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Original Article

Exitence of touchdown solutions to MEMS control problem

Dang Le Anh Truong[1]

[1] *Faculty of Education, An Giang Univercity, An Giang, Viet Nam.*

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Abstract - *In this paper, we consider the following nonlocal model*

$$\begin{cases} \partial_t u = \Delta u + \dfrac{\lambda f}{(1-u)^2\left(1+\alpha\displaystyle\int_\Omega \frac{dx}{1-u}\right)^2} \ in \ \Omega\times(0,T), \\ u = 0 \ on \ \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \ for \ \Omega. \end{cases}$$

The model is a well-known one to study micro-electromechanical systems (MEMS) devices. We prove the existence of touchdown solutions in the genera cases that Ω is an arbitrary bounded domain and λ, α are positive numbers and f is not constant.

1. Introduction

In this paper, we are interested in the following nonlocal equation of Dirichlet type

$$\begin{cases} \partial_t u = \Delta u + \dfrac{\lambda f}{(1-u)^2\left(1+\alpha\displaystyle\int_\Omega \frac{dx}{1-u}\right)^2} \ in \ \Omega\times(0,T), \\ u = 0 \ on \ \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \ for \ \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary, f is a nonnegative continuous function on $\bar{\Omega}$ and λ, α are positive constants, and $u : \Omega\times[0,T) \to [0,1)$.

The model (1.1) is well-known for modelling the motion of some elastic menbrannes which is usually found in Micro-Electro Mechanical Systems (MEMS) devices included in a variety of electronic devices such as microphones, transducers, sensors, actuators and so on. We kindly refer the readers to [1], [2], [3], [4] and the references therein for more detail. Here, we briefly mention that such a MEMS device actually contains an elastic membrane which is hanged above a rigid ground plate connected in series with a fixed voltage source and a fixed capacitor. In (1.1), λ is the ratio of the reference electrostatic force and the reference elastic force, f is the varying dielectric properties of the divice and α denotes the ratio of the fixed capacitance and the reference capacitance of the device. One focus on the distance (deflection) between the elastic membrane and the rigid ground which depends on time and its maximum is less than 1, see in (1.1).

The Cauchy problem of (1.1) is locally well posed in $C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ where $Q_T = \Omega\times[0,T)$, see in [2]. In addition, the authors in [5] also proved that the existence of the solution can be continued as long as u stricly remains less than 1. We refer to [5] where the authors proved the existence of the global solution to equation (1.1). We observe that the '1' is critical for the deflection of the membrane, since the devices will breaks down. The singularity formation actually occurs $u(\cdot;t)$ touches down '1' at some point in Ω and in finite time T i.e.

$$\liminf_{t\to T}\left[\min_{x\in\bar{\Omega}}\left(1-u(x,t)\right)\right]=0. \tag{1.2}$$

In particular, we call the phenomenon by touchdown in finite time. The phenomenon has gotten a lot of attention in recent decades from mathematicians and engineering. We firstly mention to [6] where the authors gave a lower bound to the touchdown behavior asymptotic. In addition, the studies [1] showed the touchdown rate when the touchdown phenomenon occurs. In particular in [5], the authors showed the existence of touchdown solutions to (1.1) provided that Ω is a bounded domain with a smooth boundary satisfying $\left|\Omega\right|\le\frac{1}{2}$, $f(.)\equiv 1$, and $\lambda>0$, $\alpha=1$. Motivated in this result, we aim to show in this paper the extension to the cases where Ω is a bounded domain with a smooth boundary arbitrarily given, α, $\lambda>0$. Besides that, we also handle to the case where f is not constant.

