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Original Article

Independent Semitotal Bondage number of a Graph

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Abstract - *A subset S of vertices of a graph G with no isolated vertices is an independent semitotal dominating set if S is an independent dominating set, and for each u ∈S there is a v ∈S at a distance exactly two. The independent semitotal domination number of a graph is the minimum size of an independent semitotal dominating set of vertices in G and is denoted by $\gamma_{it2}(G)$. This paper initiates the study of independent semitotal bondage number of graphs. The independent semitotal bondage number denoted by $b_{it2}(G)$, is the minimum number of edges whose removal from the graph increases the independent semitotal domination number.*

Keywords - *Semitotal domination, Independent semitotal domination, Bondage number, Semitotalbondage number, Independent semitotal bondage number.*

1. Introduction

In this paper simple, finite, undirected graphs are considered. For graph theoretic terminology, refer to the book by G.Chartrand and Lesniak [1]. Let $G = (V, E)$ be a graph. "A set S of vertices of G is a dominating set if for each $v \in V - S$ there is a u inS such that $uv \in E(G)$. The minimum and maximum cardinality of a minimal dominating set of G are the domination number $\gamma(G)$ and upper domination number $\Gamma(G)$ respectively". For further information on domination, refer to [5] and [6]. "A dominating set S of G is an independent dominating set if $< S >$ consists of isolate vertices. $i(G)$, the independent domination number of G is the minimum cardinality of a minimal independent dominating set."

"The maximum cardinality of a maximal independent set of vertices of G is the independence number ofG, denoted by $\beta_0(G)$." Cockayne.E.J. et.al. [2] established a domination chain as "$ir(G) \le \gamma(G) \le i(G) \le \beta_0(G) \le \Gamma(G) \le IR(G)$"

The above chain was extended using isolate domination parameters $\gamma0(G)$ and $\Gamma0(G)$ by Sahul Hamid.I. and Balamurugan.S. [12] as "$ir(G) \le \gamma(G) \le \gamma0(G) \le i(G) \le \beta0(G) \le \Gamma0(G) \le \Gamma(G) \le IR(G)$"

The concept of independent semitotal domination number of a graph is introduced by S.V.Padmavathi and J.Sabari Manju [11] motivated from semitotal domination by Goddard et.al.[4].

"In an educational institution the teachers are considered as vertices and two teachers are joined by an edge if they share a class. This forms a non-complete connected setup. In order to pass information to students, minimum number of teachers who don't share classes are chosen. This forms an independent dominating set. Also when two teachers chosen are on-duty, it is necessary to have a common back-up so that information can be passed to students through the back-up teacher. Thus in this setup, independent semitotal domination is applied."

"Aset S of vertices of G is an independent semitotal dominating set if S is an independent dominating set and for each $u \in S$ there is a $v \in S$ at a distance exactly two. The minimum and maximum cardinality of a minimal independent semitotal dominating set are called the independent semitotal domination number $\gamma_{it2}(G)$ and upper independent semitotal domination number $\Gamma_{it2}(G)$ of G respectively."[11]

This parameter exactly lies between $i(G)$ and $\beta_0(G)$ of a graph. Then the domination chain can be extended as

$$\text{“}ir(G) \le \gamma(G) \le \gamma_0(G) \le i(G) \le \gamma_{it2}(G) \le \Gamma_{it2}(G) \le \beta_0(G) \le \Gamma_0(G) \le \Gamma(G) \le IR(G)\text{”.}$$

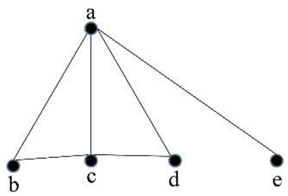

Fig.1 Graph with $i(G) < \gamma_{it2}(G) < \beta_0(G)$

$i(G) = |\{a\}| = 1$
$\gamma_{it2}(G) = |\{c,e\}| = 2$
$\beta_0(G) = |\{b,d,e\}| = 3$

J.F. Fink et.al. defined the "bondage number of a graph" in[3]. " The bondage number b(G) of a nonemptygraph G is the minimum cardinality among all sets of edges F for which $\gamma(G-F) > \gamma(G)$."

Kartal et. al. defined "semitotal bondage number of a graph" in [7]. Likewise, the concept of independent semitotal bondage number of a graph is defined here.

"The independent Semitotal Bondage number of a graph G with no isolated vertices is the minimum number of edges whose removal from G increases the independent semitotal domination number."

For a graph G, if there exists a $W \subset E(G)$ such that
(i) there is no isolated vertex in G-W
(ii)$\gamma_{it2}(G-W) > \gamma_{it2}(G)$ then W is independent semitotal bondage edge set.

