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Original Article

Inverse Fair Restrained Domination in the Corona of Two Graphs

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Abstract *- Let G be a connected simple graph. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S. A fair dominating set $S \subseteq V(G)$ is a fair restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a fair dominating set $S \subseteq V(G)$ is a fair restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. Let D be a minimum fair restrained dominating set of G. A fair restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an inverse fair restrained dominating set of G with respect to D. The inverse fair restrained domination number of G denoted by $\gamma_{frd}^{-1}(G)$ is the minimum cardinality of an inverse fair restrained dominating set of G. An inverse fair restrained dominating set of cardinality $\gamma_{frd}^{-1}(G)$ is called $\gamma_{frd}^{-1}(G)$-set. In this paper, the researchers investigate the concept and give some important results on inverse fair restrained dominating sets under the corona of two graphs.*

Keywords *- Dominating set, Fair dominating set, Fair restrained dominating set, Inverse fair restrained dominating set, Corona of two graphs.*

1. Introduction

A Swiss mathematician Leonhard Euler first presented the fundamental thought of graphs in eighteenth era. His endeavors and inevitable answer for the popular Konigsberg bridge problem portrayed is ordinarily cited as root of theory of graph [35].

Accordingly, one of the quickest developing area in theory of graph is the domination [36]. The analysis of dominating set in theory of graph was initiated by Claude Berge in 1958 and Oystein Ore in 1962. Berge wrote in his book, *Theory of Graphs and Its Applications*, about the “coefficient of external stability” referring to the domination number of a graph while on the other hand, Ore first used the term "domination" in his book entitled *Theory of Graphs*, respectively. Since then, the domination in graphs became an area of study by many researchers.

A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

In 2011, Caro, Hansberg and Henning [14] introduced fair domination and k-fair domination in graphs. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called $\gamma_{fd}(G)$-set. Some studies on fair domination in graphs were found in the paper [15, 16].

Meanwhile, the restrained domination in graphs was introduced by Telle and Proskurowski [17] indirectly as a vertex partitioning problem. Accordingly, a set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of a restrained dominating set of G, denoted by $\gamma_r(G)$, is called the restrained domination number of G. A restrained dominating set of cardinality $\gamma_r(G)$ is called γ_r-set. Restrained domination in graphs was also found in the papers [18, 19, 20, 21, 22, 23, 24, 25].

The other variance of domination is the fair restrained dominating set which can be found in [26]. A fair dominating set $S \subseteq V(G)$ is a *fair restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The minimum cardinality of a fair restrained dominating set of G, denoted by $\gamma_{frd}(G)$, is called the fair restrained domination number of G. A fair restrained dominating set of cardinality $\gamma_{frd}(G)$ is called $\gamma_{frd}(G)$-set.

Another variance of domination is the inverse dominating set. Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of inverse dominating set is called an *inverse domination number* of G and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called γ^{-1}-set of G. Inverse domination in graphs is found in [27, 28, 29, 30, 31, 32].

These variance of dominations led the researcher to first introduce inverse fair restrained domination in graphs in [33]. Let D be a minimum fair restrained dominating set of G. A fair restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an *inverse fair restrained dominating set* of G with respect to D. The *inverse fair restrained domination number* of G denoted by $\gamma_{frd}^{-1}(G)$ is the minimum cardinality of an inverse fair restrained dominating set of G. An inverse fair restrained dominating set of cardinality $\gamma_{frd}^{-1}(G)$ is called γ_{frd}^{-1}-set.

In this paper, the researchers investigate the concept and give some important results on inverse fair restrained dominating sets under the corona of two graphs. For the general terminology in graph theory, readers may refer to [34].

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called vertices and the cardinality $|V(G)|$ of $V(G)$ is the order of G. The elements of $E(G)$ are called edges and the cardinality $|E(G)|$ of $E(G)$ is the size of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set. The *closed neighborhood* of X in

G is the set. When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$].

$$N_G[X] = \bigcup_{v \in X} N_G(v) = \bigcup_{v \in X} N_G(X) \cup X \qquad N_G(X) = \bigcup_{v \in X} N_G(v)$$

2. Results

Let G and H be graphs of order m and n, respectively. The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy H^v of H in the corona of G and H is denoted by $v + H^v$.

