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Original Article

Generalized Derivations on $M_{n\times N}$(C)

K.L. Kaushik[1], Praveen Sharma[2]

[1]*Department of Mathematics, Aggarwal college, Ballabgarh, Haryana, India.*
[2]*Department of Mathematics, Shri Vishwakarma Skill University, Haryana, India.*

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Abstract *- In this paper we initiated the study of generalized derivations on $M_{n\times n}(C)$. We characterize all the generalized derivations on $M_{n\times n}(C)$ and proved the sufficient conditions when the product of two generalized derivations is again a derivation.*

Keywords *- Generalized derivation, Lie product, Multiplication operator, Generalized inner derivation.*

1. Introduction

Derivation on different algebras and Banach algebras are studied widely across the world from the 20th century. It generalizes the features and properties of the differentiation operator. The study of derivation on algebras are started back from the 1940s. In 1960 Shoichiro Sakai [6] proved that any derivation on $M_{n\times n}$(C) is a bounded linear operator on $M_{n\times n}$(C) i.e. continuous. Also in 1966 [7] he proved that all the derivations on $M_{n\times n}$(C) must be inner i.e. there is no outer derivation possible on $M_{n\times n}$(C). We generalize the definition of derivation and inner derivation on $M_{n\times n}$(C) by getting the motivation from the definition of generalized derivation given by Havala [2], Aydin[3] who defines a generalized derivation on a ring R is an additive map f : R → R such that there exists a ring derivation δ : R→R and satisfies

$$f(xy) = f(x)y + x\delta(y) \ \forall \ x,y \in R.$$

Then after that in the work [8,9]. We characterize the generalized derivations on $M_{n\times n}$(C). In [4] Bresner find few conditions in which the product of two generalized derivation on a ring R will again be a generalized derivation on R. We also find the condition on any two generalized derivations such that their product is again a generalized derivation. We also proved Let A be any commutative sub-algebra of $M_{n\times n}$(C). Then GD(A) is an algebra.

2. Overview

Let $M_{n\times n}$(C) be the set of all n × n matrices over C with an identity I. Let δ denotes any derivation on $M_{n\times n}$(C) i.e. δ : $M_{n\times n}$(C) → $M_{n\times n}$(C) is a linear map which satisfies $\delta(XY) = \delta(X)Y + X\delta(Y)$. A derivation δ is said to be inner if there exists an element A in $M_{n\times n}$(C) such that $\delta(X) = [A,X] = AX - XA \ \forall \ X \in M_{n\times n}(C)$ where [.,.] is the Lie product.

After giving the definition of generalized derivations and generalized inner derivations. In Theorem 3.4 we first proved that all the generalized derivation f on $M_{n\times n}$(C) is of the form $f = M_P + [A,.]$.

where $P,A \in M_{n\times n}$(C) and M_P is the multiplication operator on $M_{n\times n}$(C) i.e. all the generalized derivations on $M_{n\times n}$(C) are generalized inner derivations on $M_{n\times n}$(C). Let GD($M_{n\times n}$(C)) be the set of all the generalized derivations on $M_{n\times n}$(C). Theorem 3.8 we proved the product of two generalized derivations $f_1 = M_P + [A,.]$ and $f_2 = M_L + [B,.]$ on $M_{n\times n}$(C) can again be a generalized derivation iff the following two conditions holds

- $f_1(X)[B,Y] + f_2(X)[A,Y] = 0.$
- $[P,X][B,Y] + [A,X][B,Y] + [B,X][A,Y] + [A,[B,X]]([A,Y] - Y) = 0 \ \forall \ X,Y \in M_{n\times n}(C).$

Then in Theorem 3.4 we proved Let A be any commutative sub-algebra of $M_{n\times n}$(C).

Then GD(A) is an algebra.