If at least one such independent semitotal bondage edge set can be found for a graph G, independent semitotal bondage number can be defined and denoted by bit2(G)=min{|W | : W is an independent semitotal bondage edge set of G}.

Otherwise, bit2(G) = ∞

Proposition 1.1. [11]

"(i) $\gamma_{it2}(P_n) = \left\lceil \frac{2n}{5} \right\rceil$

(ii) $\Gamma_{it2}(P_n) = \left\lceil \frac{n}{2} \right\rceil$

(iii) $\gamma_{it2}(C_n) = \Gamma_{it2}(C_n) = \left\lceil \frac{2n}{5} \right\rceil$

(iv) $\gamma_{it2}(W_{n+1}) = \gamma(C_n) = \left\lfloor \frac{2n}{5} \right\rfloor = \left\lceil \frac{n}{3} \right\rceil$

(v) $\Gamma_{it2}(W_{n+1}) = \left\lfloor \frac{n}{2} \right\rfloor$

(vi) $\gamma_{it2}(K_{r1,r2....rn}) = r_i$ where r_i is the minimum vertices of a partite set V_i
(i.e) $r_i = \min(r_1,r_2,\ldots r_n)$
(vii) $\Gamma_{it2}(K_{r1,r2....rn}) = r_j$ where $r_j = \max(r_1,r_2,\ldots r_n)$."

2. Independent Semitotal Bondage Number of some Standard Graphs

Observation 2.1. For paths P_2, P_3, P_4, $b_{it2}(P_2) = b_{it2}(P_3) = b_{it2}(P_4) = \infty$.

Theorem 2.2 For path P_n where $n \geq 5$, $b_{it2}(P_n) = \begin{cases} 2 & n \equiv 3 \pmod 5 \\ 1 & Otherwise \end{cases}$

Proof. Case(i) Let $n \neq 3 (mod 5)$
Let $w_1, w_2, ... w_{n-1}$ be the edges of P_n.
Let $W = \{w_3\}$. Then $P_n - W$ has two components P_3 and P_{n-3}.
$\gamma_{it2}(P_n - W) = \gamma_{it2}(P_3) + \gamma_{it2}(P_{n-3})$
$= 2 + \left\lceil \frac{2(n-3)}{5} \right\rceil$
$> \left\lceil \frac{2n}{5} \right\rceil$
$= \gamma_{it2}(P_n)$
$\therefore b_{it2}(P_n) \leq 1$

Suppose $b_{it2}(P_n) < 1$. Then $b_{it2}(P_n) = 0$ and $W = \varphi$.
Now $\gamma_{it2}(P_n - W) = \gamma_{it2}(P_n)$ which is a contradiction.
Thus, $b_{it2}(P_n) = 1$.

Case(ii) Let $n \equiv 3 (mod 5)$
Let $w_1, w_2, ... w_{n-1}$ be the edges of P_n.
Let $W = \{w_3, w_5\}$. Then $P_n - W$ has three components P_3, P_2 and P_{n-5}
$\gamma_{it2}(P_n - W) = \gamma_{it2}(P_3) + \gamma_{it2}(P_2) + \gamma_{it2}(P_{n-5})$
$= 2 + 1 + \left\lceil \frac{2(n-5)}{5} \right\rceil$
$> \left\lceil \frac{2n}{5} \right\rceil$
$= \gamma_{it2}(P_n)$
$\therefore b_{it2}(P_n) \leq 2$
Suppose $b_{it2}(P_n) < 2$. Let $b_{it2}(P_n) = 1$.
Then $W = \{w_i\}$ and $i \neq 1, n-1$
Then, $P_n - W$ has two components P_r, P_s
$\gamma_{it2}(P_n - W) = \gamma_{it2}(P_r) + \gamma_{it2}(P_s)$ where $2 \leq r, s \leq n-2$ and $r + s = n$
$\gamma_{it2}(P_n - W) = \gamma_{it2}(P_n)$ which is a contradiction.
Let $b_{it2}(P_n) = 0$ Then $W = \varphi$.
Now $\gamma_{it2}(P_n - W) = \gamma_{it2}(P_n)$ which is a contradiction.
Thus, $b_{it2}(P_n) = 2$.

Observation 2.3. *For cycles C_3 and C_4,* $b_{it2}(C_3) = 1$
$b_{it2}(C_4) = \infty$

Theorem 2.4. *For Cycle C_n where $n \geq 5$,* $b_{it2}(C_n) = \begin{cases} 3 & n \equiv 3 (\text{mod } 5) \\ 2 & otherwise \end{cases}$

Proof. Let 'e' be any edge in C_n.
Then $C_n - \{e\} = P_n$
By theorem 2.2 $b_{it2}(C_n) = \begin{cases} 3 & n \equiv 3 (\text{mod } 5) \\ 2 & otherwise \end{cases}$

Observation 2.5. *For wheel W_4,* $b_{it2}(W_4) = 1$.

