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Original Article

The Rotation Number in Moser Generic Diffeomorphisms of the Annulus

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Abstract *- In this paper, we will study the rotation number on the sphere, and we will show that under certain conditions we can guarantee the existence of an essential hyperbolic periodic point in Moser generic diffeomorphism of the annulus.*

Keywords *- Rotation number, Moser generic, Hyperbolic, Prime ends, Periodic orbits.*

1. Introduction

When we study dynamic systems, the evolution law is generally given by a function f and the space of times can be discrete ($T = \mathbb{Z}$) or continuous($T = \mathbb{R}$). We are interested in knowing the future behavior of this system ([1], [4]), but we also want information about the existence of periodic orbits of the system and about the behavior in terms of its stability of it. Even though it is known that for a dynamical system with a law of evolution given by a diffeomorphism f on $\mathbb{R}$ there is no nontrivial recurrence and, there can only be periodic points of period one or two, the same in the case of the circle $\mathbb{S}^1$ where there is non-trivial recurrence and we can find periodic points of any period. Thus, it is interesting to study the dynamics of diffeomorphisms on $\mathbb{S}^1$ and other spaces such as the sphere, the ring, the torus, etc.

Currently, arduous research is being done on a part of dynamical systems that has to do with the existence of generic properties. A property P is said to be generic in F if there exists a residual set $A \subset F$, such that every element of A satisfies the property P. We know that one of the most well-known and important generic properties is given by the Kupka-Smale theorem, and other properties were given by Robinson, Pixton, and Franks, among others.

One of the tools used in dynamical systems to study the existence of periodic orbits is the rotation number. In this work, we will revise some important results about rotation number in the circle, annulus, and sphere. Furthermore, we are going to relate these concepts with generic properties in diffeomorphisms defined on the sphere and the ring, and then we will prove a result about the existence of essential hyperbolic periodic points in Moser generic diffeomorphisms defined on the annulus.

2. Preliminaries

Next, we define the stable and unstable sets for a closed orbit of a vector field f defined on a manifold M, (see [1] and [2]).

Definition 1. Let $f: M \to M$ a vector field, and γ be a closed orbit of f.The stable and unstable sets of γ are given by

$$W^s(\gamma) = \{y \in M \mid \omega(y) = \gamma\} \text{ and } W^u(\gamma) = \{y \in M \mid \alpha(y) = \gamma\},$$

where, ω and α are the omega and alpha limits.

Definition 2. The tangent space to a manifold M at the point p, is given by

$$TM_p = \{\mathrm{v}_p \mid \mathrm{v}_p \text{ is a vector of a differential curve in } M \text{ that passes through p}\}$$

The tangent bundle of M is noted by TM and defined as

$$TM = \{(p, \mathbf{v}) \mid p \in M \quad y \quad \mathrm{v} \text{ is a tangent vector in p}\} = \bigcup_{p \in M} TM_p$$

Definition 3. Let S be a submanifold of N and $f: M \to N$ be a mapping of class C^r with $r \geq 1$. We say that f is transversal to S at a point $p \in M$ if: $f(p) \notin S$ or $TN_{f(p)} = Df_p(TM_p) \oplus TS_{f(p)}$.

We say that f is transverse to S, denoted by $f \pitchfork S$, if f is transverse to S at every point in M. When M is also a submanifold of N, $f = i$ is the inclusion, and i is transverse to S , we say that the manifolds M, S are manifolds that intersect transversely in N. (See [3])

We state two key results of transversality. The proof of these results can be review in [4]

Theorem 1. Let γ be a closed hyperbolic orbit of a field f of class C^r on M. Let V be a small open neighborhood of γ. The $W^s(\gamma)$ and $W^u(\gamma)$ are submanifolds of class C^r de M, futhermore $W^s(\gamma)$ is transverse to $W^u(\gamma)$ and $W^s(\gamma) \cap W^u(\gamma) = \gamma$.

Theorem 2. Let f be a field whose singularities and close orbits are hyperbolic. Give $T > 0$, there exist a finite number of closed orbits of f with period less than or equal to T. In particular, f has a maximum number of enumerable closed orbits.

