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Original Article

Modification of the *Inner* Cross Theorem on Crossed Quadrilateral

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Abstract ***-*** *This article discusses the inner development of Cross's Theorem on crossed quadrilaterals. By doing two steps, namely expanding the square leads inward once and twice. The proof is based on the rule of sines and the rule of cosines. The achieved result for a one-time expansion is* $||L\triangle BFG - L\triangle DKJ| = L\ \square ABCD$ and $|L\triangle LAE - L\triangle ICH| = L\ \square ABCD$ *and for double expansion, the difference between* $L\ \square EFON$and$L\ \square IJSR$*is equal to* $2L\square ABCD$*and the difference between* $L\ \square KLMT$and$L\ \square GHQP$ *is zero.*

Keywords - *Crossed Quadrilateral, Sine and Cosine, Cross Theorem.*

1. Introduction

In geometry, one of the theorems about the shape of planes is the Cross's Theorem. The Cross's Theorem states that a triangle *ABC* constructs its sides from squares facing outwards, and then the points of adjacent squares are combined to form a new triangle with the same area as triangle *ABC* [1]. The proof is [2, 3].

Cross's theorems are discussed and expanded, including Cross's theorem for quadrilaterals and triangles using quadrilaterals [4-7]. Another proof of Cross's theorem is performed using a broad concept [3]. Second, the expansion of Cross's theorem on quadrilaterals using the rules of sine and cosine [6]. In [7], we used the concept of congruence to describe the modification of triangular cross sets [8]. In addition, Cross's theorem is expanded to convex and non-convex quadrilaterals contained in [9], and Cross's theorem for convex and non-convex quadrilaterals states that the squares on each side of a convex quadrilateral are joined outwards. is extended twice by if a non-convex quadrilateral and its adjacent vertices form four new quadrilaterals, then the total area of the opposing convex quadrilaterals is $4L\square ABCD$, and the difference in the area of the opposing non-convex quadrilaterals is $4L\square ABCD$.

In triangles there are many geometric concepts, such as the area of a triangle [12-15] also on the quadrilateral has been widely discussed in the Napoleonic theorem in [16-18], Euler's quadrilateral Theorem [24] and Cross Theorem [6]. The idea of partitioning and expansion for inner also refers to [19-22].

Furthermore, quadrilaterals can be classified and grouped into convex quadrilaterals, non-convex quadrilaterals, and crossed quadrilaterals [9]. The crossed quadrilateral has the uniqueness that the area of the crossed quadrilateral is the difference between the areas of the two triangles formed and will have zero areas if the areas of the triangles formed are the same [9, 10]. Seeing that there is a between from Cross's Theorem developed on crossed quadrilaterals, this article discusses the inner development of Cross's Theorem on crossed quadrilaterals which are expanded twice leading inward as evidenced by using the rules of sines and cosines.

2. Literature Review

This literature review explains some basic definitions and supporting theories related to the cross theorem in crossed quadrilaterals. The discussion begins with the Cross's Theorem on the triangle which can be seen in [1].

Theorem 1. It is known that any $\triangle ABC$, on each side of the triangle is constructed $\square ABIH$, $\square BCGF$, $\square CAED$ pointing outwards. If the lines EH, IF,and$DG$are drawn $\triangle AEH$, $\triangle BIF$,and$\triangle CDG$ will be formed with each area equal to $\triangle ABC$, the illustration can be seen in Figure 1

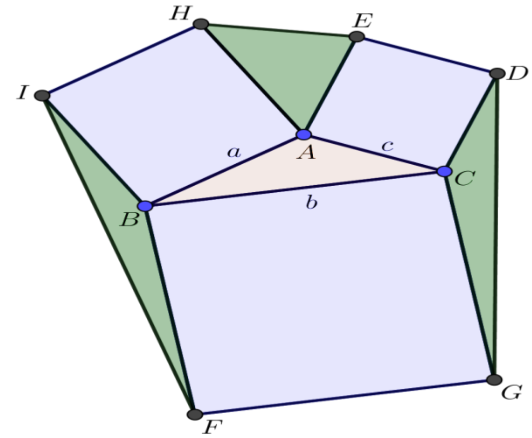

Fig. 1 Cross′ s Theorem on Triangles.