2. Main results

Lemma 2.1. *Let u be a classical solution to (1.1) on $\left[0,T\right]$ for some $T>0$ with $f(.)\equiv 1$ and α, $\lambda>0$. Then, the energy functional*

$$E(t)=\frac{1}{2}\int_\Omega\left|\nabla u\right|^2dx+\frac{\lambda}{\alpha\left(1+\alpha\int_\Omega\frac{dx}{1-u}dx\right)}, \tag{2.1}$$

satisfies

$$\frac{dE}{dt}=-\int_\Omega u_t^{\,2}(x,t)dx<0 \text{ for all } t\in(0,T). \tag{2.2}$$

Proof. Let us consider u be a classical solution to (1.1) that belongs to $C^{2,1}(Q_T)\cap C(\bar{Q}_T)$. By using (1.1) and (2.1), we arrive at

$$\frac{dE}{dt}=\int_\Omega\nabla u.\nabla u_t dx-\lambda\frac{\int_\Omega\frac{u_t}{(1-u)^2}dx}{\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}=-\int_\Omega\Delta u.u_t dx-\lambda\frac{\int_\Omega\frac{u_t}{(1-u)^2}dx}{\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}$$

$$=-\int_\Omega u_t.\left(u_t-\frac{\lambda}{(1-u)^2\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}\right)dx-\lambda\frac{\int_\Omega\frac{u_t}{(1-u)^2}dx}{\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}$$

$$=-\int_\Omega u_t^{\,2}(x,t)dx<0, \text{ for all } t\in\left(0,T\right),$$

which concludes (2.2) and the lemma follows.

Remark 2.2. *From* (2.2), *it is easy to check that:*

$$0\le\int_0^T\int_\Omega u_t^{\,2}(x,t)dxdt=E(0)-E(T)\le E(0)<\infty. \tag{2.3}$$

Theorem 2.3. *Let Ω be a bounded domain with smooth boundary arbitrarily given, $f\equiv 1$ and $\lambda,\alpha>0$. Then, there exist initial data u_0 such that the solution u to (1.1) touches down in finite time provided the corresponding initial energy is chosen sufficiently small.*

Proof. Since the problem (1.1) is locally well-posed in $C^{2,1}(Q_T)\cap C(\bar{Q}_T)$ (classical solution), we argue by contradiction that for all $u_0\in C^2(\Omega)\cap C(\bar{\Omega})$, it has a global solution u on $(0,+\infty)$. Let us define

$$A(t)=\int_\Omega u^2(x,t)dx. \tag{2.4}$$

Note that $u(x,t)\in[0,1)$ for all (x,t) in $\overline{\Omega}\times[0,+\infty)$, we have then

$$A(t)=\int_\Omega u^2(x,t)dx\le\int_\Omega dx=|\Omega| \text{ for all } t\ge 0. \tag{2.5}$$

Multiplying equation (1.1) by u and integrating over Ω, we obtain

$$\frac{1}{2}\frac{dA}{dt}=\int_\Omega u.u_t dx=\int_\Omega u\left[\Delta u+\frac{\lambda}{(1-u)^2\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}\right]dx$$

$$=\int_\Omega u\left[\Delta u+\frac{\frac{\lambda}{(1-u)^2}}{\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}\right]dx=\int_\Omega u\Delta u dx+\lambda\frac{\int_\Omega\frac{u}{(1-u)^2}dx}{\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}.$$

Using integration by parts in combining the condition $u\,|_{\partial\Omega\equiv 0}$, we have

$$\int_\Omega u\Delta u dx=-\int_\Omega|\nabla u|^2dx.$$

Consequently,

$$\frac{dA}{dt}=-\int_\Omega|\nabla u|^2dx+\lambda\frac{\int_\Omega\frac{u}{(1-u)^2}dx}{\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}. \tag{2.6}$$

From (2.1), the energy dissipation formula (2.2) and relation (2.6), we express as follows

$$\frac{1}{2}\frac{dA}{dt}=-2E(t)+\frac{2\lambda}{\alpha\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)}+\lambda\frac{\int_\Omega\frac{u}{(1-u)^2}dx}{\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}$$

$$\ge-2E(0)+\lambda\frac{2\left(1+\alpha\int_\Omega\frac{1-u}{(1-u)^2}dx\right)+\alpha\int_\Omega\frac{u}{(1-u)^2}dx}{\alpha\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}$$

$$=-2E(0)+\lambda\frac{2+\alpha\int_\Omega\frac{2-u}{(1-u)^2}dx}{\alpha\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2}. \tag{2.7}$$

