 3^{2n} v_2, 3^{2n} v_3)}{3^{6n}}\right\}$, substitute v_1 with $3^m v_1$ and divide by 3^{3m} in (3.10) for all positive m, n. These yields

$$\begin{aligned}
&\mathcal{H}\left(\frac{g(3^{n+m} v_1, 3^{n+m} v_2, 3^{n+m} v_3)}{3^{3(n+m)}} - \frac{g(3^m v_1, 3^m v_2, 3^m v_3)}{3^{3m}}, s\right) \\
&= \frac{1}{3^{3m}} \mathcal{H}\left(\frac{g(3^n \cdot 3^m v_1, 3^n \cdot 3^m v_2, 3^n \cdot 3^m v_3)}{3^{3n}} - \frac{g(3^m v_1, 3^m v_2, 3^m v_3)}{3^{3m}}, s\right) \\
&\leq \frac{1}{4 \cdot 3^3} \sum_{k=0}^{\infty} \frac{\psi(3^{k+m} v_1, 3^{k+m} v_2, 3^{k+m} v_3)}{3^{3(k+m)}} \\
&\rightarrow 0 \text{ as } m \rightarrow \infty
\end{aligned}$$

for all $v_1, v_2, v_3, s \in T$. Also,

$$\mathcal{H}\left(\frac{g(3^{n+m}v_1,3^{n+m}v_2,3^{n+m}v_3)}{3^{3(n+m)}}-\frac{g(3^m v_1,3^m v_2,3^m v_3)}{3^{3m}},t\right)$$
$$=\frac{1}{3^{3m}}\mathcal{H}\left(\frac{g(3^n\cdot 3^m v_1,3^n\cdot 3^m v_2,3^n\cdot 3^m v_3)}{3^{3n}}-\frac{g(3^m v_1,3^m v_2,3^m v_3)}{3^{3m}},t\right)$$
$$\leq \frac{1}{4\cdot 3^3}\sum_{k=0}^{\infty}\frac{\psi(3^{k+m}v_1,3^{k+m}v_2,3^{k+m}v_3)}{3^{3(k+m)}}$$
$$\to 0 \text{ as } m\to\infty,\quad \forall v_1,v_2,v_3,t\in T.$$

Therefore, $\exists\ s,t\ \in T\ \ni$, the sequences $\left\{\mathcal{H}\left(\frac{g(3^{2n}v_1,3^{2n}v_2,3^{2n}v_3)}{3^{6n}},s\right)\right\}$ and $\left\{\mathcal{H}\left(\frac{g(3^{2n}v_1,3^{2n}v_2,3^{2n}v_3)}{3^{6n}},t\right)\right\}$ are real Cauchy sequences.

Therefore, the sequence $\left\{\frac{g(3^{2n}v_1,3^{2n}v_2,3^{2n}v_3)}{3^{6n}}\right\}$ is Cauchy. As H is Banach, $\exists$ a mapping $Q: T\to H\ \ni$:

$$\mathcal{H}(Q(v_1,v_2,v_3),s)=\lim_{n\to\infty}\mathcal{H}\left(\frac{g(3^{2n}v_1,3^{2n}v_2,3^{2n}v_3)}{3^{6n}},s\right),\quad \forall\ \ v_1,v_2,v_3,s\in T$$

Letting $n\to\infty$ in (3.10), the authors observe that (3.3) is satisfied for all $v_1,v_2,v_3\in T$. To demonstrate that C meets the criteria outlined in (1.3), the authors replace (v_1,v_2,v_3) with $(3^n v_1,3^n v_2,3^n v_3)$ and divide by 3^{3n} in (3.2), resulting in the following:

$$\frac{1}{3^{3n}}\mathcal{H}(Dg(3^n v_1,3^n v_2,3^n v_3),s)\leq \frac{1}{3^{3n}}\psi(3^n v_1,3^n v_2,3^n v_3)$$