Furthermore, Cross's theorem is developed on non-convex quadrilaterals which show that the difference in the area of the new quadrilaterals facing each other is equivalent to four times the area of the quadrilateral $ABCD$ and can be seen in [8].

Theorem 2. It is known that ΔAEL, ΔBFG, ΔICH,and$\Delta DJK$are made of triangles $ABCD$Cross Theorem. Besides that $\square ELMN$, $\square GFOP$, $\square JKTS$,and $\square ADKL$ are constructed as a rectangle pointing outer on the sides of the triangles EL, FG, HI,andJK. Connect the corner squares to form a shape plane figure and have area $L\ \square EFON - L\ \square IJSR = 4L\square ABCD$ and $L\ \square KLMT - L\ \square GHQP = 4L\square ABCD$can be seen in the image below.

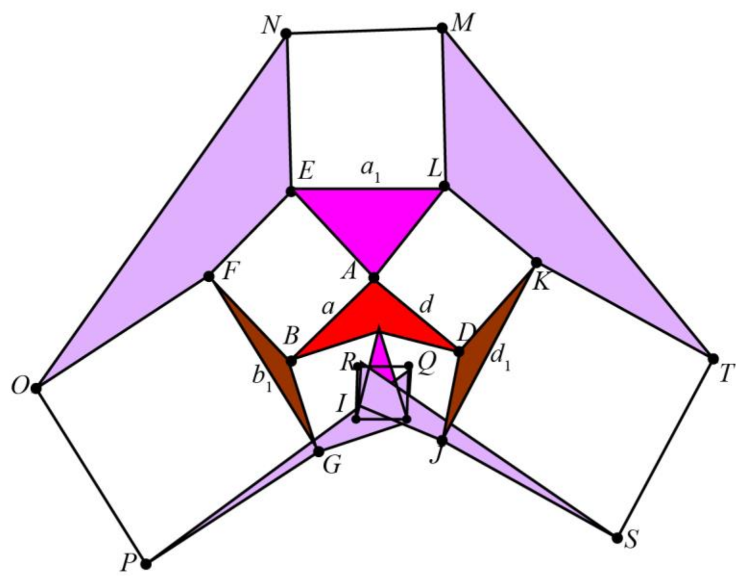

Fig. 2 Modification of the Cross's Theorem on Non-Convex Quadrilateral.

Cross' Theorem in quadrilaterals has been developed in convex quadrilaterals and non-convex quadrilaterals, in this case, Cross's theorem is developed in crossed quadrilaterals and shown in [9, 23, 25].

Theorem 3. Suppose quadrilateral $ABCD$ is any crisscrossed quadrilateral with P being the point whereAB and BC intersect. so that the area of the crossed quadrilaterals is obtained, as shown in Figure 3.