On the one hand, using Holder and Young inequalities, we arrive at

$$\left(1+\alpha\int_\Omega\frac{dx}{1-u}\right)^2=1+2\alpha\int_\Omega\frac{dx}{1-u}+\left(\alpha\int_\Omega\frac{dx}{1-u}\right)^2\le 1+\alpha\left(\int_\Omega dx\right)^2+(\alpha^2+\alpha)\left(\int_\Omega\frac{dx}{1-u}\right)^2.$$

On the other hand, we use Jensen inequality to obtain

$$\left(\int_{\Omega} \frac{dx}{1-u}\right)^2 \leq \int_{\Omega} \frac{dx}{(1-u)^2}.$$

Hence, we get

$$\left(1+\alpha\int_{\Omega} \frac{dx}{1-u}\right)^2 \leq 1+\alpha\left|\Omega\right|^2 + (\alpha^2+\alpha)\left|\Omega\right|\int_{\Omega} \frac{dx}{(1-u)^2}.$$

Accordingly to (2.7), we have

$$\frac{1}{2}\frac{dA}{dt} \geq -2E(0) + \lambda \frac{2+\alpha\int_{\Omega} \frac{2-u}{(1-u)^2}dx}{\alpha\left(1+\alpha\left|\Omega\right|^2 + (\alpha^2+\alpha)\left|\Omega\right|\int_{\Omega} \frac{dx}{(1-u)^2}\right)}. \tag{2.8}$$

We note that $u \in \left[0,1\right)$, we have

$$2-u \geq 1,$$

which yields

$$\int_{\Omega} \frac{2-u}{(1-u)^2}dx \geq \int_{\Omega} \frac{dx}{(1-u)^2} \geq \int_{\Omega} dx = \left|\Omega\right|.$$

Thus, we deduce from (2.8) that

$$\begin{aligned} \frac{1}{2}\frac{dA}{dt} &\geq -2E(0) + \lambda \frac{2+\alpha\int_{\Omega} \frac{2-u}{(1-u)^2}dx}{\alpha\left(1+\alpha\left|\Omega\right|^2 + (\alpha^2+\alpha)\left|\Omega\right|\int_{\Omega} \frac{dx}{(1-u)^2}\right)} \\ &\geq -2E(0) + \lambda \frac{2+\alpha\int_{\Omega} \frac{dx}{(1-u)^2}}{\alpha\left(1+\alpha\left|\Omega\right|^2 + (\alpha^2+\alpha)\left|\Omega\right|\int_{\Omega} \frac{dx}{(1-u)^2}\right)}. \end{aligned} \tag{2.9}$$

Now, we introduce

$$f(X) = \frac{\lambda(2+\alpha X)}{\alpha\left(1+\alpha\left|\Omega\right|^2 + (\alpha^2+\alpha)\left|\Omega\right|X\right)},$$

and the following hold.

+ Case 1: If $\left|\Omega\right| \in \left[\frac{\alpha+1-\sqrt{\alpha^2+\alpha+1}}{a};\frac{\alpha+1+\sqrt{\alpha^2+\alpha+1}}{a}\right]$, we have then

$$f(X) \geq \frac{\lambda}{(\alpha^2+\alpha)\mid\Omega\mid}, \text{ for all } X \geq 0. \tag{2.10}$$

+ Case 2: If $\left|\Omega\right| > \frac{\alpha+1+\sqrt{\alpha^2+\alpha+1}}{a}$ or $\left|\Omega\right| < \frac{\alpha+1-\sqrt{\alpha^2+\alpha+1}}{a}$ **,** we have then

$$f(X) \geq \frac{2\lambda}{\alpha\left(1+\alpha\left|\Omega\right|^2\right)}, \text{ for all } X \geq 0. \tag{2.11}$$