for all $v_1,v_2,v_3,s\in T$. As the authors let n into infinity in the previous condition with utilizing the concept of $C(v_1,v_2,v_3)$, the authors can conclude that C satisfies (1.3) for all $v_1,v_2,v_3\in T$. To demonstrate the uniqueness of C, consider another cubic mapping $R(v_1,v_2,v_3)$ that adheres to both (1.3) and (3.3). In this case, the authors have:

$$\mathcal{H}(C(v_1,v_2,v_3)-R(v_1,v_2,v_3),s)$$
$$=\frac{1}{3^{3n}}\mathcal{H}(C(3^n v_1,3^n v_2,3^n v_3)-R(3^n v_1,3^n v_2,3^n v_3),s)$$
$$\leq\frac{1}{3^{3n}}\{\mathcal{H}(C(3^n v_1,3^n v_2,3^n v_3)-g(3^n v_1,3^n v_2,3^n v_3),s)$$
$$+\mathcal{H}(g(3^n v_1,3^n v_2,3^n v_3)-R(3^n v_1,3^n v_2,3^n v_3),s)\}$$
$$\leq\frac{2}{4\cdot 3^3}\sum_{k=0}^{\infty}\frac{\psi(3^{k+n}v_1,3^{k+n}v_2,3^{k+n}v_3)}{3^{3(k+n)}}$$
$$\to 0\text{ as } n\to\infty,\forall\ v_1,v_2,v_3,s\ \in\ T,$$

the uniqueness of C is established. As a result, the theorem is valid when $j=1$. Now, substituting v_1 by $\frac{v_1}{3}$ in (3.5), the authors reach

$$\mathcal{H}\left(g(v_1,v_2,v_3)-27g\left(\frac{v_1}{3},\frac{v_2}{3},\frac{v_3}{3}\right),s\right)\leq\frac{\psi\left(\frac{v_1}{3},0,0\right)}{4}\tag{3.12}$$

for all $v_1,v_2,v_3\in T$. The remaining part of the proof follows a parallel structure to the case when $j=1$. Consequently, the theorem is valid if $j=-1$. The theorem is now proven. □

Corollary 2. *Let $\lambda,k\in[0,\infty)$. Consider a mapping $g: T\to H$ that meets:*

$$\mathcal{H}(Dg(v_1,v_2,v_3),t)\leq\begin{cases}\lambda, & \\ \lambda\|v_1\|^z+\|v_2\|^z+\|v_3\|^z\}, & \text{if } z\neq 3;\\ \lambda\|v_1\|^z\|v_2\|^z\|v_3\|^z+\{\|v_1\|^{3z}+\|v_2\|^{3z}+\|v_3\|^{3z}\}, & \text{if } z\neq 1;\end{cases}$$

for all $v_1,v_2,v_3,t\in T$. Consequently, a unique cubic function $C: T\to H$ is established, $\ni$:

$$\mathcal{H}(g(v_1,v_2,v_3)-C(v_1,v_2,v_3),t) \leq \begin{cases} \frac{\lambda}{4|3^3-1|}, \\ \frac{\lambda\|v_1\|^z}{4|3^3-3^z|}, \\ \frac{\lambda\|v_1\|^{3z}}{4|3^3-3^{3z}|} \end{cases}$$

for all $v_1 \in T$.

4. Conclusion

Our results shed light on the complex interaction between its parts and characteristics, allowing us to pinpoint the precise locations where stability is most prominent. Our findings have important practical implications that go well beyond pure mathematics. Evaluating stability in generalized 2-normed spaces might improve our comprehension of mathematical structures in many areas of science and engineering. These findings can potentially encourage the creation of more robust models and solutions to solve pressing societal issues.

In a nutshell, this research represents a significant step forward in mathematical stability theory, demonstrating how thorough investigation may be used to reveal the stability characteristics of difficult equations. By extending our understanding of Ulam stability into the domain of generalized 2-normed spaces, the authors contribute to the dynamic landscape of mathematics, paving the way for fresh avenues of exploration and practical application.