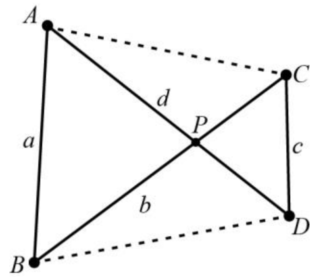

Fig. 3 Crossed Quadrilateral

3. Result Research Methodology

The methodology for proving the inner development of Cross's Theorem on a crossed quadrilateral are the construction of the $ABCD$ crossed quadrilateral in Figure 3 above, from an arbitrary crossed quadrilateral $ABCD$is constructed towards the inside of the equilateral quadrilateral from the points $AB, BC, CD,$and DA has the give a point Pat the point cut $BC$and$AD$with $AB = a, BC = b, CD = c,$and$AD = d$, it will form an equilateral quadrilateral, namely $ABFE, BCHG, CDJI,$and$DALK$with$AB = AE = BF = EF = a, BC = BG = CH = GH = b, \rightleftarrows CD = CI = DJ = JI = c,andAD = AL = DK = KL = d.$ Then a triangle is formed from the points $\Delta AEL, \Delta BFG, \Delta ICH,and\Delta DJK$ with $EA = AL = LE = a, FG = BG = GF = b, IC = CH = HI = c,andDK = KJ = JD = d$. Then it was expanded once again from the lines $EL, FG, HI,$andJK from $\square ELMN$, $\square GFOP$, $\square JKTS$,and $\square ADKL$. Then a line is drawn from point $N$to point O, point P to Q a point R to point S, and from point T to point M. So as to form four new quadrilaterals namely $\square EFON$, $\square GHQP$, $\square IJSR$,and $\square KLMT$. Furthermore, for each quadrilateral $\square EFON$, $\square GHQP$, $\square IJSR$,and $\square KLMT$ formed into two triangles, namely the $EFON$quadrilateral to become the $NFO$and NEF triangle, then the $GHQP$ quadrilateral to become the HGP and QHP triangles, then the $IJSR$quadrilateral to become the JIR and RJS triangle and for the $KLMT$ quadrilateral to become the TKL and TLM triangles. Furthermore, By applying the sine and cosine rules, it is proved that difference in the area of a $\square EFON$and $\square IJSR$ is equivalent to twice the area of $\square ABCD$, and the difference between the areas of a $\square KLMT$and $\square GHQP$ is *zero*.

4. Discussion

The results and discussion will prove the inner development of Cross's Theorem in crossed quadrilaterals using the rules of sines and cosines.

Theorem 4. Suppose quadrilateral $ABCD$ is crossed quadrilateral. Furthermore, the construction of $ADKL, ABFE, BCHG,$and$CDJI$leads inward. If the lines $EL, FG, HI,$andJK are drawn, $\Delta AEL, \Delta BFG, \Delta ICH,and\Delta DJK$ will be formed with $|L\Delta BFG - L\Delta DKJ| = L \square ABCD$ and $|L\Delta LAE - L\Delta ICH| = L \square ABCD$.

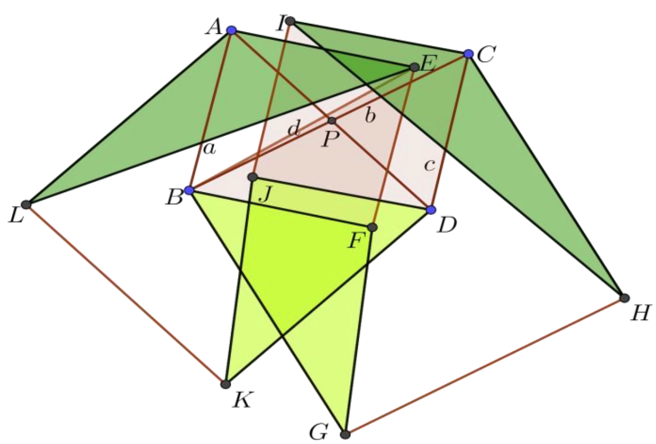

Fig. 4 *Inner* Cross's Theorem on Crossed Quadrilateral.

Proof: Assumption $AB = AE = BF = EF = a, BC = BG = CH = GH = b, \rightleftarrows CD = CI = DJ = JI = c,andAD = AL = DK = KL = d.$ and suppose $\angle BAD = \alpha, \angle CBA = \beta, \angle BCD = \gamma,and\angle ADC = \delta$. Then, using Theorem 3 and the sine rule, it will be proved $|L\Delta BFG - L\Delta DKJ| = L \square ABCD$ and $|L\Delta LAE - L\Delta ICH| = L \square ABCD$. By using trigonometry to the area of quadrilateral $ABCD$, it is found that the area of $\Delta BAD = 1/2\, ad \sin \alpha$, $\Delta ABC = 1/2\, ab \sin \beta$, $\Delta BCD =$

$1/2\, bc \sin\gamma$,and$\Delta ADC = 1/2\, cd \sin\delta$. Furthermore, based on Figure 4, the ΔLAE is obtained

$$L\Delta LAE = \frac{1}{2} ad \sin(180^o - \alpha)$$
$$L\Delta LAE = L\Delta BAD \tag{1}$$