Definition 4. A field of vector f of class C^r on M is said if the critical elements of f (singularities and closed orbits) are hiperbolic and if p_1 and p_2 are critical elements of f, then $W^s(p_1)$ and $W^u(p_2)$ are transverse.

Definition 5. A set A is *residual* in a space X if there a countable number of open and dense sets $\{U_j\}_{j=1}^{\infty}$ in X such that $A \subseteq \cap_{j=1}^{\infty} U_j$.

Theorem 3. [*Kupka-Smale*] The set Kupka-Smale fields is a residual subset of the set of all fields of class C^r.

Proof. See [4].

2.1. Ergodic Theory

Ergodic theory is a mathematical discipline that studies dynamical systems together with invariant measures. We will give some basic definitions and some important results that we will use throughout the work. (See [5])

Definition 5. A measure μ is said to be invariant by the transformation $f: M \to M$ if

$$\mu(E) = \mu\big(f^{-1}(E)\big) \quad \text{for every measurable set} \quad E \subset M.$$

The proof of the following theorems can be seen in [1] y [4].

Theorem 4. [*Poincaré recurrence*] Let $f: M \to M$ be a measurable transformation and μ a finite invariant. Let $E \subset M$ be any measurable set with $\mu(E) > 0$. Then, μ-almost every point $x \in E$ has some iterate $f^n(x)$, $n \geq 1$ which is also in E.

Theorem 5. [*Existence of invariant measures*] Let $f: M \to M$ be a continuous transformation on a compact metric space. Then there exists at least one invariant probability measure for f.

Given a compact metric space M, we can define $m_1(M)$ as the set of Berelian probability measures on M ans we can endow this space with a topology whose neighborhoods are given as follow

We take a measure $\mu \in m_1(M)$, a finite set $F = \{\phi_1, \phi_2 \ldots, \phi_N\}$ of continuous functions $\phi_j : M \to \mathbb{R}$, and a number $\epsilon > 0$, we define

$$V(\mu, F, \epsilon) = \{\nu \in m_1(M) \mid \left|\int \phi_j d\nu - \int \phi_j d\mu\right| < \epsilon \quad \text{for every} \quad \phi_j \in F,\}$$

The topology generated by these sets is called weak topology. One of the most important theorems of ergodic theory is Birkhoff's ergodic theorem which states the following.

Theorem 6 [*Birkhoff's Ergodic*] Let $f: M \to M$ be a measurable transformation and μ a probability measure invariant by f. For all integrable function φ: M → ℝ, the limit

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(f^j(x)\right)$$

there exist for μ-almost every point $x \in M$. Furthermore,

$$\int \tilde{\varphi}(x) d\mu(x) = \int \varphi(x) d\mu(x).$$

Proof. See [5].

3. The Rotation Number

The rotation number has been a concept widely studied by multiple authors, among these we can mention John Franks. He studied about the rotation number in one and two dimensions, in the study of geodesics, of circle and ring homeomorphisms and Moser diffeomorphisms. (See [6] to [11]). Handel studied properties of the rotation number in ring homeomorphisms ([13]), Herman obtained properties of the diffeomorphisms defined on the circle using rotation number ([14]), Novo and Núñez studied the rotation number in Hamiltonian ([15]), Zhang and Obaya got important results about the existence of periodic orbits using the rotation number ([16]) Pavani and Veldhuizen made numerical approximations of the rotation number ([17], [18] and [19]), Lamberd studied the rotation number and Lyapunov exponents for two-dimensional maps ([20], [21]). Other authors who obtained results about the existence of periodic orbits of different types of mappings; Johnson, Puel, Bates, Feng, Jaroslaw and Navas (See [20] to [26])

The rotation number can also be defined on the ring and on the sphere. In the next section, we will work on some important properties and results of the rotation number in the ring. The rotation number on the sphere has also been studied and requires elements of algebraic topology, Caratheodory theory and prime ends, Letschetz index, etc. For further information, see [27] to [31]

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. the unit circle. There exists a natural projection $\pi\colon \mathbb{R} \to \mathbb{S}^1$, given by $\pi(t) = t \bmod 1$ or $\pi(t) = e^{2\pi i t}$. We are going to assume that f is an orientation-preserving homeomorphism, that is,

$$x \leq y \bmod 1 \quad then \quad f(x) \leq f(y) \bmod 1.$$

To $F\colon \mathbb{R} \to \mathbb{R}$ that satisfies: $\pi \circ F = f \circ \pi,$ we will call it a *lifting* of f. This function is monotone increasing and $F(x+1) = F(x) + 1$.