From (2.10) and (2.11), we arrive at

$$\lambda \frac{2+\alpha \int_{\Omega} \frac{dx}{(1-u)^2}}{\alpha\left(1+\alpha\left|\Omega\right|^2+(\alpha^2+\alpha)\left|\Omega\right|\int_{\Omega}\frac{dx}{(1-u)^2}\right)}$$
$$= f(X)\,|_{X=\int_{\Omega}\frac{dx}{(1-u)^2}} \geq \min\left(\frac{2\lambda}{\alpha\left(1+\alpha\left|\Omega\right|^2\right)},\frac{\lambda}{(\alpha^2+\alpha)\left|\Omega\right|}\right) := X_0 > 0.$$

Hence, we derive from (2.9) that

$$\frac{1}{2}\frac{dA}{dt} \geq -2E(0) + X_0. \tag{2.12}$$

Now, we choose u_0 such that $E(0) \leq \frac{X_0}{4}$, then we have

$$\frac{1}{2}\frac{dA}{dt} \geq \frac{-2X_0}{4} + X_0 = \frac{X_0}{2} > 0. \tag{2.13}$$

Integrating (2.13), we obtain

$$\int_0^t dA \geq \int_0^t X_0 dt,$$

which arrives at

$$A(t) \geq X_0 t + A(0) \geq X_0 t.$$

However, we combine with (2.5) that

$$\left|\Omega\right| \geq A(t) \geq X_0 t, \tag{2.14}$$

provided that t is a existence time of the solution. In other words, we have

$$t \leq \frac{\left|\Omega\right|}{X_0}.$$

This fact is contradict to our assumption that the solution is global. Thus, the solution must to touche down in finite time. Finally, we get the conclusion of the theorem. □

Next, we consider the problem (1.1) with $\alpha = 0,\ \lambda = 1$ and f be a continuous function that

$$\begin{cases} \partial_t u = \Delta u + \dfrac{f}{(1-u)^2} \ in\ \Omega\times(0,T), \\ u = 0\ on\ \partial\Omega\times(0,T), \\ u(x,0) = u_0(x)\ for\ \Omega, \end{cases} \tag{2.15}$$

We aim at proving the existence of touchdown solutions for the problem (2.15).

Lemma 2.4. *Let u be a classical solution to (2.15) on $\left[0,T\right]$ for some $T > 0$ Then, the energy functional*

$$E(t) = \frac{1}{2}\int_{\Omega}\left|\nabla u\right|^2 dx - \int_{\Omega}\frac{f}{1-u}dx, \tag{2.16}$$

satisfies

$$\frac{dE}{dt} = -\int_{\Omega} u_t^{\,2}(x,t)dx < 0. \tag{2.17}$$

Proof. It is quite the same as proof of Lemma 2.1 that we have

$$\begin{aligned}\frac{dE}{dt} &= \int_{\Omega} \nabla u.\nabla u_t dx - \int_{\Omega} \frac{u_t f}{(1-u)^2} dx = -\int_{\Omega} \Delta u.u_t dx - \int_{\Omega} \frac{u_t f}{(1-u)^2} dx \\ &= -\int_{\Omega} u_t \left(u_t - \frac{f}{(1-u)^2} \right) dx - \int_{\Omega} \frac{u_t}{(1-u)^2} dx \\ &= -\int_{\Omega} u_t(x,t)^2 \, dx < 0.\end{aligned}$$

Thus, the conclusion of the lemma follows.

Now, we have the following result.

Theorem 2.5. *Let Ω be a bounded domain with smooth boundary, arbitrarily given, and f be a continuous function on $\overline{\Omega}$ satisfying*

$$\int_{\Omega} f dx > 0.$$

Then, there exists initial data $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$ such that the solution to (2.15) touches down in finite time, provided the corresponding initial energy is chosen sufficiently small.