In the same way, with attention $\Delta BFG, \Delta ICH$,andΔDKJobtained

$$L\Delta BFG = L\Delta ABC, \tag{2}$$

$$L\Delta CHI = L\Delta BCD \text{ and} \tag{3}$$

$$L\Delta DKJ = L\Delta ADC \tag{4}$$

By using Theorem 3 and equations (2) dan (4) will be obtained $L\ \square ABCD$ is

$$|L\Delta ABP - L\Delta CDP| = L\ \square ABCD,$$
$$|L\Delta BFG - L\Delta DKJ| = L\ \square ABCD \tag{5}$$

In the same way, by using Theorem 3 and equations (1) dan (3) will be obtained $L\ \square ABCD$ is

$$|L\Delta LAE - L\Delta ICH| = L\ \square ABCD \tag{6}$$

By looking at equations (5) and (6) then Theorem 4 is proven. ■

Next, the construction is done by constructing a square leading to the inner sides of each triangle in Theorem 4. To be clearer about this modification of Cross's Theorem in Theorem 5.

Theorem 5. Let $\Delta AEL, \Delta BFG, \Delta ICH$,and$\Delta DJK$ be triangles formed from $\square ABCD$ Cross' Theorem on crossed quadrilaterals. Furthermore, the construction of $\square ELMN$, $\square GFOP$, $\square JKTS$,and $\square ADKL$ leads inward on the sides of triangular EL, FG, HI,andJK. When connecting the vertices of adjacent squares, namely points $N$and O, P and Q, R and S as well as $T$and M, four quadrilaterals will be formed that have an area relationship with the crossed quadrilaterals $ABCD$, namely $|L\square EFON - L\square IJSR| = 2.L\square ABCD$ and $|L\square KLMT - L\square GHQP| = 0$, the illustration can be seen in the image below.

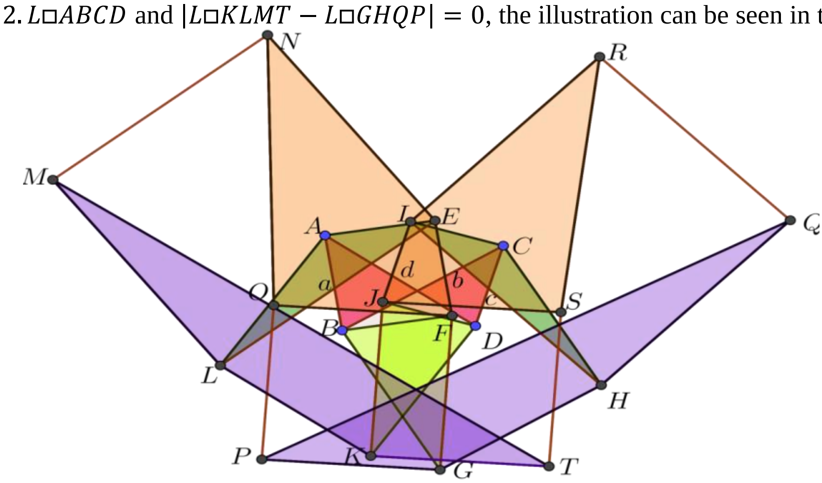

Fig. 5 Modification of the *Inner* Cross's Theorem on Crossed Quadrilateral.

Proof. Let $AB = AE = BF = EF = a, BC = BG = CH = GH = b, \rightleftarrows CD = CI = DJ = JI = c$,and$AD = AL = DK = KL = d$. Next, let the sides $EL = EN = LM = MN = a_1, FG = FO = GP = OP = b_1, HI = HQ = IR = QR = c_1$,and$JK = JS = KT = ST = d_1$. With attention to $\Delta AEL, \Delta BFG, \Delta ICH$,and$\Delta DJK$, so obtained

$$a_1^2 = a^2 + d^2 + 2ad \cos \quad \alpha \tag{7}$$

$$b_1^2 = a^2 + b^2 - 2ab \cos \quad \beta \tag{8}$$

$$c_1^2 = b^2 + c^2 + 2bc \cos \quad \gamma \tag{9}$$

$$d_1^2 = c^2 + d^2 - 2cd \cos \quad \delta \tag{10}$$

The first step is to find $L\ \square EFON$, done by dividing $\square EFON$ into ΔNFOandΔNEF, the illustration can be seen in Figure 6.