Definition 7. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism in $\mathbb{S}^1$ and F: $A \to A$ a lifting of f. We define the *translation number* as

$$\rho_0(F, x) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}.$$

Definition 8. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism and $F\colon \mathbb{R} \to \mathbb{R}$ a lifting of f, we define the *rotation number of* f as $\rho(f) = \rho_0(F) \bmod 1$.

We have the following result about the period orbit and rotational number ([1], [32], [33])

Theorem 7. The rotation number is rational if and only if f in $\mathbb{S}^1$ has a periodic point.

Of the set of all orientation-preserving diffeomorphisms, we are interested in an important class called Morse-Smale diffeomorphisms.

Definition 9. An orientation-preserving diffeomorphism f in $\mathbb{S}^1$ is *Morse-Smale* if it has a rational rotation number, and all its periodic points are hyperbolic.

This class of diffeomorphisms satisfies two very important properties: They are structurally stable and any diffeomorphisms in $\mathbb{S}^1$ can be approximated by Morse-Smale diffeomorphisms.

Now, we define the rotation number in the annulus. Many results obtained about the dynamics of homeomorphisms defined in the annulus are motivated historically by mechanical systems and geometric problems. One of the famous problems in physics is the three bodies problem ([34] to [39]). This problem consists of determining at any instant the velocities and the positions of three bodies of any mass, subjected to each other to an attraction mutual and starting from given speeds and positions. Although it is known that it is possible to give a formula for a similar system with two masses, Poincaré showed that there is no formula for the problem posed with three masses. Euler posed a weaker problem called the restricted three body problem, which consists in assuming that the mass of one of the three bodies is negligible with respect to the other two, which can be applied, for example, to the earth moon-sun system. ([40] and [41])

Poincaré studied this problem arduously and found many results. One of them is the following

Theorem 8. [Poincaré] Given the annulus $0 < a \leq r \leq b$ in the plane, θ in polar coordinates, and T an injective transformation, continuous and area-preserving of the annulus itself, which takes points from $r = a$ and sends them into points at $r = b$. Then there exist at least two points on the ring invariant by the map T.

Proof. See [40]

We denote the annulus as $\mathbb{A} = \mathbb{S}^1 \times I \; where \; I = [0,1]$.

Definition 10. Let $\widetilde{\mathbb{A}} = \mathbb{R} \times I$. A homeomorphism $F: \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ is called a *lifting* of f if it satisfies the following relation.

$$f(\tilde{\pi}(x, y)) = \tilde{\pi}(F(x, y))$$

where, $\pi: \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ is given by $\tilde{\pi}(x, y) = (\pi(x), y)$.

Definition 11. Let be $f: \mathbb{A} \to \mathbb{A}$ be an orientation-preserving and boundary-preserving homeomorphism and let $F: \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lifting of f. The *translation number in the annulus* of a point $\omega = (x, y) \in \widetilde{\mathbb{A}}$ under F, is

$$\rho_0(F, \omega) = \lim_{n \to \infty} \frac{p_1(F^n(\omega) - \omega)}{n}$$

where, $p_1: \widetilde{\mathbb{A}} \to \mathbb{R}$ is the projection to the first coordinate, that is, $p_1(x, y) = x$.

We might think that a homeomorphism in the ring satisfies similar properties to the homeomorphisms in the circle, this is partly true, many of these properties are conserved but others are not, for example, for a homeomorphism in the ring it does not always exist $\rho_0(\omega, F)$ and when it exists it is rarely independent of the point ω.