Remark 2.1 *For any connected graph G and graph H, $V(G)$ is a minimum fair dominating set in $G \circ H$.*

The following results are needed for the characterization of an inverse fair dominating set in the corona of two graphs.

Lemma 2.2 *Let G and H be nontrivial connected graphs. If $S = \bigcup_{v \in V(G)} V(H^v)$ and $D = V(G)$, Then S is an inverse fair restrained dominating set of $G \circ H$.*

Proof. Since $D = V(G)$, D is a minimum fair restrained dominating set of $G \circ H$ by Remark 2.1. *If* $S = \bigcup_{v \in V(G)} V(H^v)$, then $S = V(G \circ H) \setminus D$ is a dominating set of $G \circ H$ is clear. Let $x, y \in V(G \circ H) \setminus S = V(G)$.

Then $|N_{G \circ H}(x) \cap S| = |V(H)| = |N_{G \circ H}(y) \cap S|$. Hence, S is a fair dominating set of $G \circ H$. Further, for every $v \in V(G \circ H) \setminus S = V(G)$, there exists $v' \in V(G)$ and $u \in S$ such that $vv' \in E(G \circ H)$ and $vu \in E(G \circ H)$. This means that S is a restrained dominating set of $G \circ H$, that is, S is a fair restrained dominating set of $G \circ H$. Since D is a minimum fair restrained dominating set of G∘H and $S = V(G \circ H) \setminus D$, it follows that S is an inverse fair restrained dominating set of $G \circ H$.

Lemma 2.3 *Let G and H be nontrivial connected graphs. If $S = \bigcup_{v \in V(G)} S_v$ where S_v is an $|S_v|$-fair dominating set of H^v for each $v \in V(G)$ and $D = V(G)$, then S is an inverse fair restrained dominating set of $G \circ H$ with respect to D.*

Proof. Since $D = V(G)$, D is a minimum fair restrained dominating set of $G \circ H$. Let $x, y \in V(H^v) \setminus S_v$ for each $v \in V(G)$. Since S_v is an $|S_v|$-fair dominating set of H^v for each $v \in V(G)$,

$$|N_{G\circ H}(x) \cap S| = |N_{H^v}(x) \cap S_v| = |S_v| = |N_{H^v}(y) \cap S_v| = |N_{G\circ H}(y) \cap S|$$

for all $x, y \in V(H^v) \setminus S_v$ for all $v \in V(G)$ and

$$|N_{v+H^v}(v) \cap S_v| = |S_v| = |N_{v+H^v}(y) \cap S_v|$$

for all $v \in V(G)$ and $y \in V(H^v) \setminus S_v$.

Hence, S is a fair dominating set of $G \circ H$. For each $u \in V(G \circ H) \setminus S$, there exists $x \in S$ and $v \in V(G \circ H) \setminus S$ such that $ux, uv \in E(G \circ H)$, that is, S is a fair restrained dominating set of $G \circ H$. Since D is a minimum fair restrained dominating set of $G \circ H$ and $S \subseteq V(G \circ H) \setminus D$, it follows that S is an inverse fair restrained dominating set of $G \circ H$ with respect to D. ■

2.4. *Let G and H be nontrivial connected graphs. If $S = V(G) \cup (\bigcup_{v \in V(G)} S_v$ where S_v is a fair restrained dominating set of H^v and $D_v = \{x\}$ is a dominating set of Hv with $S_v \cap D_v = \emptyset$ for all $v \in V(G)$, then S is an inverse fair restrained dominating set of $G \circ H$.*

Proof. Since $D_v = \{x\}$ is a dominating set of H^v, $D = \bigcup_{v \in V(G)} D_v$ is a dominating set of $G \circ H$. Since H is a nontrivial connected graph, $V(H^v) \setminus D_v \neq \emptyset$ for each $v \in V(G)$. Let $y \in V(H^v) \setminus D_v$. Then

$$|N_{G\circ H}(v) \cap D| = |N_{v+H^v}(v) \cap D_v| = |N_{v+H^v}(y) \cap D_v| = |N_{G\circ H}(y) \cap D|$$

for all $v, y \in V(G \circ H) \setminus D$. Hence, D is a fair dominating set of $G \circ H$. Since $V(G)$ is a minimum fair dominating set in $G \circ H$ and

$$\begin{aligned} |D| &= \left|\bigcup_{v \in V(G)} D_v\right| \\ &= \sum_{v \in V(G)} |D_v| \\ &= |V(G)||D_v| \\ &= |V(G) \cdot 1 = |V(G)| \end{aligned}$$