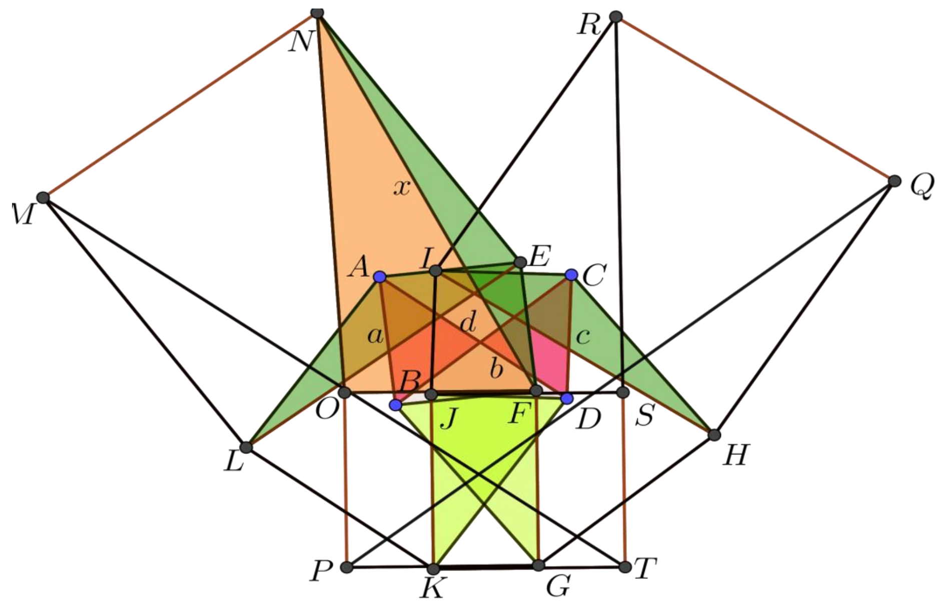

Fig. 6 illustration of $EFON$ divided into two triangles

Based on $\Delta AEL, \Delta BFG,$and ΔEFN using the obtained sine rule, in Fig. 6

$$sin\angle AEL = \frac{d\ sin\ \alpha}{a_1} \quad (11)$$

$$sin\angle BFG = \frac{b\ sin\ \beta}{b_1} \quad (12)$$

$$sin\angle EFN = \frac{d\ sin\ \alpha}{x} \quad (13)$$

Based on $\Delta AEL, \Delta BFG,$and ΔEFN in Figure 6, using the cosine rule obtained

$$cos\angle AEL = \frac{a_1^2+a^2-d^2}{2aa_1} \quad (14)$$

$$cos\angle BFG = \frac{a^2+b_1^2-b^2}{2ab_1} \quad (15)$$

$$cos\angle EFN = \frac{a^2-x^2-a_1^2}{2ax} \quad (16)$$

Then use the resulting cosine law to find the value of x

$$x^2 = 2a^2 + 2a_1^2 - d^2 \quad (17)$$

Using the sine rule on ΔNEF, obtained

$$L\Delta NEF = \frac{1}{2}aa_1\ sin\big(90° + (90° - \angle AEL)\big)$$

$$L\Delta NEF = \frac{1}{2}aa_1\ sin(180° - \angle AEL) \quad (18)$$

Next, substitute the value $sin\angle AEL$ in equation (11) into equation (20) to obtain

$$L\Delta NEF = L\Delta LAE, \quad (19)$$

Using the sine rule on ΔNFO, obtain

$$L\Delta NFO = \frac{1}{2}b_1 x\ sin(90° - 90° + \angle BFG - \angle EFN)$$

$$L\Delta NFO = \frac{1}{2}b_1 x\ sin(\angle BFG - \angle EFN)$$

$$L\Delta NFO = \frac{1}{2}b_1 x(sin\angle BFG\ cos\angle EFN - cos\angle BFG\ sin\angle EFN) \quad (20)$$