We have the following proposition. We can see the proof in [1].

Proposition 1. Let $f: \mathbb{A} \to \mathbb{A}$ be a orientation-preserving homeomorphism and boundary components and let $F: \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lifting of f.

- If $\rho_0(F, \omega)$ exists, then $\rho_0(F^m, \omega) = m\rho_0(F, \omega)$ for all $m \in \mathbb{N}$.
- If $\rho_0(F, \omega,)$ exists, them $\rho_0(F + (m, 0), \omega) = \rho_0(F, \omega) + m$ for all $m \in \mathbb{N}$

Definition 12. Let $f: \mathbb{A} \to \mathbb{A}$ be a orientation-preserving homeomorphism and boundary components and let $F: \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lifting of f. The rotation number of f in the annulus is defined by $\rho(f) = \pi(\rho_0(F, \omega))$, where $\pi: \mathbb{R} \to \mathbb{S}^1$ is the natural projection to the first coordinate component.

Theorem 9. [Franks] Let $f: \mathbb{A} \to \mathbb{A}$ be a boundary components-orientation-area-preserving diffeomorphism. If f has at least one periodic point, then it has infinitely many such points.

Proof. See [10], [33].

2.2. Rotation Number in the Sphere

To define the rotation number in the sphere, we need to give some previous result

Definition 13. Si Ω is a open set of $\mathbb{R}^2$ and $p \in \Omega$ es an isolated fixed point of a continuous function $h: \Omega \to \Omega$, the *Lefschetz index* of h at p, denoted by $Ind(h, p)$, is the number of turns in the vector vector field $h(z) - z$ of any simple closed curve that is around and fairly close to p.

Theorem 10. [Lefschetz- Hopf] Let $f: M \to M$ be a continuous function, then

$$\sum_{x \in Fix(f)} I\,(x, f) = L(f),$$

Where, $Fix(f)$ is a fixed-point set of f, and $L(f)$ is the Lefschetz index.

Proof. See [42].

There are several ways to describe the Lefschetz number. (See [42], [27])

Definition 14. Let M be a compact manifold, $f: M \to M$ a continuous mapping and f^* the morphism induced by the homology group (rational coefficients). The Lefschetz number of f is defined as

$$\Lambda_f = \sum_{i \geq 0} (-1)^i \, Tr(f_i^*)$$

Now, we have the following theorem (See proof in [42])

Theorem 11. Let M be a compact manifold and $f: M \to M$ a continuous mapping. If $\Lambda_f \neq 0$ then f has a fixed point.

Theorem 12. Let V, W be two open and connected sets of $\mathbb{R}^2$ that with has the origin 0 and let $h: V \to W = h(V)$ be an orientation-preserving homeomorphism which has the origin as an isolated fixed point. Then

$$Ind(h, 0) \in \{-1,0,1\}.$$

Proof. See [27].

Now, we can see some previous definition to define the rotation number in the sphere.

Definition 15. A set $K \subset \mathbb{A}$ is called an *essential continuum* if it is compact, connected and its complement has two components that we denote as

$$\Omega_k^+; Upper \qquad \Omega_k^-: Lower$$

Let $f: \mathbb{S}^2 \to \mathbb{S}^2$ a homeomorphism and $\Omega \subset \mathbb{S}^2$ a simply connected non-trivial invariant domain by f, that is, $f(\Omega) = \Omega$. We can extend f a homeomorphism in the compactification of prime ends ([43], [44]) $\hat{\Omega} \cong \Omega \cup \mathbb{S}^1$ de Ω. If $f|_\Omega$ preserves orientation, then its restriction to $\mathbb{S}^1$ preserves orientation and we can define the rotation number as element of $\mathbb{S}^1$. We can denote the rotation number as ρ_Ω.

Theorem 13. Let f be an orientation-preserving homeomorphism of $\mathbb{S}^2$ and Ω a non-trivial simply connected invariant domain. Let's suppose.

- The rotation number is rational $\rho_\Omega = \frac{p}{q} \in \mathbb{Z}$.
- There is a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ that does not contain the positive orbit and negative of any non-errant open set.