It follows that D is also a minimum fair dominating set of $G \circ H$. Since G and H are nontrivial connected graphs, $V(G \circ H) \setminus D \neq \emptyset$. Let $u \in V(G \circ H) \setminus D$. Then there exists $u' \in V(G \circ H) \setminus D$ and $x \in D$ such that $uu', uv \in E(G \circ H)$. Hence D is a restrained dominating set of $G \circ H$, that is, D is a minimum fair restrained dominating set of $G \circ H$.

Now, S_v is a fair restrained dominating set of H^v implies that $V(v + \langle S_v \rangle)$ is a fair restrained dominating set of $v + H^v$ for all $v \in V(G)$. Thus,

$$S = V(G) \cup (\bigcup_{v \in V(G)} S_v) = \bigcup_{v \in V(G)} V(v + \langle S_v \rangle)$$

is a fair restrained dominating set of $G \circ H$.

Since $S_v \cap D_v = \emptyset$ for all $v \in V(G)$,

$$\begin{aligned} S \cap V &= \left(V(G) \cup (\bigcup_{v \in V(G)} S_v \right) \cap \left(\bigcup_{v \in V(G)} D_V \right) \\ &= \left(V(G) \cap \left(\bigcup_{v \in V(G)} D_v \right) \right) \cup \left(\left(\bigcup_{v \in V(G)} S_v \right) \cap (\bigcup_{v \in V(G)} D_v \right) \right) \\ &= \left(\bigcup_{v \in V(G)} (V(G) \cap D_v \right) \cup \left(\bigcup_{v \in V(G)} (S_v \cap D_v) \right) \\ &= \left(\bigcup_{v \in v(G)} \emptyset \right) \cup \left(\bigcup_{v \in V(G)}, \emptyset \right) \text{since } V(G) \cap D_v = \emptyset \\ &= \emptyset \end{aligned}$$

Thus, $S \cap D = \emptyset$. Let $S \subset V(G \circ H) \setminus D$. Since D is a minimum fair restrained dominating set, it follows that S is an inverse fair restrained dominating set of $G \circ H$ with respect to D. ■

Lemma 2.5 *Let G and H be nontrivial connected graphs. If* $S = \bigcup_{v \in V(G)} S_v$ *where* S_v *is an* $|S_v|$*-fair dominating set of* H^v *and* $D_v = \{x\}$ *is a dominating set of* H^v *with* $S_v \cap Dv = \emptyset$ *for all* $v \in V(G)$*, then* S *is an inverse fair restrained dominating set of* $G \circ H$.

Proof : Suppose that $D_v = \{x\}$ is a dominating set of H^v. Let $D = \bigcup_{v \in V(G)} D_v$. By similar reasoning that is used in the proof of

Lemma 2.4, D is a minimum fair restrained dominating set of $G \circ H$. Let $x, y \in V(H^v) \setminus S_v$ for each $v \in V(G)$. Since S_v is an $|S_v|$-fair dominating set of H^v for each $v \in V(G)$,

$$|N_{G \circ H}(x) \cap S| = |N_{H^v}(x) \cap S_v| = |S_v| = |N_{H^v}(y) \cap S_v| = |N_{G \circ H}(xy) \cap S|$$

for all $x, y \in V(H^v) \setminus S_v$ and $|N_{G \circ H}(v) \cap S| = |N_{v+H^v}(v) \cap S_v| = |S_v|$ for all $v \in V(G)$. Thus, for all $u, v \in V(G \circ H) \setminus S$,

$$|N_{G \circ H}(v) \cap S| = |N_{v+H^v}(v) \cap S_v| = |N_{v+H^v}(u) \cap S_v| = |N_{G \circ H}(u) \cap S|$$