Theorem 2.6. *For Wheel W_n where $n \geq 5$, $b_{it2}(W_n) = \begin{cases} 3 & n = 3m + 2 \\ 2 & Otherwise \end{cases}$ where m=1,2..*

Proof. Case (i) Let $n \neq 3m + 2$.
Label the edges which are incident with the central vertex as $f_1, f_2, f_3, ...f_{n-1}$ and edges in the outercycle as $e_1, e_2, e_3,$

$...e_{n-1}$.
Let $E' = \{e_1, e_2\}$. Then $W_n - E' = F_{n-1} + P_2$
$\gamma_{it2}(W_n - E') = \gamma_{it2}(F_{n-1}) + \gamma_{it2}(P_2)$
$=\gamma_{it2}(P_{n-2}) + \gamma_{it2}(P_2)$

$$= \left\lceil \frac{2(n-2)}{5} \right\rceil + 1$$

$$> \left\lceil \frac{n}{3} \right\rceil$$

$=\gamma_{it2}(W_n)$
$b_{it2}(W_n) \leq 2$.
Suppose $b_{it2}(W_n) < 2$ Let $b_{it2}(W_n) = 1$
Let e_i be any edge in the outer cycle.
Then $W_n - \{e_i\} = F_n$
$\gamma_{it2}(W_n - \{e_i\}) = \gamma_{it2}(F_n)$
$= \gamma(P_{n-1})$
$$= \left\lceil \frac{n-1}{3} \right\rceil$$
$$< \left\lceil \frac{n}{3} \right\rceil$$
$= \gamma_{it2}(W_n)$

This is a contradiction.
Let $b_{it2}(W_n) = 0$ Then $E' = \varphi$. Now $\gamma_{it2}(W_n - E') = \gamma_{it2}(W_n)$ which is a contradiction.

Therefore $b_{it2}(W_n) = 2$ when $n \neq 3m + 2$
case(ii) Let n = 3m+2
Let $E' = \{e_1, e_2, e_3\}$ Then $W_n - E' = F_{n-2} + P_2 + P_2$
$\gamma_{it2}(W_n - E') = \gamma_{it2}(F_{n-2}) + \gamma_{it2}(P_2) + \gamma_{it2}(P_2)$
$= \gamma(P_{n-3}) + \gamma_{it2}(P_2) + \gamma_{it2}(P_2)$
$$= \left\lceil \frac{n-1}{3} \right\rceil + 1 + 1$$

$$> \left\lceil \frac{n}{3} \right\rceil$$
$= \gamma_{it2}(W_n)$

Therefore $b_{it2}(W_n) \leq 3$
Suppose $b_{it2}(W_n) < 3$ and let $b_{it2}(W_n) = 2$
Let $E' = \{e_1, e_2\}$ Then $W_n - E' = F_{n-1} + P_2$
$\gamma_{it2}(W_n - E') = \gamma_{it2}(F_{n-1}) + \gamma_{it2}(P_2)$
$= \gamma(P_{n-2}) + \gamma_{it2}(P_2)$
$$= \left\lceil \frac{n-2}{3} \right\rceil + 1$$
$$< \left\lceil \frac{n}{3} \right\rceil$$
$= \gamma_{it2}(W_n)$

This is a contradiction.
Let $b_{it2}(W_n) = 1$
Let $E' = \{e_1\}$ Then $W_n - E' = F_n$
$\gamma_{it2}(W_n - E') = \gamma_{it2}(F_n)$
$= \gamma(P_{n-1})$

$= \left\lceil \frac{n-1}{3} \right\rceil$

$< \left\lceil \frac{n}{3} \right\rceil$

$= \gamma_{it2}(W_n)$
This is a contradiction.
Let $b_{it2}(W_n) = 0$ Then $E' = \varphi$. Now $\gamma_{it2}(W_n - E') = \gamma_{it2}(W_n)$ which is a contradiction. Therefore $b_{it2}(W_n) = 3$ when $n = 3m + 2$

Observation 2.7. $b_{it2}(K_n) = 1$

Observation 2.8. $b_{it2}(K_{1,n}) = \infty$, $b_{it2}(K_{2,2}) = \infty$

Theorem 2.9. *For* $m, n > 1, b_{it2}(K_{m,n}) = \begin{cases} m+n & m \text{ and } n \neq 2 \\ \max\{m,n\} & m \text{ or } n = 2 \end{cases}$

3. Some Results

Theorem 3.1. *For a complete graph K_n, let 'W' be an edge bondage set then* $b_{it2}(K_n - W) = 2$
Proof. For a complete graph K_n, $b_{it2}(K_n) = 1$
Therefore $|W| = 1$.
In $K_n - W$ graph, there is a pair of non-adjacent vertices x and y.
Now choose any one vertex say 'u' and remove its adjacency with x and y.
The resultant graph has three mutually non-adjacent vertices x,y and u.
The rest of the vertices have degree n-1.
so these three vertices form a γ_{it2}-set.
Therefore $b_{it2}(K_n - W) = 2$.