Then $\partial\Omega$ contains a fixed point of f^q.

Proof. Taking f^q is the time of f, we can assume that the rotation number ρ_Ω is equal to zeroes. It follows that thre is a fixed point $z \in \mathbb{S}^1$ of the extended homeomorphism in the compactification of prime ends of Ω. From the theory of prime ends we can find a sequence of open arcs $(\gamma_n)_{n=1}^{\infty}$ in Ω whose diameters tend to zero and they can be extended to closed arcs in the compactification of prime ends with point in $\mathbb{S}^1$. Furthermore, they are next to z.

These arcs have the property that their closure $\overline{\gamma_n}$ converges to z in the compactification of prime ends when $n \to \infty$ and each of then divide Ω in two simply connected domains. We denote Ω_n as the smaller neighborhood (z is in the Ω_n in the compactification of prime ends). We want to show that $f(\gamma_n) \cup \gamma_n \neq 0$. If the arc γ_n does not find its image under F, the fact that z is a fixed point implies that one of the two inclusions $f(\Omega_n) \subset \Omega_n$ or $f^{-1}(\Omega_n) \subset \Omega_n$ In the first case, we can find an open set V contained in $\Omega_n - f(\Omega_n)$ which is not errant anh it has a positive orbit contained in Ω_n. In the second case we can build a non-errant domain whose negative orbit is contained in Ω_n. Then those conditions contradict the hypothesis, therefore we conclude that $f(\gamma_n) \cup \gamma_n \neq \emptyset$.

Since the diameters of the arcs γ_n tend to zero, then any point in $\partial\Omega$ is a fixed point of f.

From the previous theorem the following corollary

Corollary. Let $f: \mathbb{A} \to \mathbb{A}$ be a area-preserving homeomorphism homotopic to the identity and K an essential continuum invariant. Suppose that $\rho_K^+ = \frac{p}{q}$. Then K contains a fixed point of f^q and this fixed point is in $\partial\Omega_K^+$. The same for ρ_K^-.

3. Generic Properties

In this section we will discuss some generic properties of the set of diffeomorphisms on $\mathbb{S}^2$ that preserve an element of area ω. We will state the main generic properties, we will use them to define a Moser generic diffeomorphism, and we will list some of its characteristics. For the study of these diffeomorphisms, we will base ourselves on the results already obtained and the properties of the Lefschetz index.

We denote by $Diff(r, \omega, M)$ the set of all C^r-diffeomorphisms on M that preserve an area element ω. This set is a Baire, that is, if $A \subset Diff(r, \omega, M)$ is a set (A contains the enumerable intersection of a family of dense and open subsets) the A is dense in $Diff(r, \omega, M)$

Now, we will define a generic property.

Definition 16. We will say that P is a generic property for the diffeomorphisms $Diff_{\omega}^{r}(M)$ if there exist a residual set $A \subset Diff(r, \omega, M)$such that every function $f \in A$ satisfies the property P.

We list the most important generic properties.

- P1: All periodic points are elliptical or hyperbolic ([1] and [3])
- P2: All elliptic points are Moser Stable ([45])
- P3: For two periodic and hyperbolic points x , y the intersection of the stable manifold of x, $W^s(x)$, and the unstable manifold of y, $W^u(y)$ is transversal ([46])
- P4: For any hyperbolic periodic point x, if Γ_1 and Γ_2 are two branches of the stable and unstable manifolds of x respectively, then $\overline{\Gamma_1} = \overline{\Gamma_2}$ ([1] and [3]).
- P5: In the sphere $\mathbb{S}^2$, if Γ_1 is a branch of the stable manifold of a periodic hyperbolic point x y Γ_2 is a branch of the unstable manifold then $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ ([1])

The following results show important properties of an area-preserving diffeomorphism and satisfy some of the generic properties stated above.

Proposition 2. Suppose that f is an area-preserving diffeomorphism of an open and connected subset $\Omega \subset \mathbb{S}^2$ and let $K \subset \Omega$ be an invariant, compact and connected set. If Γ is a branch of a hyperbolic periodic point, then $\Gamma \subset K$ ó Γ is completely contained in a single component of $\Omega - K$.

