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Original Article

Some results on Spectral radius and Distance energy of a Wheel graph

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Abstract

The Wheel graph on $n+1$ vertices is denoted by $W_{1,n}$ and is a graph obtained from the cycle graph C_n by joining all vertex of C_n to a new vertex called the center. In this paper, we compute the best possible bounds for the spectral radius and the distance energy of the wheel graph.
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1 Introduction

In 2007, Bo Zhou gives upper and lower bounds for the largest eigenvalue of the distance matrix in terms of the number of vertices, the sum of squares of the distance between all unordered pairs of vertices of the tree [3]. In 2009, G. Indulal gives sharp bounds of the greatest D eigenvalues and the distance energy[8]. In 2008, Gopalapillai Indulal, Ivan Gutman, and A. Vijaykumar proved the formula for the distance energy of wheel graph and is given by $E_D(W_{1,n}) = 2(n-2+\sqrt{n^2-3n+4}$ [6].

In this article, section 2 contains the preliminaries and notations related to the wheel graph and the distance energy of the wheel graph. In section3, we compute the wheel graph's upper and lower bound of distance energy and gave the best possible upper and lower bound for the spectral radius of the wheel graph.

2 Preliminaries and Notations

Definition 2.1. Distance matrix[10]
Let G be a connected graph with vertex set V and edge set E. The distance matrix D of G is defined as $D = [d_{ij}]$, where d_{ij} is the distance between the vertex v_i and v_j in G.

Definition 2.2. Distance energy[10]
The distance energy $E_D(G)$ is defined as the sum of absolute eigenvalues of the distance matrix of G,

$$E_D(G) = \sum_{i=1}^{i=n} |\mu_i|$$

Definition 2.3. Wheel Graph[9]
A wheel W_n, $n \geq 3$, is a graph obtain by joining all vertices of cycle C_n to a further vertex called the center. Thus W_n contains $n+1$ vertices and $2n$ edges. The edges incident to the center we call spokes and the edges not incident to the central vertex we call rim edges.

3 Bounds of the distance energy and the spectral radius of wheel graph

In this section, we will investigate the bounds of distance energy of the wheel graph as well as the bounds of the spectral radius of the wheel graph.

Theorem 3.1. *If $W_{1,n}$ is the wheel graph on $n+1$ vertices then*

$$\sum_{i=1}^{i=n} \mu_i^2 = 4n + 2\sum_{i=1,i<j,d(u_i,v_j)\neq 1}^{i=n} d(u_i, v_j)^2.$$

Proof. The edges in $W_{1,n}$ consist of the union of edges in cycle on n vertices and edges in star graph on $n+1$ vertices. But the cycle on n vertices contains n edges and the star graph on $n+1$ vertices consists of n edges. Therefore the number of edges in $W_{1,n}$=$2n$. Since, the sum of squares of the eigenvalues of $A_D(W_{1,n})$ is the trace of $A_D(W_{1,n})^2$.
$\sum_{i=1}^{i=n} \mu_i^2$
$= \sum_{i=0}^{i=n} \sum_{j=0}^{j=n} d_{ij}d_{ji}$
$= \sum_{i=0}^{i=n} d_{ii}^2 + \sum_{i=0,i\neq j}^{i=n} d_{ij}d_{ji}$
$= \sum_{i=0,i\neq j}^{i=n} d_{ij}d_{ji}$, since $d_{i,i}$=0 for the wheel graph.
$= 2\sum_{i=0,i<j}^{i=n} d_{u_i,v_j}$
$= 2[2n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2]$, since wheel graph contains $2n$ edges.

$= 4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2$.

Therefore, $\sum_{i=1}^{i=n} \mu_i^2 = 4n + 2\sum_{i=1,i<j,d(u_i,v_j)\neq 1}^{i=n} d(u_i, v_j)^2$. □

Theorem 3.2. *If $W_{1,n}$ is the wheel graph on $n+1$ vertices then*

$\sqrt{4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2} \leq E_D(W_{1,n}) \leq \sqrt{(n+1)[4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2]}$.