Theorem 3.2. *For a tree T, bit2(T) = 1 iff there is an edge xy in T such that $x \in S$ and either 'x' is a central vertex of a star or 'x' is the only internal private neighbour for a 'u' in S*
Proof. Let T be a tree and S be $\gamma_{it2} - set$.
Assume $b_{it2}(T) = 1$.
Let xy be the edge such that $\gamma_{it2}(T - xy) > \gamma_{it2}(T)$
Let T_1 and T_2 be the two components of $T - xy$.
Let $x \in T_1$ and $y \in T_2$.
Suppose x and y do not belong to S, then removing the edge xy does not increase $|S|$.
Therefore x or $y \in$ S (say x).
Suppose x is not a central vertex of a star and since $\gamma_{it2}(T - xy) > \gamma_{it2}(T)$, then x is an internal private neighbour of a $u \in S$ (ie) $ipn_{=2}(u) = x$.
Conversely, let x be a central vertex of a star in $T - xy$, then there is no vertex in S at a distance two for x in T_1.
Therefore $\gamma_{it2}(T - xy) > \gamma_{it2}(T)$
Suppose $'x'$ is the only internal private neighbour for a $'u'$ in S, then this also implies $\gamma_{it2}(T - xy) > \gamma_{it2}(T)$.

Theorem 3.3. *Let G_1 and G_2 be vertex disjoint graphs with $b_{it2}(G_1) = b_{it2}(G_2) = \infty$. Let G be a graph obtained by joining uv such that $u \in G_1$ and $v \in G_2$. Then $b_{it2}(G) \neq \infty$*

Observation 3.4. *There exists a tree T with bit2(T) = k, for any non negative integer 'k'.*

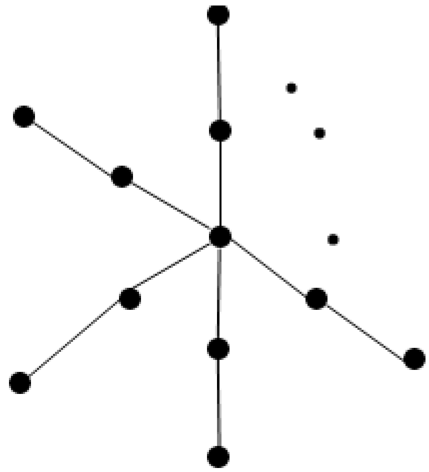

Fig. 2 Tree with 2k+3 vertices

Observation 3.5. *There exists a graph G with $b_{it2}(G)=k$, for any non negative integer 'k'.*

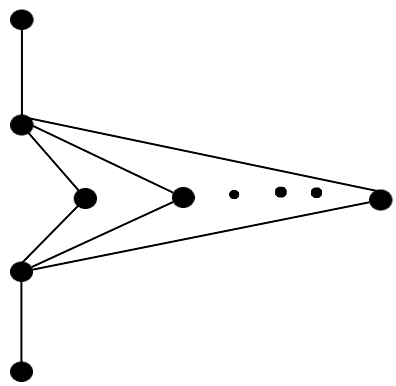

Fig. 3 Graph with $k+4$ vertices

Theorem 3.6. *For a graph G, if $\gamma_{t2}(G) = \gamma_{it2}(G)$,then $b_{t2}(G) = b_{it2}(G)$.*

But the converse is not true.

4. Conclusion

"A set S of vertices of G is an independent semitotal dominating set if S is an independent dominating set and for each $u \in S$ there is a $v \in S$ at a distance exactly two. The minimum and maximum cardinality of aminimal independent semitotal dominating set are called the independent semitotal domination number $\gamma_{it2}(G)$ and upper independent semitotal domination number $\Gamma_{it2}(G)$ of G respectively." This new parameter lies between $i(G)$ and $\beta_0(G)$.

This γ_{it2} is introduced and presented in Online International workshop on Domination in Graphs organised by Indian Institute of Technology, Ropar and Academy of Discrete Mathematics and Applications during14-16 November 2021 and "Independent Semitotal Domination in Graphs" paper has been communicated [11]

In the sequence ,Independent Semitotal Domination Excellent graphs is presented in Online National Conference on Emerging Trends in Discrete Mathematics organised by Hemachandracharya North Gujarat University during 17-19 February 2022 and the results have been communicated.

Here Independent Semitotal Bondage number of G is studied. Independent Semitotal Bondage number of Special graphs is in process

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