Hence, S is a fair dominating set of $G \circ H$. Clearly, for every $u \in V(G \circ H) \setminus S$, theres exists $x \in S$ and $u' \in V(G \circ H) \setminus S$ such that $uu', ux \in E(G \circ H)$. Thus, S is restrained dominating set of $G \circ H$, that is, S is a fair restrained dominating set of $G \circ H$. Since $S_v \cap D_v = \emptyset$ for all $v \in V(G)$,

$$\begin{aligned} S \cap D &= \left(\bigcup_{v \in V(G)} S_v \right) \cap \left(\bigcup_{v \in V(G)} D_v \right) \\ &= \bigcup_{v \in V(G)} (S_v \cap D_v) \\ &= \bigcup_{v \in V(G)} (\emptyset) \\ &= \emptyset \end{aligned}$$

Thus, $S \cap D = \emptyset$. Let $S \subset V(G \circ H) \setminus D$. Since D is a minimum fair restrained dominating set, it follows that S is an inverse fair restrained dominating set of $G \circ H$ with respect to D. ■

The following result, shows the characterization of an inverse fair restrained dominating set in the corona of two graphs.

Theorem 2.6 *Let G and H be nontrivial connected graphs. A nonempty subset S of $V(G \circ H)$ is an inverse fair restrained dominating set of $G \circ H$ with respect to D if and only if for each $v \in V(G)$, one of the following is satisfied:*

$(i)\ D = V(G)\ and\ S = \bigcup_{v\in V(G)} S_v\ where$

$a)\ S_v = V(H^v), or$

$b)\ S_v\ is\ an\ |S_v| - fair\ dominating\ set\ of\ H^v$

$(ii)\ D = \bigcup_{v\in V(G)} D_v\ where\ D_v = \{x\} is\ a\ dominating\ se\ of\ H^v\ and$

$a)\ S = V(G), or$

$b)\ S = V(G) \cup \left(\bigcup_{v\in V(G)} S_v\right) where\ S_v\ is\ a\ fair\ dominating\ set\ of\ H^v\ and$

$c)\ S_v \cap D_v = \emptyset\ c)\ S = \bigcup_{v\in V(G)} S_v\ where\ S_v\ is\ an\ |S_v| - fair\ dominating\ set\ of\ H^v\ and$

$S_v \cap D_v = \emptyset$

Proof : Supposed that a nonempty subset S of $V(G \circ H)$ is an inverse fair restrained dominating set of $G \circ H$ with respect to D. Then D is a γ_{frd}-set of $G \circ H$ such that $S \cap D = \emptyset$. Consider the following cases:

Case 1: Supposed that $D = V(G)$. Since $S \cap D = \emptyset$, let $S = \bigcup_{v \in V(G)} S_v$ such that $S_v \subseteq V(H^v)$ and $S_v \neq \emptyset$. If $S_v = V(H^v)$ for each $v \in V(G)$, then the proof of statement $(i)a)$ is done. Suppose that $S_v \neq V(H^v)$. Let $x \in V(H^v) \setminus S_v$ for each $v \in V(G)$. Since S is a fair dominating set of $V(G \circ H)$, S_v must be a fair dominating set of H^v for each $v \in V(G)$. This means that $|N_{H^v}(x) \cap S_v| = |N_{H^v}(y) \cap S_v|$ for each $x, y \in V(H^v) \setminus S_v$. Since S is a fair dominating set of $G \circ H$, $|N_{v+H^v}(x) \cap S_v| = |S_v|$ for all $x \notin S$. This implies that for all $x, y \notin S_v$, $|N_{H^v}(x) \cap S_v| = |N_{H^v}(y) \cap S_v| = |S_v|$, that is, S_v is an $|S_v|$-fair dominating set of H^v for each $v \in V(G)$. This proves the statement $(i)b)$.

Case 2. Suppose that D 6= V (G). If D V (G), then D is not a dominating set of $G \circ H$ contrary to the definition of D. Thus, must be $D \subseteq V(G \circ H) \setminus V(G)$. If $D = V(G \circ H) \setminus V(G)$, then $D = \bigcup_{v\in V(G)} V(H^v)$. Since H is nontrivial, $|V(H)| \geq 2$ and

$$
\begin{aligned}
|D| &= \left|\bigcup_{v\in V(G)} V(H^v)\right| \\
&= \sum_{v\in V(G)} V(H^v) \\
&= |V(G)||V(H)| \geq |V(G)| \cdot 2 > |V(G)|
\end{aligned}
$$