Proof. The proof is by contradiction. Suppose that Γ has a non-empty intersection with K and with $\Omega - K$. Let Ω_0 be a component of $\Omega - K$ containing a point in Γ. Since every component of $\Omega - K$ is a periodic domain then, Ω_0 must be. We can guarantee the existence of an open arc $\alpha \subset \Gamma \cup \Omega_0$ such that the end points of the curve are in K.

If $i > 0$, then $f^i(\alpha)$ and α must be disjoint, since α is a branch (stable or unstable), also $f^i(\alpha)$ cannot contain the endpoints of α.

Let V be the component of $\mathbb{S}^2 - K$ that contains Ω_0. As V is simply connected α separates it into two simply connected nonempty domains, V_1 y V_2, each of which has a non-empty intersection with Ω_0. Therefore, α separates Ω_0 into two connected components

$$\Omega_1 = \Omega_0 \cup V_1 \qquad \Omega_2 = \Omega_0 \cup V_2,$$

since Ω_0 is periodic, there exists $k > 0$ such that $f^k(\Omega_0) = \Omega_0$, as f is area-preserving, then $\Omega_1 \times \Omega_2$ is a non-errant set for $f^k \times f^k$.

That is, there exists an integer $n \geq 1$, such that.

$$f^{nk}(\Omega_1) \cap \Omega_1 \neq \emptyset \qquad f^{nk}(\Omega_2) \cap \Omega_2 \neq \emptyset,$$

Furthermore, the closure in Ω_0 de $f^{nk}(\Omega_1)$ is not contained in $\Omega_1$and intersects α. Similarly, α intersects $f^{nk}(\Omega_2)$ in Ω_0. From where it can be concluded that α intersects its image through f^{nk} which is impossible, that is, Γ cannot have empty intersection with K and with $\Omega - K$. In conclusion, if Γ belongs to $\Omega - K$ this must be in a single component.

Definition 17. A set S is a *minimal set* for f if it is closed, non-empty and invariant by f and if furthermore B is a subset of S closed, non-empty and invariant, then $S = B$.

A trivial example of a minimal set is the periodic orbits.

Definition 18. A periodic point z is *Moser stable* if it admits a fundamental system of neighborhoods that are closed disks D such that $f|_{\partial D}$ is a minimal set.

Theorem 14. Let $f \in Diff(r, \omega, M)$ be a function that satisfies $P1$, $P2$ properties and it does not have connections between sinks. Then

- The rotation number of any periodic domain simply with non-trivial connection is irrational,
- The branches of a hyperbolic point have the same closure.

Proof. See [1]

The following result was proved in [3] and gives us important properties of the stable and unstable varieties of a hyperbolic periodic point in $\mathbb{S}^2$.

Theorem 15. For all $r \geq 1$, there exists a generic set G_r in $Diff(r, \omega, M)$ such that for all f in G_r the following properties are satisfied.

- If z is a periodic point, any stable branch Γ of z has a non-empty cross-section with any unstable branch Γ' of z. Furthermore $\overline{\Gamma} = \overline{\Gamma}'$.
- If z and z' are two distinct hyperbolic periodic points such that z' belongs to the closure of a branch of z, then any stable branch Γ de z finds traversally any unstable branch Γ' de z'. Also, $\overline{\Gamma} = \overline{\Gamma}'$.

Any hyperbolic periodic point has a fundamental system of neighborhoods that are closed disks whose boundary is contained in $W^s(z) \cup W^u(z)$. This property and the λ-lemma are fundamental in the proof of the previous theorem.

Now, we are going to consider a diffeomorphism f of class C^r area-preserving defined in $\mathbb{S}^2$ that satisfies $P1$, $P2$ y $P3$, that is

- There are no degenerate periodic points.
- Every elliptic point is Moser stable.
- If x and $y$are hyperbolic periodic points, then $W^s(x)$ and $W^u(y)$ are transversal, and $\Gamma^s(x) \cap \Gamma^u(x) \neq \emptyset$.