Proof. As a matter of fact, the argument is similar to Theorem 2.3. By contradiction, we asume that for all initial data $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$, and $u_0 \in (0,1)$ for all $x \in \overline{\Omega},$ the correspoding solution, u is global in time that

$$u(x,t) \in (0,1) \text{ for all } (x,t) \in \Omega \times (0,+\infty).$$

Now, we define

$$A(t) = \int_{\Omega} u^2(x,t)dx,$$

which implies

$$\frac{dA}{dt} = \int_{\Omega} 2u.u_t dx.$$

Multiplying equation (2.15) by u and integrating over $\Omega,$ we arrive at

$$\begin{aligned}\frac{1}{2}\frac{dA}{dt} &= \int_{\Omega} u\left(\Delta u + \frac{f}{(1-u)^2} \right) dx = \int_{\Omega} u\Delta u dx + \int_{\Omega} \frac{uf}{(1-u)^2} dx \\ &= -2E(t) + \int_{\Omega} \frac{f}{1-u} dx + \int_{\Omega} \frac{uf}{(1-u)^2} dx = -2E(t) + \int_{\Omega} \frac{f}{(1-u)^2} dx \\ &\geq -2E(t) + \int_{\Omega} f dx = -2E(t) + \lambda_0,\end{aligned}$$

where $\lambda_0 = \int\limits_{\Omega} f dx > 0.$

Now, we choose u_0 such that $E(0) \le \frac{\lambda_0}{4}$, then we have

$$\Rightarrow \frac{1}{2}\frac{dA}{dt} \ge \frac{-2\lambda_0}{4} + \lambda_0 = \frac{\lambda_0}{2} > 0. \quad (2.18)$$

Integrating (2.18), we obtain

$$\int_0^t dA \ge \int_0^t \lambda_0 dt,$$

which arrives at

$$A(t) \ge \lambda_0 t + A(0) \ge \lambda_0 t.$$

However, we combine with (2.5) that:

$$\left|\Omega\right| \ge A(t) \ge \lambda_0 t.$$

provided that t is a existence time of the solution. In other words, we have

$$t \le \frac{\left|\Omega\right|}{\lambda_0}.$$

This fact is contradict to our assumption that the solution is global. Thus, the solution must to touche down in finite time. Finally, we get the conclusion of the theorem. □

3. Discussion

In this paper, we obtained two main results that Theorems 2.3 and 2.5. With Theorem 2.3, we generated the result in [6, Theorem 4.2] that we showed the existence of a touchdown solution when Ω is an arbitrary bounded domain with smooth boundary. In Theorem 2.5, we handled the nontrivial case where f is not constant. Up to our knowledge, there is no result of touchdown existence for the non-constant one before.

References

[1] J.A. Pelesko and A.A. Triolo, *"Nonlocal problems in MEMS device control, Journal of Engineering Mathematics",* pp. 345-366, 2001.

[2] N. I. Kavallaris and T. Suzuki, *"Non-local partial differential equations for engineering and biology",* Mathematics for Industry (Tokyo): Springer, Cham, 2018, p. xix+300.

[3] Jong-Shenq Gou and Bei Hu, *"Quenching rate for a nonlocal problem arising in the micro-electro mechanical system",* Journal of Differential Equations, pp. 3285-3311, 2018.

[4] Jong-Shenq Gou và Philippe Souple, *"No touchdown at zero points of the permittivity profile for the MEMS problem",* Siam J. Math. Anal, pp. 614-625, 2015.

[5] Jong-Shenq Gou and N. I. Kavallaris, *"On a nonlocal parabolic problem arising in electrostatic MEMS control, Discrete and Continuous dynamical systems"*, pp. 1723-1746, 2012.

[6] Jong-Shenq Gou, Bei Hu and C. J. Wang, *"A nonlocal quenching problem arising in a micro-electro mechanical system",* Quart. Appl. Math., p. 25–734, 2009.