that is, $|D| > |V(G)|$. By Remark 2.1, $V(G)$ is a minimum fair dominating set of $G \circ H$ contradicts to our assumption that $D \subset \gamma_{fd}$- set of $G \circ H$. This implies that $D \neq (G \circ H) \setminus V(G)$. Thus, $D \subset V(G \circ H) \setminus V(G)$. Let $D = \bigcup_{v\in V(G)} V(D_v)$ where $D_v \subset V(H^v)$ for all $v \in V(G)$. Since D and $V(G)$ are minimum fair dominating sets of $G \circ H$, $|D| = |V(G)|$. Thus,

$$|V(G)| = |D| = |\bigcup_{v\in V(G)} D_v| = \sum_{v\in V(G)} |D_v| = |V(G)||D_v|,$$

where $D_v \subset V(H^v)$ for all $v \in V(G)$. This implies that $|D_v| = 1$. Since D is a dominating set of $G \circ H$, $D_v = \{x\}$ must be a dominating set of H^v for all $v \in V(G)$. Thus, $D = \cup_{v\in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v. Now, let $S \subseteq$

$V(G \circ H) \setminus D$. If $S = V(G \circ H) \setminus D$ then S is not a restrained dominating set of $(G \circ H)$ because $V(G \circ H) \setminus S = D$ is the union of isolated vertices of $(G \circ H)$. Thus, $S \neq V(G \circ H) \setminus D$, that is,

$$S \subset V(G \circ H) \setminus D = V(G) \cup (\bigcup_{v \in V(G)} V(H^v) \setminus D_v)$$

, then the proof of $(ii)a)$ is done. Consider that $V(G) \subset S$ and let $S_v \subset V(H^v)$ for each $v \in V(G)$ such that $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$. If $S = V(G)$. If S_v is not fair dominating set of H^v for each $v \in V(G)$, then S is not a fair dominating set of $G \circ H$ contrary to the definition of S. Thus, S_v must be a fair dominating set of H^v for each $v \in V(G)$. Since D is a minimum fair dominating set of $G \circ H$ and S is an inverse fair dominating set of $\circ H$, $S \cap D = \emptyset$ and $V(G) \cap D_v = \emptyset$ where $D_v \subset V(H^v)$ for all $v \in V(G)$,

$$\begin{aligned}
S \cap D &= \left((V(G) \cup (\bigcup_{v \in V(G)} S_v) \right) \cap \left(\bigcup_{v \in V(G)} D_v \right) \\
&= \left(V(G) \cap (\bigcup_{v \in V(G)} D_v) \right) \cup \left((\bigcup_{v \in V(G)} S_v) \cap (\bigcup_{v \in V(G)} D_v) \right) \\
&= \left(\bigcup_{v \in V(G)} (V(G) \cap D_v) \right) \cup \left(\bigcup_{v \in V(G)} (S_v \cap D_v) \right) \\
&= \left(\bigcup_{v \in V(G)} \emptyset \right) \cup \left(\bigcup_{v \in V(G)} (S_v \cap D_v) \right) \\
&= \left(\bigcup_{v \in V(G)} (S_v \cap D_v) \right) = \emptyset
\end{aligned}$$

This implies that $S_v \cap D_v = \emptyset$ for all $v \in V(G)$. This proves statement $(ii)b)$. Further, consider that $V(G) \not\subset S$ and let $S_v \subset V(H^v) \setminus D_v$ for each $v \in V(G)$ such that $S = \bigcup_{v \in V(G)} S_v$. If S_v is not a fair dominating set of H^v for each $v \in V(G)$, then S is not a fair dominating set of $G \circ H$. Thus, S_v must be a fair dominating set of H^v for each $v \in V(G)$. However, since $v \in V(G)$ is not an element of S, $|N_{v+H^v}(v) \cap S_v| = |S_v|$. Hence, for all $x, y \in V(G \circ H) \setminus S$, $|N_{v+H^v}(vx) \cap S_v| = |N_{v+H^v}(y) \cap S_v| = |S_v|$, that is, S_v is an $|S_v|$-fair dominating set of H^v for each $v \in V(G)$ with $S_v \cap D_v = \emptyset$ by similar computations above. This proves statement $(ii)c)$.