3. Generalized Derivations

3.1. Definition(Generalized Derivations)

A linear map $f : M_{n\times n}(\mathrm{C}) \to M_{n\times n}(\mathrm{C})$ is said to be a generalized derivation on a $M_{n\times n}(\mathrm{C})$ if there exists a derivation δ on $M_{n\times n}(\mathrm{C})$ such that

$$f(AB) = f(A)B + B\delta(A)\ \forall\ A,B \in M_{n\times n}(\mathrm{C}).$$

Remark 3.2. if $f = \delta$ then clearly it is the definition of a derivation on $M_{n\times n}(\mathrm{C})$. i.e. all the derivations on $M_{n\times n}(\mathrm{C})$ are generalized derivations.

3.3. Definition(Generalized Inner Derivations)

A linear map $f : M_{n\times n}(\mathrm{C}) \to M_{n\times n}(\mathrm{C})$ is said to be a generalized inner derivation on a $M_{n\times n}(\mathrm{C})$ if there exists an inner derivation δ_A on $M_{n\times n}(\mathrm{C})$ such that $f(XY) = f(X)Y + Y\,\delta_A(X)\ \forall\ X,Y \in M_{n\times n}(\mathrm{C})$.

where $\delta_A(X) = [A,X] = AX - XA$

Theorem 3.4. *Let f be a generalized derivation on* $M_{n\times n}(\mathrm{C})$ *then it is of the form*

$$f = M_P + [A,.]$$

where $P,A \in M_{n\times n}(\mathrm{C})$ *and* M_P *is the multiplication operator on* $M_{n\times n}(\mathrm{C})$.
i.e. all generalized derivations on $M_{n\times n}(\mathrm{C})$ *are generalized inner derivations.*

Proof. For any $A \in M_{n\times n}(\mathrm{C})$ we have,

$$\begin{aligned} f(A) &= f(I.A) \\ &= f(I).A + I.\delta(A) \quad (1) \end{aligned}$$

As δ is a derivation on $M_{n\times n}(\mathrm{C})$, then it must be inner, so there exists $A \in M_{n\times n}(\mathrm{C})$ such that

$$\delta(X) = [A,X] = AX - XA\ \forall\ X \in M_{n\times n}(\mathrm{C}) \quad (2)$$

From (1) and (2)

$$\begin{aligned} f(X) &= f(I).x + [A,X] \\ &= M_{f(I)}(X) + [A,X]\ \forall\ X \in M_{n\times n}(\mathrm{C}) \end{aligned}$$

Put $\alpha = f(I)$ and hence the result.

Remark 3.5. If A is a commutative sub-algebra of $M_{n\times n}(\mathrm{C})$ then all the derivations on A must be 0. But this need not true for the generalized derivations i.e. some non zero generalized derivations are possible on A. Here we will discuss an example. Let A be the set of all the commutative $n \times n$ matrices over C then A is a commutative sub-algebra of $M_{n\times n}(\mathrm{C})$ with identity. Let $0 \neq B \in$ A be any non zero matrix then we define f : A→A by

$$f(X) = BX$$

then f is a non zero generalized derivation on A

Remark 3.6. The set of all the generalized derivations on $M_{n\times n}(\mathrm{C})$ need not to be closed under multiplication.
let $f_1 = M_P + [A,.]$ and $f_2 = M_L + [B,.]$ be any two generalized derivations on $M_{n\times n}(\mathrm{C})$ where $A \neq 0$ and B is not in the centralizer of $M_{n\times n}(\mathrm{C})$ then

$$\begin{aligned} (f_1 f_2)(XY) &= f_1(f_2(XY)) \\ &= f_1(f_2(X).Y + X.[L,Y]) \\ &= f_1(f_2(X).Y) + f_1(X.[L,Y]) \end{aligned}$$

$$= (f_1 f_2)(X).Y + f_2(X).[A,Y] + f_1(X).[L,Y] + X.f_1([L,Y])$$

which is not a generalized derivation as

$$f_2(I).[A,Y] + f_1(I).[B,Y] = L.[A,Y] + P.[B,Y] \neq 0$$

for some Y in $M_{n\times n}(\mathrm{C})$.
Now we will discuss under what conditions the product of two generalized derivation is again a derivation.