[7] P. Esposito and N. Ghoussoub, *"Uniqueness of solutions for an elliptic equation modeling MEMS",* Methods Appl. Anal., 15

[8] N. Ghoussoub and Y. Guo, *"On the partial differential equations of electrostatic MEMS devices: Stationary case",* SIAM J. Math. Anal., 38 (2006/07), 1423{1449.

[9] N. Ghoussoub and Y. Guo, *"On the partial differential equations of electrostatic MEMS devices. II: Dynamic case",* NoDEA Nonlinear Di_. Eqns. Appl., 15 (2008), 115{145.

[10] Y. Guo, Z. Pan and M. J. Ward, *"Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties",* SIAM J. Appl. Math., 66 (2005), 309{338.

[11] Y. Guo, *"On the partial differential equations of electrostatic MEMS devices. III: Refined touchdown behavior",* J. Diff. Eqns., 244 (2008), 2277{2309.

[12] Y. Guo, *"Global solutions of singular parabolic equations arising from electrostatic MEMS",* J.Diff. Eqns., 245 (2008), 809{844.

[13] Z. Guo and J. Wei, *"Asymptotic behavior of touch-down solutions and global bifurcations for an elliptic problem with a singular*

nonlinearity", Comm. Pure Appl. Anal., 7 (2008), 765{786.

[14] G. Flores, G. Mercado, J. A. Pelesko and N. Smyth, *"Analysis of the dynamics and touchdown in a model of electrostatic MEMS",* SIAM J. Appl. Math., 67 (2006/07), 434{446.

[15] N. I. Kavallaris, T. Miyasita and T. Suzuki, *"Touchdown and related problems in electrostatic MEMS device equation",* NoDEA Nonlinear Di_. Eqns. Appl., 15 (2008), 363{385.

[16] N. I. Kavallaris, A. A. Lacey, C. V. Nikolopoulos and D. E. Tzanetis, *"A hyperbolic non-local problem modelling MEMS technology",* Rocky Mountain J. Math., 41 (2011), 505{534.

[17] J. A. Pelesko and D. H. Bernstein, *"Modeling MEMS and NEMS",* Chapman & Hall/CRC,Boca Raton, FL, 2003.

[18] J. S. Guo, *"Recent developments on a nonlocal problem arising in the micro-electro mechanical system",* Tamkang J. Math. 45 (2014) 229–241

[19] M. Fila, J. Hulshof, *"A note on the quenching rate",* Proc. Amer. Math. Soc. 112 (1991) 473–477.

[20] G. Flores, G. Mercado, J.A. Pelesko, N. Smyth, *"Analysis of the dynamics and touchdown in a model of electrostatic MEMS",* SIAM J. Appl. Math. 67 (2007) 434–446.

[21] M.R. Boyd, S.B. Crary and M.D. Giles, *"A heuristic approach to the electromechanical modeling of MEMS beams",* Technical Digest Solid-State Sensor and Actuator Workshop (1994) pp. 123–126.

[22] D. Bernstein, P. Guidotti and J.A. Pelesko*, "Analytical and numerical analysis of electrostatically actuated mems devices",* Proceeding of Modeling and Simulation of Microsystems 2000 (2000) pp. 489–492.

[23] M.T.A. Saif, B.E. Alaca and H. Sehitoglu, *"Analytical modeling of electrostatic membrane actuator micro pumps",* IEEE J. Microelectromech. Syst. 8 (1999) 335–344

[24] J.-S. Guo, B.-C. Huang, *"Hyperbolic quenching problemwith damping in the micro-electromechanical system device",* Discrete Contin. Dyn. Syst. Series B, 19 (2014), 419-434.

[25] H.A. Levine, *"Quenching, nonquenching, and beyond quenching for solution of some parabolic equations",* Ann.Mat. Pura Appl., 155 (1989), 243–260.