Definition 19. A subset $\mathbb{A}_0 \subset \mathbb{S}^2$ is said to be a *ring domain,* if it is an open, connected set and its complement has exactly two components.

In other words, a ring domain is a set homeomorphic to an open ring. If $\mathbb{A}_0$ is a ring domain f-invariant, a essential continuum $K \subset \mathbb{A}_0$ is a set that separates the two components of the complement of $\mathbb{A}_0$.

Theorem 16. Suppose $z \in \mathbb{A}_0$ is a hyperbolic fixed point of f. Let $U(z)$ be the union of all closed topological disks in $\mathbb{A}_0$ hose boundary consists of a finite number of branches of z, then.

- $U(z)$ is open, connected, and invariant by f containing to z and its branches.
- If z is nonessential, then $U(z)$ is simply connected.
- If z is nonessential, then $U(z)$ is an essential annulus domain in $\mathbb{A}_0$.
- If $V \subset \mathbb{A}_0$ is any other open connected invariant set that is simply connected or an essential annulus domain, then $U(z) \subset V$, or $V \subset U(z)$, or $U(z) \cup V \neq \emptyset$.

Let $f: \mathbb{A}_0 \to \mathbb{A}_0$ be an area-preserving diffeomorphism of an annular domain, which extends continuously to the closure of $\mathbb{A}_0 \subset \mathbb{S}^2$. We consider a hyperbolic periodic point z of f. We assume for f the property $P3$.

Definition 20. If there is an essential closed curve in $\mathbb{A}_0$ consisting of a finite number of branch segments of z we say that z is essential, otherwise we say that z is non-essential.

It follows immediately that a hyperbolic periodic orbit Θ consists of points that are all essential or all nonessential.

Proposition 3. Suppose that $z \in \mathbb{A}_0$ is an essential hyperbolic fixed point of f. Then the union of any stable branch and any unstable branch of z contains an essential simple closed curve and a point on each component of the complement of this curve.

Proof. From the definition of an essential hyperbolic fixed point, we know that there is a curve $\beta \in \mathbb{A}_0$ consisting of branches of z that intersect transversely. The fact that any branch of a fixed point has a non-empty transversal intersection with any unstable branch tells us that this curve can be chosen as the union of any pair of branches, one stable and one unstable. Let α_0 be the boundary of one of the components of the complement of β that intersects a component of $\mathbb{S}^2 - \mathbb{A}_0$ and let α be the boundary of the component of the complement of α_0 that contains the other component of $\mathbb{S}^2 - \mathbb{A}_0$. Then α consists of branch segments of z that intersect transversely and is simple. So, it separates the two components of $\mathbb{S}^2 - \mathbb{A}_0$ and is therefore essential. Since α is in the interior of $U(z)$, the branches of z contain points of each of the components of α

4. Moser Generic Diffeomorphisms

Definition 21. Suppose that f is an area-preserving diffeomorphism of an annular domain $\mathbb{A}_0$ and let $P(q)$ be the set of periodic points of f with period q. It is said that f is a *Moser generic diffeomorphisms* if it satisfies the generic properties $P1$, $P2$, $P3$ y and also the Jordan curves that enclose neighborhoods of any periodic elliptic point have rotation number on not constant in any neighborhood of the elliptic point.

Because of the Kupka-Smale theorem ([46]) and the results in [1] and [3], we have the following result.

Theorem 17. If M is the sphere or the annulus, then the set of Moser generic diffeomorphism maps on M is a residual subset of the set $Diff(r,\omega,M)$.

For Moser generic diffeomorphisms we have the following result proved by Franks in [33]

Proposition 4. If p is a hyperbolic fixed point of a Moser generic diffeomorphism in $\mathbb{S}^2$, then

$$\overline{W^s(p)} = \overline{W^u(p)}$$

Now, we will prove the main result.