For the converse, suppose that statement (i) is satisfied. Then $D = V(G)$ and $S = \bigcup_{v \in V(G)} S_v$. Consider statement $(i)a)$. Then $S_v = V(H^v)$, that is $S = \bigcup_{v \in V(G)} V(H^v)$. By Lemma 2.2, S is an inverse fair restrained dominating set of $G \circ H$. Consider statement $(i)b)$. Then S_v is an $|S_v|$-fair dominating set of H^v, that is, $S = \bigcup_{v \in V(G)} S_v$. By Lemma 2.3, S is an inverse fair restrained dominating set of $G \circ H$

Suppose that statement (ii) is satisfied. Then $D = \bigcup_{v \in V(G)} D_v$ where $D_v = \{x\}$ is a dominating set of H^v. Consider statement $(ii)a)$. Then $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$ where S_v is a fair restrained dominating set of H^v and $S_v \cap D_v = \emptyset$. By Lemma 2.4, S is an inverse fair restrained dominating set of $G \circ H$. Consider statement (ii)b). Then $S = \bigcup_{v \in V(G)} S_v$ where S_v is an $|S_v|$-fair dominating set of H^v and $S_v \cap D_v = \emptyset$. By Lemma 2.5, S is an inverse fair restrained dominating set of $G \circ H$. This complete the proofs. ■

The next result is an immediate consequence of Theorem 2.6

Corollary 2.7 *Let G and H be nontrivial connected graphs with $|V(G)| = m$ and $|V(H)| = n$, and $k = |S_v|$ where S_v is a γ_{fd}-set of H^v for all $v \in V(G)$. Then*

$$\gamma_{frd}^{-1}(G \circ H) = \begin{cases} m, if\ \gamma(H) = 1 \\ km, if\ \gamma(H) \neq 1 \end{cases}$$

Proof: Suppose that a nonempty subset S of $V(G \circ H)$ is an inverse fair restrained dominating set of $G \circ H$. Then $\gamma_{frd}^{-1}(G \circ H) \leq |S|$. Consider the following cases:

Case 1. Suppose that $\gamma(H) = 1$. Then $S = V(G)$, by Theorem 2.6$(ii)a$). This implies that $\gamma_{frd}^{-1}(G \circ H) \leq |S| = |V(G)|$. Note that $V(G)$ is a γ_{fd}-set of $G \circ H$ by Remark 2.1. Further, for each $v \in V(G)$, $|D_v| = \gamma(H^v) = 1$ and $D = \bigcup_{v \in V(G)} D_v$. Thus,

$$|D| = |\bigcup_{v \in V(G)} D_v| = \sum_{v \in V(G)} |D_v| = |V(G)| \cdot |D_v| = |V(G)| \cdot 1 = |V(G)|.$$

That is, D is also a γ_{fd}-set of $G \circ H$. Thus,

$$|V(G)| = |D| \leq \gamma_{frd}^{-1}(G \circ H) \leq |S| = |V(G)|.$$

Hence, $\gamma_{frd}^{-1}(G \circ H) = |V(G)| = m.$

Case 2. Suppose that $\gamma(H) \neq 1$. Let S_v be a minimum k-fair dominating set of H^v for all $v \in V(G)$ where $k = |S_v| \geq 2$ (since $\gamma(H) = 2$). By Theorem 2.6 $(i)b)$, $D = V(G)$ and $S = \bigcup_{v \in V(G)} S_v$. Then,

$$|S| = |\bigcup_{v \in V(G)} S_v| = \sum_{v \in V(G)} |S_v| = |V(G)|| |S_v| = mk > |V(G)| = |D|.$$

Thus, $|S| > |D|$. Since $|D| = \gamma_{fd}(G \circ H) \leq \gamma_{frd}^{-1}(G \circ H) \leq |S|$ for all inverse fair dominating set S, it follows that the minimum inverse fair restrained dominating set of such S is a γ_{frd}^{-1}-set $G \circ H$, that is, $|S| = \gamma_{frd}^{-1}(G \circ H)$. Hence, $\gamma_{frd}^{-1}(G \circ H) = |S| = km$. ■

3. Conclusion and Recommendations

In this work, the fair restrained domination in the join of two paths of order $n \geq 2$ were characterized and the exact fair restrained domination number resulting from this binary operation of two paths were computed. This study will result to new research such as bounds and other binary operations of two graphs. Other parameters involving the inverse fair restrained domination in graphs may also be explored. Finally, the characterization of a fair restrained domination in graphs and its bounds is a promising extension of this study.

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