Because the value obtained from the equation (8), (12), (13), (15), (16), and (17) then the equation (20) is obtained

$$L\Delta NFO = 2L\Delta BFG - L\Delta LAE + \frac{bd\, sin(\beta+\alpha)}{2} \quad (21)$$
$$L\Delta EFON = 2L\Delta BFG + \frac{bd\, sin(\beta+\alpha)}{2} \quad (22)$$

Next, we will look for the value of $L\ \square IJSR$, done by dividing $\square IJSR$ into ΔJIRandΔRJS the illustration can be seen in Figure 7.

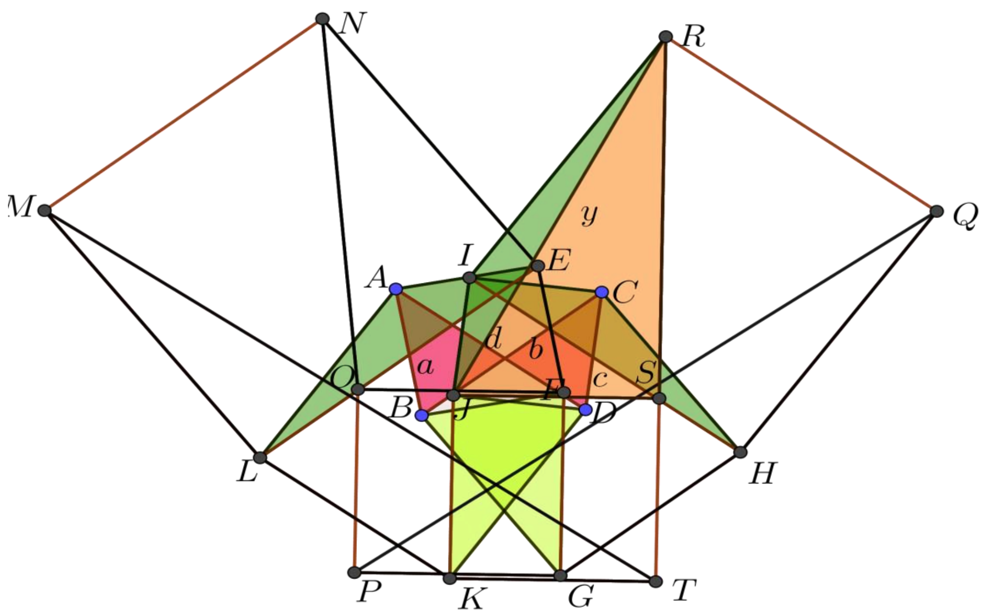

Fig. 7 illustration of $IJSR$ divided into two triangles.

With the same steps as before then $L\Delta JIR$, $L\Delta RJS$,and $L\ \square IJSR$,

$$L\Delta JIR = L\Delta HIC \quad (23)$$
$$L\Delta RJS = 2L\Delta DKJ - L\Delta ICH + \frac{bd\, sin(\delta+\gamma)}{2} \quad (24)$$
$$L\Delta IJSR = 2L\Delta DKJ + \frac{bd\, sin(\delta+\gamma)}{2} \quad (25)$$

The results obtained from equations (22) and (25) will be shown $|L\square EFON - L\square IJSR| = 2.L\square ABCD$ obtained

$$|L \square EFON - L \square IJSR| = \left|\left(2L\Delta BFG + \frac{bd\ \sin(\beta+\alpha)}{2}\right) - \left(2L\Delta DKJ + \frac{bd\ \sin(\delta+\gamma)}{2}\right)\right|$$

$$|L \square EFON - L \square IJSR| = \left|2L \square ABCD + \frac{1}{2}bd\left(2\cos\frac{(\beta+\alpha+\gamma+\delta)}{2}\cdot 0\right)\right|$$

$$|L \square EFON - L \square IJSR| = |2L \square ABCD + 0|$$
$$|L \square EFON - L \square IJSR| = 2L \square ABCD \quad (26)$$

The next step, with the same steps for $\square KLMT$ and $\square GHQP$, the The illustration can be seen in the image below.