Proof. By definition of the distance energy of a graph,
$E_D(W_{1,p}) = \sum_{i=0}^{i=n} |\mu_i|$
By using the statement of Cauchy-Schwarz inequality
$(\sum_{i=1}^{i=n} x_i y_j)^2 \leq (\sum_{i=1}^{i=n}(x_i)^2)(\sum_{i=1}^{i=n}(y_i)^2))$
Replacing $x_i = 1$ and $y_i = |\mu_i|$, we get.
$(\sum_{i=1}^{i=n} |\mu_i|)^2 \leq (\sum_{i=1}^{i=n} 1)(\sum_{i=1}^{i=n} \mu_i^2))$
$= (n+1)[4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2]$
Therefore,

$$E_D(W_{1,n}) \leq \sqrt{(n+1)[4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2]} \tag{1}$$

Now, consider the inequality,
$(\sum_{i=0}^{i=p} |x_i|)^2 \geq \sum_{i=0}^{i=n} x_i^2$
Replacing x_i by μ_i, we get.
$(\sum_{i=0}^{i=p} |\mu_i|)^2 \geq \sum_{i=0}^{i=n} \mu_i^2$
$(E_D(W_{1,n}))^2 \geq [4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2]$

$$E_D(W_{1,n}) \geq \sqrt{[4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2]} \tag{2}$$

Combining the above two equations, we get.
$\sqrt{4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2} \leq E_D(W_{1,n}) \leq \sqrt{(n+1)[4n + \sum_{i=0,i<j}^{i=n} d(u_i, v_j)^2]}$.
□

Theorem 3.3. *If G is a wheel graph and μ is the greatest eigenvalue of the distance matrix of G then*

$$\mu(G) \geq (n-3) + \sqrt{n^2 - 3n + 9}$$

Proof. Let $V = \{v_0, v_1, v_2, ..., v_n\}$ be the vertex set of graph G.
Consider the distance matrix of graph G.

$D(G) = \begin{bmatrix} d_{00} & d_{01} & ... & d_{02} & d_{0n} \\ d_{10} & d_{11} & ... & d_{12} & d_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n0} & d_{n1} & ... & d_{n2} & d_{nn} \end{bmatrix}$

Let $X = (x_0, x_1, x_2, ..., x_n)^T$ be the eigenvector of $D(G)$ corresponding to the greatest eigenvalue $\mu(G)$.
$D(G)X = \mu(G)X$.

$$\begin{bmatrix} d_{00} & d_{01} & ... & d_{02} & d_{0n} \\ d_{10} & d_{11} & ... & d_{12} & d_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n0} & d_{n1} & ... & d_{n2} & d_{nn} \end{bmatrix} \times \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \mu(G) \times \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$$

Since the graph G is wheel graph its diameter is 2. hence $d_{ij} = 0, 1, 2$. Let $A = v_0$ and $B = v_1, v_2, v_3, ..., v_n$ be the partition of vertex set of graph G. take $x_i = \min x_k$ and $x_j = \min x_k$, $k \in A, x_k \in X$.
from equation (1), we have,
$\mu \times x_i = = \sum_{k=0}^{k=n} d_{ik} x_k$
$\mu(G)x_i \geq nx_i$, since $\sum_{k=0}^{k=n} d_i k = 1$ and $x_i$is minimum of x_k.
Also, $\mu \times x_j = = \sum_{k=0}^{k=n} d_{jk} x_k$

$$\mu(G)x_j = \sum_{d_{jk}=1} d_{jk} x_k + \sum_{d_{jk}=2} d_{jk} x_k, \tag{3}$$

since each row of $D(G)$ contains only three ones and $(p-3)$ two.
From equation (3),
$\mu(G)x_j \geq 3x_j + 2(n-3)x_j$
$\mu(G)x_j - 2(n-3)x_j \geq 3x_j$

$$(\mu(G) - 2(n-3))x_j \geq 3x_j \tag{4}$$

From equations (3) and (4)
$\mu(G)(\mu(G) - 2(n-3))x_i x_j \geq 3n x_i x_j$
$\mu(G)(\mu(G) - 2(n-3)) \geq 3n$
$\mu(G)(\mu(G) - 2(n-3)) - 3n \geq 0$
solving this quadratic equation, we get,
$\mu(G) \geq \frac{2(n-3) + \sqrt{4(n-3)^2 - 4(-3n)}}{2}$
$= (n-3) + \sqrt{n^2 - 6n + 9 + 3n}$
$= (n-3) + \sqrt{n^2 - 3n + 9}$