Theorem 3.7. *Let* $f_1 = M_P + [A,.]$ *and* $f_2 = M_L + [B,.]$ *be any two generalized derivations on* $M_{n\times n}$(C *and* $\alpha \in$C. *Then* $\alpha f_1 + f_2$ *is again a generalized derivation on* $M_{n\times n}(\mathrm{C})$ *of the form* $M_{\alpha P+L} + [\alpha A + B,.]$

Proof. Let $X,Y \in M_{n\times n}$(C), Then

$$(\alpha f_1 + f_2)(XY) = \alpha f_1(XY) + f_2(XY)$$
$$= \alpha M_P(XY) + \alpha[A.XY] + M_L(XY) + [B,XY]$$
$$= \alpha P(XY) + \alpha A(XY) - \alpha(XY)A + L(XY) + B(XY) - (XY)B$$
$$= (\alpha P + L)(XY) + (\alpha A + B)(XY) - (XY)(\alpha A + B) = M_{\alpha P+L}(XY) + [\alpha A + B,XY].$$

Hence $\alpha f_1 + f_2$ is again a generalized derivation on $M_{n\times n}(\mathrm{C})$ of the form $M_{\alpha P+L} + [\alpha A + B,.]$.

Theorem 3.8. *Let* $f_1 = M_P + [A,.]$ *and* $f_2 = M_L + [B,.]$ *be any two generalized derivations on* $M_{n\times n}(\mathrm{C})$. *Then following are equivalent*

(a) $f_1 f_2 \in \mathrm{GD}(M_{n\times n}(\mathrm{C}))$
(b) $f_1(X)[B,Y]+f_2(X)[A,Y] = 0$ *and* $[P,X][B,Y]+[A,X][B,Y]+[B,X][A,Y]+[A,[B,X]]([A,Y] - Y) = 0 \ \forall\ X,Y \in M_{n\times n}(\mathrm{C})$

Proof.

$$\begin{aligned}(f_1 f_2)(XY) &= f_1(f_2(XY))\\ &= f_1(f_2(X).Y + X.[B,Y])\\ &= f_1(f_2(X).Y) + f_1(X.[B,Y])\\ &= (f_1 f_2)(X).Y + f_2(X).[A,Y] + f_1(X).[B,Y] + X.f_1([B,Y])\end{aligned}$$

Now let $\delta(Y) = f_1([B,Y])$

$$\begin{aligned}\delta(XY) &= f_1([B,XY])\\ &= P[B,XY] + [A,[B,XY]]\\ &= P[B,X]Y + PX[B,Y] + [A,[B,X]Y + X[B,Y]]\\ &= P[A,X]Y + PX[B,Y] + [A,[B,X]Y] + [A,X[B,Y]]\\ &= P[B,X]Y + PX[B,Y] + [A,[B,X]][A,Y] + [B,X][A,Y] + [A,X][B,Y] + X[A,[B,Y]]\end{aligned}$$

Also

$$\begin{aligned}\delta(X)Y + X\delta(Y) &= f_1([B,X])Y + Xf_1([B,Y])\\ &= P[B,X]Y + [A,[B,X]]Y + XP[B,Y] + X[A,[B,Y]]\end{aligned}$$

Then for all $X,Y \in M_{n\times n}$(C)

$$\delta(XY) = X\delta(Y)+\delta(X)Y \iff [P,X][B,Y]+[A,X][B,Y]+[B,X][A,Y]+[A,[B,X]]([A,Y]-Y) = 0$$

Hence the result.

Theorem 3.9. *Let* A *be any commutative sub-algebra of* $M_{n\times n}(\mathrm{C})$. *Then* GD(A) *is an algebra.*

Proof. Using Theorem 3. 7 we have GD(A) is a subspace of GD($M_{n\times n}$(C)). As A is a commutative sub-algebra of $M_{n\times n}(\mathrm{C})$. Then using Theorem 3.8 we have the result.

4. Conclusion

We characterize all the generalized derivations on $M_{n\times n}(C)$ and proved conditions in which the product of two generalized derivation on a ring R will again be a generalized derivation on R. We also find the condition on any two generalized derivations such that their product is again a generalized derivation. We also proved Let A be any commutative sub-algebra of $M_{n\times n}(C)$. Then GD(A) is an algebra.

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