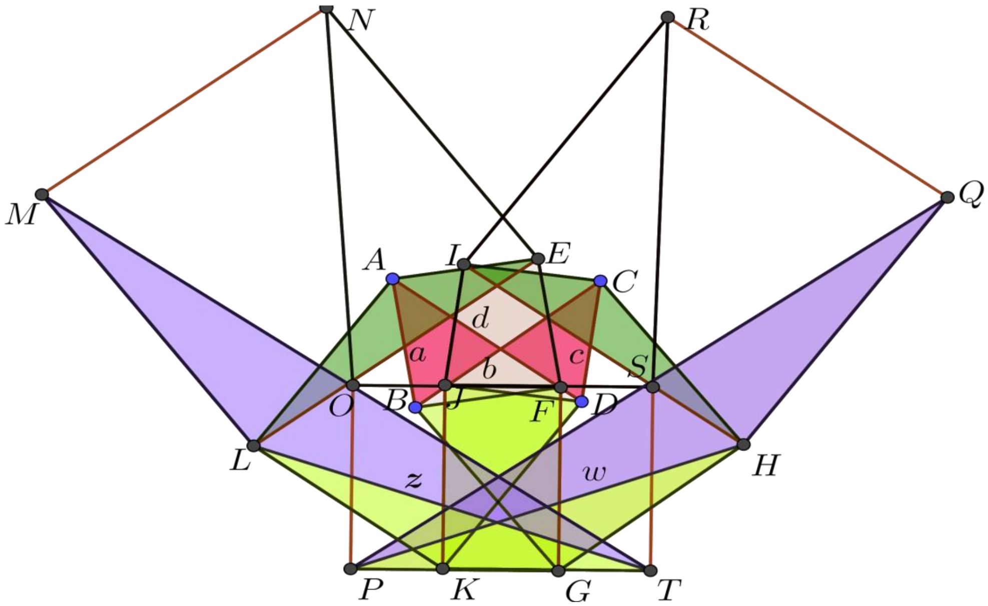

Fig. 8 illustration of $\square GHQP$ and $\square KLMT$ divided into two triangles.

With the same steps as before then $L\Delta HGP, L\Delta QHP$,and $L\ \square GHQP$

$$L\varDelta HGP = L\varDelta FGB \tag{27}$$

$$L\varDelta QHP = L\varDelta FGB + 2L\varDelta ICH + \frac{ac\sin(\beta-\gamma)}{2} \tag{28}$$

$$L\varDelta GHQP = 2L\varDelta FGB + 2L\varDelta ICH + \frac{ac\sin(\beta-\gamma)}{2} \tag{29}$$

With the same steps as before then $L\Delta TKL, L\Delta TLM$,and $L\ \square KLMT$

$$L\varDelta TKL = L\varDelta DKJ \tag{30}$$

$$L\varDelta TLM = 2L\varDelta DKJ + L\varDelta LAE + \frac{ac\sin(\delta-\alpha)}{2} \tag{31}$$

$$L\varDelta KLMT = 2L\varDelta DKJ + 2L\varDelta LAE + \frac{ac\sin(\delta-\alpha)}{2} \tag{32}$$

In the same, The results obtained from equations (29) and (32) will be shown $|L \square GHQP - L \square KLMT| = 0$ obtain

$$|L \square GHQP - L \square KLMT| = 0 \tag{33}$$

Therefore, theorem 5 is proved based on equations (26) and (33). ■

5. Conclusion

From the inner development of Cross's Theorem on crossed quadrilaterals, there is a relation between the area of the new plane that is formed and the initial rectangle that was given. The development of the inner Cross’s Theorem on a crossed quadrilateral which is expanded once and twice leads to the inner. The results obtained from the inner development of the Cross’s Theorem on crossed quadrilaterals are that the difference between $L\ \square EFON$and$L\ \square IJSR$ is equal to $2L\square ABCD$and the difference between $L\ \square KLMT$and$L\ \square GHQP$ is *zero*.

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