Throughout this work, the authors have ventured into the captivating domain of mathematical stability theory, specifically focusing on the stability of a cubic equation within the Hyers-Ulam framework. The equation in question, characterized by its intricate arrangement of terms and coefficients, reflects the complexity of the mathematical phenomena explored. By scrutinizing this equation within the context of generalized 2-normed spaces, the authors have extended the frontiers of mathematical analysis to encompass a broader spectrum of areas and structures.

Conflicts of Interest

The authors assert that they own neither financial nor personal interests in the results of this study. This research is conducted without any financial or personal relationships that could potentially influence or bias the research, analysis, or findings. The authors have no affiliations or competing interests in AI detection, similarity, or subjects. This work aims to advance the mathematical stability theory field and is free from any conflicts in those areas. The authors are committed to upholding the highest academic and scientific integrity standards in the presentation and interpretation of our research.

References

[1] J. Aczel, and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989. [Google Scholar] [Publisher Link]

[2] Tosio Aoki, "On the Stability of the Linear Transformation in Banach Spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64-66, 1950. [CrossRef] [Google Scholar] [Publisher Link]

[3] M. Arunkumar, "Solution and Stability of Arun-Additive Functional Equations," *International Journal of Mathematical Sciences and Engineering Applications (IJMSEA)*, vol. 4, no. 3, pp. 33–46, 2010. [Google Scholar] [Publisher Link]

[4] Stefan Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, 2002. [CrossRef] [Google Scholar] [Publisher Link]

[5] P. Gavruta, "A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431-436, 1994. [CrossRef] [Google Scholar] [Publisher Link]

[6] Govindan Vediyappan, Alina Alb Lupaș, and Samad Noeiaghdam, "Stability of Additive Functional Equation Originating from Characteristic Polynomial of Degree Three," *Symmetry*, vol. 14, no. 4, pp. 1-14, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[7] D.H. Hyers, "On the Stability of the Linear Functional Equation," *Proceedings of the National Academy of Sciences (PNAS)*, vol. 27, no. 4, pp. 222-224, 1941. [CrossRef] [Google Scholar] [Publisher Link]

[8] Donald H. Hyers, George Isac, and Themistocles M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Springer, vol. 34, 1998. [Google Scholar] [Publisher Link]

[9] Kil-Woung Jun, and Hark-Mahn Kim, "The Generalized Hyers-Ulam-Rassias Stability of a Cubic Functional Equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 867-878, 2002. [CrossRef] [Google Scholar] [Publisher Link]

[10] Kil-Woung Jun, and Hark-Mahn Kim, "On the Hyers-Ulam-Rassias Stability of a General Cubic Functional Equation," *Mathematical Inequalities & Applications*, vol. 6, no. 2, pp. 289-302, 2003. [CrossRef] [Google Scholar] [Publisher Link]

[11] Soon-Mo Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, 2001. [Google Scholar] [Publisher Link]

[12] Zofia Lewandowska, "Linear Operators on Generalized 2–Normed Spaces," *Bulletin Mathématique De La Société Des Sciences Mathématiques De Roumanie*, vol. 42, no. 4, pp. 353–368, 1999. [Google Scholar] [Publisher Link]

[13] John M. Rassias, "On Approximately of Approximately Linear Mappings by Linear Mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126-130, 1982. [CrossRef] [Google Scholar] [Publisher Link]

[14] J.M. Rassias, "On Approximately of Approximately Linear Mappings by Linear Mappings," *Bulletin Des Sciences Mathematiques*, vol. 108, pp. 445-446, 1984.

[15] Themistocles M. Rassias, "On the Stability of the Linear Mapping in Banach Spaces," *Proceedings of the American Mathematical Society*, vol. 72, pp. 297-300, 1978. [Google Scholar] [Publisher Link]

[16] Stanislaw M. Ulam, *Problems in Modern Mathematics (Science Editions)*, 1st ed., Wiley, New York, 1964. [Google Scholar] [Publisher Link]