Theorem 18. Let f be a Moser generic diffeomorphism in $\mathbb{A} = \mathbb{S}^1 \times [0,1]$ that has no periodic points on the boundary of $\mathbb{A}$ and K an essential continuum invariant. If $\rho = \frac{p}{q}$ $(p, q$ relative primes) is in $\rho(K)$, then, the set of periodic points of f is not empty, and there exist an essential hyperbolic periodic point in K.

Proof. Since ρ is rational it follows immediately that f has a periodic point in K. In particular, if $\rho = 0$ there is a fixed point. We will show that there is an essential hyperbolic periodic point.

We denote $\hat{f}$ as the extension of f to the compactification of prime ends of $\mathbb{A}$, that is, the compactification obtained by adding two copies of $\mathbb{S}^1$ to the top and bottom of the annulus. The set $\widehat{\mathbb{A}}$ is homeomorphic to the sphere $\mathbb{S}^2$. We will denote as U^+ y U^- the components of $\widehat{\mathbb{A}}\backslash K$, these turn out tobe simply connected invariant domains since K is connected. From theorem 14 we can affirm that U^+ y U^- have irrational rotation number. Thus, we can find a closed disk $D^+ \subset U^+$ that contains all fixed point of $\hat{f}$ that are in U^+ y such that $I(\hat{f}, D^+) = 1$. Similarly, we can find a $D^- \subset U^-$ with the same property.

Since the set of periodic points of f is non-empty, by hypothesis, it must be finite and as a consequence of the Lefschetz formula not all fixed points can be elliptical.

Let z be a non-essential hyperbolic fixed point. Let z be a non-essential hyperbolic fixed point. We can associate to every nonessential hyperbolic fixed point za simply connected and invariant domain $U(z)$ that is formed by the union of topological disks formed by branches of the stable and unstable variety of z (theorem 16 and 17).

There exists a finite number of domains $U(z)$ that are disjoint and contain all non-essential fixed points of f . We can assume with rotation number equal to zero. In each of these domains we are going to choose a disk $\mathbb{D} \subset U(z)$ that contains all the fixed points in $U(z)$ and such that $I(f, \mathbb{D}) = 1$. For each domain $U(z)$ we have a disk with this property which we are going to denote as:

$$(\mathbb{D}_i)_{1 \leq i \leq k} \qquad k \geq 0.$$

Let us also consider the finite family of fixed points of f that are not in the interior of the $\mathbb{D}_i$, and they are essential,

$$\left(z_j\right)_{1<j<l} \qquad l \geq 0.$$

The idea is to prove that $l > 0$, thus we could guarantee that there are essential hyperbolic periodic points.

By the Lefschetz formula we have

$$\sum_{\omega \in Fix(\hat{f})} I\left(\hat{f}, \mathbb{D}_\omega\right) = 2$$

that is,

$$\sum_{j=1}^{l} I\left(\hat{f}, z_j\right) + \sum_{i=1}^{k} I\left(\hat{f}, \mathbb{D}_i\right) + I\left(\hat{f}, D^+\right) + I\left(\hat{f}, D^-\right) = 2$$

since $f = \hat{f}$ en $\mathbb{A}$ then, $I\left(\hat{f}, \mathbb{D}\right) = I(f, \mathbb{D}_i) = 1$

Thus,

$$\sum_{j=1}^{l} I\left(f, z_j\right) + k = 0,$$

and

$$\sum_{j=1}^{l} I\left(f, z_j\right) = -k,$$

since f preserves area and orientation $I\left(f, z_j\right) = \pm 1$, for all j and $I\left(f, z_j\right) = 1$ when z_j is elliptic. Therefore $l > 0$, then, to have equality there must be some z_i such that $I(f, z_i) \leq 0$, this z is hyperbolic and essential, since z is not in any of the $\mathbb{D}_i$. This completes the proof.

5. Conclusion

The rotation number plays a fundamental role in the study of diffeomorphisms defined in the circle, the annulus, and the sphere. Particularly, when the rotation number is rational, the existence of periodic orbits can be guaranteed under certain additional conditions depending on the dimension of space.

Finally, using the rotation number, it is possible to prove the existence of essential hyperbolic periodic points in the Moser generic diffeomorphisms defined in the ring.

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