$\mu(G) \geq (n-3) + \sqrt{n^2 - 3n + 9}$

$\square$

Theorem 3.4. *If G is a wheel graph and* $\mu(G)$ *is the largest eigenvalue of the matrix then* $\mu(G) \leq d(n-3) + \sqrt{d^2n^2 - 3d^2n + 9d^2}$*, where* d *is the diameter of the wheel graph.*

Proof. Let G be a wheel graph and $V = \{v_0, v_1, v_2, ..., v_n\}$. Consider the distance matrix of graph G.

$D(G)=\begin{bmatrix} d_{00} & d_{01} & ... & d_{02} & d_{0n} \\ d_{10} & d_{11} & ... & d_{12} & d_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n0} & d_{n1} & ... & d_{n2} & d_{nn} \end{bmatrix}$

Let $X = (x_0, x_1, x_2, ..., x_n)^T$ be the eigenvector of $D(G)$ corresponding to the greatest eigenvalue $\mu(G)$.
$D(G)X = \mu(G)X$.
Let $X = (x_0, x_1, x_2, ..., x_n)^T$ be the eigenvector of $D(G)$ corresponding to the greatest eigenvalue $\mu(G)$.
$D(G)X = \mu(G)X$.

$$\begin{bmatrix} d_{00} & d_{01} & ... & d_{02} & d_{0n} \\ d_{10} & d_{11} & ... & d_{12} & d_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n0} & d_{n1} & ... & d_{n2} & d_{nn} \end{bmatrix} \times \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \mu(G) \times \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$$

Let $A = v_0$ and $B = v_1, v_2, v_3, ..., v_n$ be the partition of vertex set of graph G. take $x_i = \max x_k$ and $x_j = \max x_k$, $k \in A, x_k \in X$.
from equation (1), we have,
$\mu(G)\times x_i = \sum_{k=0}^{k=n} d_i k x_k$
$\mu(G)x_i \leq pdx_i$, since $\sum_{k=0}^{k=n} d_i k = 1$ and $x_j$is maximum of x_k.
Also, $\mu\times x_j = \sum_{k=0}^{k=n} d_j k x_k$

$$\mu(G)x_j = \sum_{d_{jk}=1} d_{jk}x_k + \sum_{d_{jk}=2} d_{jk}x_k, \tag{5}$$

since each row of $D(G)$ contains only three ones and $(n-3)$ two.
From equation (3),
$\mu(G)x_j \leq 3dx_j + 2d(n-3)x_j$
$\mu(G)x_j - 2(n-3)x_j \leq 3x_j$

$$(\mu(G) - 2d(p-3))x_j \leq 3dx_j \tag{6}$$

From equations (3) and (4)
$\mu(G)(\mu(G) - 2d(n-3))x_i x_j \leq 3dnx_i x_j$
$\mu(G)(\mu(G) - 2(n-3)) \leq 3dn$
$\mu(G)(\mu(G) - 2(n-3)) - 3n \leq 0$
solving this quadratic equation, we get,
$\mu(G) le \frac{2d(n-3)+\sqrt{4d^2(n-3)^2-4(-3dn)}}{2}$
$= d(n-3) + \sqrt{d^2n^2 - 6d^2n + 9d^2 + 3d^2n}$
$= d(n-3) + \sqrt{d^2n^2 - 3d^2n + 9d^2}$
$\mu(G) \leq d(n-3) + \sqrt{d^2n^2 - 3d^2n + 9d^2}$

□

4 Conclusion

In this paper, we obtained the upper and lower bound for the spectral radius and the distance energy of the wheel graph.

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